University of Annaba–"M.I. Mathematiques et Informatique LMD"
First year undergraduation **2011–2012**

Analysis

Continuity

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Exercice 1. (Study of continuity) Let f be the function defined by $f(0) = 1$ and $f(x) = x\mathrm{E}\left(\frac{1}{x}\right)$ (E denotes the integer part function).
Study the contnuity of the function f. In particular study the continuity on $x = 0$ and on the points $\{\dfrac{1}{m},\ m \in \mathbb{Z} \setminus \{0\}\}$.

Exercice 2. (Study of continuity) Let f be the function defined by

$$f(x) = \frac{x}{1+x},\ x \in]-1, +\infty[\setminus\mathbb{Q}, \tag{1}$$

and for $x = \frac{p}{q} \in \mathbb{Q}$ with $q > 0$ and the bigest divisor between p and q is one

$$f(x) = \frac{p}{p+q+1}. \tag{2}$$

Prove that f is continuous on $\mathbb{R} \setminus \mathbb{Q}$.

Exercice 3. (Maximum of continuous function) Let f be a given function defined on $\mathbb{R}^+$ into $\mathbb{R}^+$ such that

$$\lim_{x\to+\infty} f(x) = 0. \tag{3}$$

Prove that for all $a \geq 0$, there exists $b \geq a$ such that

$$f(b) = \max_{x\in[a,+\infty[} f(x). \tag{4}$$

Exercice 4. (Some examples) Study the continuity of the following functions:

1.
$$f(x) = 0,\ x < 0, \tag{5}$$
and
$$f(x) = 1,\ x \geq 0. \tag{6}$$

2.
$$f(x) = \frac{1}{x},\ x \neq 0, \tag{7}$$
and
$$f(x) = 0,\ x = 0. \tag{8}$$

3.
$$f(x) = \sin\frac{1}{x},\ x \neq 0, \tag{9}$$
and
$$f(x) = 0,\ x = 0. \tag{10}$$

Exercice 5. (Other examples) Study the continuity of the following functions:

1.
$$f(x) = \frac{1}{x^2}, \; x \neq 0, \tag{11}$$

and
$$f(x) = 1, \; x = 0. \tag{12}$$

2.
$$f(x) = \frac{x^2 - x - 2}{x - 2}, \; x \neq 2, \tag{13}$$

and
$$f(x) = 5, \; x = 2. \tag{14}$$

3.
$$f(x) = \mathrm{E}(x). \tag{15}$$

Exercice 6. (Constant functions) Let a be a given real number such that $a > 1$ and let f be a given function such that $f(ax) = f(x)$, for all $x \in \mathbb{R}$.

1. Prove that if f is a continuous function, then f is constant.
2. What about the case when f is not continuous.

Exercice 7. (Extension by continuity) Let f given by
$$f(x) = \sin\frac{1}{x^3} \exp\left(-\frac{1}{|x|}\right), \; x \neq 0. \tag{16}$$
Justify that f can be extended by continuity to $\mathbb{R}$.

Exercice 8. (Density and continuity) Let f be a given continuous function such that $f(x) = 0$ for all $x \in \mathbb{Q}$.

1. Justify that for any real x, there exists a sequence $x_n \in \mathbb{Q}$ such that $\lim\limits_{n\to+\infty} x_n = x$.
2. Use the previous item to justify that $f(x) = 0$ for all $x \in \mathbb{R}$.

Exercice 9. (Continuity) Prove that the function $4x^3 - 6x^2 + 3x - 2$ vanishes between 1 and 2.