University of Annaba–"M.I. Mathematiques et Informatique LMD"
First year undergraduation **2011–2012**

Analysis

Limits of real valued functions

Last update Monday 16th January 2012

Exercice 1. (Domain of definition) Let f and g be the following functions:

$$f(x) = \sqrt{\frac{(x-3)(x+2)}{x-1}}. \tag{1}$$

$$g(x) = \frac{x^2-16}{x-7}\sqrt{x^2-9}. \tag{2}$$

Find D_f, D_g, $D_{f\pm g}$, $D_{f/g}$, and D_{fg}.

Exercice 2. (Domain of definition) Find D_f:

1. $$f(x) = \tan(x). \tag{3}$$
2. $$f(x) = \frac{1}{\sqrt{1-|x|}}. \tag{4}$$
3. $$f(x) = \frac{1}{x(x-1)}. \tag{5}$$
4. $$\exp([f(x)]^2) = x,\ \ f(x) \geq 0. \tag{6}$$
5. $$f(x) = \frac{\sin x}{x}. \tag{7}$$

Exercice 3. (Proof of a limit) Find the limit $\lim_{x\to x_0} f(x)$ and justify your answer using $\varepsilon - \delta$

1. $$f(x) = x^2 + x + 1,\ \ x_0 = 1. \tag{8}$$
2. $$f(x) = \frac{x^3-8}{x-2},\ \ x_0 = 2. \tag{9}$$
3. $$f(x) = \sqrt{x},\ \ x_0 = 4. \tag{10}$$
4. $$f(x) = \frac{1}{x^2-1},\ \ x_0 = 0. \tag{11}$$
5. $$f(x) = \frac{x^3-1}{(x-1)(x-2)} + x,\ \ x_0 = 1. \tag{12}$$

Exercice 4. (Definition of the limit) Let $K > 0$ a given positive real. Prove that the definition of the limit $\varepsilon - \delta$ is equivalent to for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$| f(x) - l| \leq K\varepsilon \tag{13}$$

if

$$0 < |x - x_0| \leq \delta. \tag{14}$$

Exercice 5. (Definition of the limit) Find $\lim_{x\to x_0^-} f(x)$ and $\lim_{x\to x_0^+} f(x)$ if they exist. Use $\varepsilon - \delta$ proofs, where applicable, to justify your answers.

1. $$f(x) = \frac{x + |x|}{x}, \quad x_0 = 0. \tag{15}$$

2. $$f(x) = x \cos\frac{1}{x} + \sin\frac{1}{x} + \sin\frac{1}{|x|}, \quad x_0 = 0. \tag{16}$$

3. $$f(x) = \frac{|x-1|}{x^2 + x - 2}, \quad x_0 = 1. \tag{17}$$

Exercice 6. (Proof of the limit) Use the definion $\varepsilon - \delta$ of the limit to prove that

1. $$\lim_{x\to 0}(5x - 1) = -1. \tag{18}$$

2. $$\lim_{x\to x_0} x^n = x_0^n, \ \forall n \in \mathbb{N}\setminus\{0\}. \tag{19}$$

3. $$\lim_{x\to -2} \frac{x^2 + x + 1}{1 - x} = 1. \tag{20}$$

4. $$\lim_{x\to +\infty} \frac{1}{\sqrt{x}} = 0. \tag{21}$$

5. $$\lim_{x\to +\infty} \frac{1}{x^2 + x + 1} = 0. \tag{22}$$

6. $$\lim_{x\to -\infty} x^2 + x + 1 = +\infty. \tag{23}$$

Exercice 7. (Computation the limit) Compute the following limits when $x \to +\infty$:

1. $$f(x) = \frac{\cos x^2 + \ln 2x - x^3}{3x^3 + \sin x - x}. \tag{24}$$

2. $$f(x) = \frac{\ln(5x+1) - \sin x^2 + \exp(x)}{\exp(2x) + \exp(\sin x) - \ln x}. \tag{25}$$

3. $$f(x) = \frac{[x]}{x}. \tag{26}$$

4.

$$f(x) = x\ln(x+1) - x\ln(x). \tag{27}$$

5.

$$f(x) = \left(1+\frac{a}{x}\right)^x,\ a\in\mathbb{R}\setminus\{0\}. \tag{28}$$

6.

$$f(x) = x\exp\left(\frac{1}{x}\right) - x. \tag{29}$$

Exercice 8. (Operations o and O)

1. Prove that when $x\to 0$, we have
   (a) $x^2 = o(x)$.
   (b) $\frac{1}{x} = o\left(\frac{1}{x^2}\right)$.
   (c) $\cos(x) - 1 + \tan^2(x) = o(x)$.
   (d) $x\sin(x) = O(x^2)$.
   (e) $\frac{1}{1-x} = 1 + x + o(x)$.
   (f) $x^2\sin\frac{1}{x} + x^3 = o(\tan x)$.
2. Prove that when $x\to+\infty$, we have
   (a) $x\sin(x) = O(x)$.
   (b) $x^2 = O(x^3)$.
   (c) $\frac{1}{1-x} = -\frac{1}{x} + O\left(\frac{1}{x^2}\right)$.
   (d) $\ln^\alpha x = o(x^\beta)$, for all $\alpha\in\mathbb{R}$ and $\beta>0$.

Exercice 9. (Some theortical results) Let f be a given function defined on the interval $]x_0-\alpha, x_0+\alpha[$, where x_0 and $\alpha>0$ are given

1. Prove that if $\lim_{h\to 0} f(x_0+h)\in\mathbb{R}$ then $\lim_{h\to 0}(f(x_0+h) - f(x_0-h)) = 0$.
2. Provide an example in which the inverse of the previous statement is not true.

Exercice 10. (Existence of the limit) Prove that $\lim_{x\to+\infty}\cos(x)$ does not exist.

Exercice 11. (Equivalence of functions) Provide simple equivalent functions for the following ones when $x\to+\infty$ and $x\to 0$

1. $\ln(x) + \ln^2(x)$.
2. $\exp(x) + \sin(x)$.
3. $\sqrt{x+1} - \sqrt{x}$.

Exercice 12. (Equivalence of functions) Provide simple equivalent functions for the following ones

1. $\sin(x^2)$, when $x\to 0$.

2. $\ln(\cos(x))$, when $x \to 0$.
3. $\dfrac{\tan(x)\ln(x+1)}{\sqrt{x^2+x}-1}$, when $x \to 0$.
4. $\ln(x^2+x+1)$, when $x \to \infty$.
5. $\sqrt{\ln(x+1)}+\sqrt{x}$, when $x \to \infty$.