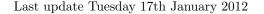
## University of Annaba–"M.I. Mathematiques et Informatique LMD" First year undergraduation

## Analysis

## Continuity



**Exercise 1.** (Study of continuity) Let f be the function defined by f(0) = 1 and  $f(x) = x \mathbb{E}\left(\frac{1}{x}\right)$  (E denotes the integer part function).

Study the continuity of the function f. In particular study the continuity on x = 0 and on the points  $\{\frac{1}{m}, m \in \mathbb{Z} \setminus \{0\}\}$ .

**Exercice 2.** (Study of continuity) Let *f* be the function defined by

$$f(x) = \frac{x}{1+x}, \ x \in ]-1, +\infty[\backslash \mathbb{Q}, \tag{1}$$

and for  $x=\frac{p}{q}\in\mathbb{Q}$  with q>0 and the bigest divisor between p and q is one

$$f(x) = \frac{p}{p+q+1}.$$
(2)

Prove that f is continuous on  $\mathbb{R} \setminus \mathbb{Q}$ .

**Exercice 3.** (Maximum of continuous function) Let f be a given function defined on  $\mathbb{R}^+$  into  $\mathbb{R}^+$  such that

$$\lim_{x \to \infty} f(x) = 0. \tag{3}$$

Prove that for all  $a \ge 0$ , there exists  $b \ge a$  such that

$$f(b) = \max_{x \in [a, +\infty[} f(x).$$
(4)

**Exercice 4.** (Some examples) Study the continuity of the following functions:

1.

$$f(x) = 0, \ x < 0, \tag{5}$$

and

$$f(x) = 1, \ x \ge 0. \tag{6}$$

$$f(x) = \frac{1}{x}, \ x \neq 0,\tag{7}$$

and

$$f(x) = 0, \ x = 0.$$
(8)

3.

$$f(x) = \sin\frac{1}{x}, \ x \neq 0,\tag{9}$$

and

$$f(x) = 0, \ x = 0. \tag{10}$$

**Exercice 5.** (Other examples) Study the continuity of the following functions:

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1.

$$f(x) = \frac{1}{x^2}, \ x \neq 0,$$
 (11)

and

$$f(x) = 1, \ x = 0. \tag{12}$$

2.

$$f(x) = \frac{x^2 - x - 2}{x - 2}, \ x \neq 2,$$
(13)

and

$$f(x) = 5, \ x = 2.$$
 (14)

3.

$$f(x) = \mathcal{E}(x). \tag{15}$$

**Exercise 6.** (Constant functions) Let a be a given real number such that a > 1 and let f be a given function such that f(ax) = f(x), for all  $x \in \mathbb{R}$ .

- 1. Prove that if f is a continuous function, then f is constant.
- 2. What about the case when f is not continuous.
- **Exercice 7.** (Extension by continuity) Let f given by

$$f(x) = \sin\frac{1}{x^3} \exp\left(-\frac{1}{|x|}\right), \ x \neq 0.$$
(16)

Justify that f can be extended by continuity to  $\mathbb{R}$ .

**Exercice 8.** (Density and continuity) Let f be a given continuous function such that f(x) = 0 for all  $x \in \mathbb{Q}$ .

- 1. Justify that for any real x, there exists a sequence  $x_n \in \mathbb{Q}$  such that  $\lim_{n \to +\infty} x_n = x$ .
- 2. Use the previous item to justify that f(x) = 0 for all  $x \in \mathbb{R}$ .

**Exercice 9.** (Continuity) Prove that the function  $4x^3 - 6x^2 + 3x - 2$  vanishes between 1 and 2.