University of Annaba–"M.I. Mathematiques et Informatique LMD"
First year undergraduation **2011–2012**

Analysis

Derivability

Last update Tuesday 17th January 2012

Exercice 1. (Study of derivability) Study the derivability of the following functions:

1. $$f(x) = \arctan(1 - x^2). \tag{1}$$

2. $$f(x) = x^2 \sin\left(\frac{1}{x}\right), \ x \neq 0, \tag{2}$$

   and

   $$f(0) = 0. \tag{3}$$

Exercice 2. (Study of derivability) Let f be a given function defined on some interval $]-a, a[$ (with $a > 0$) and satisfying, for some $k \in]0, 1[$ and $L \in \mathbb{R}$

$$\lim_{x \to 0} \frac{f(x) - f(kx)}{x} = L. \tag{4}$$

1. Prove that f has derivative at $x = 0$
2. Compute $f'(0)$.

Exercice 3. (A derivative vanishes everywhere) Consider the following function

$$f(x) = \arctan \frac{x + a}{1 - ax} - \arctan x. \tag{5}$$

1. Compute the domain of definition of f.
2. Study the derivability of f
3. Compute the derivative of f and deduce that

$$f(x) = -\arctan \frac{1}{a} + \frac{\pi}{2}, \ x \in]-\infty, \frac{1}{a}[, \tag{6}$$

$$f(x) = -\arctan \frac{1}{a} - \frac{\pi}{2}, \ x \in]\frac{1}{a}, +\infty[. \tag{7}$$

Exercice 4. (Rolle Theorem) Let f be a given differentiable function on $\mathbb{R}$ such that f vanishes on n roots. Prove that f' has at least $n - 1$ roots.

Exercice 5. (Inequalities) Prove the following inequalities

1. $$\ln(1 + x) \leq x, \ x \in]-1, +\infty[. \tag{8}$$

2.

$$\exp(x) \geq 1 + x,\ x \in \mathbb{R}. \tag{9}$$

3.

$$\sin(x) \leq x,\ x \in [0, +\infty[. \tag{10}$$

4.

$$\frac{x}{1+x^2} \leq \arctan(x) \leq x,\ x \in [0, +\infty[. \tag{11}$$

Exercice 6. (Derivative) Let f be the following function

$$f(x) = \arctan \sqrt{\frac{1-x}{1+x}}. \tag{12}$$

1. Compute the domain of definition of f.
2. Study the derivability of f
3. Justify that $f' \equiv 0$ and deduce a convenient expansion for f.
4. Justify that for all x in the domain of definition can be written as $x = \cos t$ where $t \in [0, \pi[$. Use this fact to confirm the expansion found in the previous item for f.

Exercice 7. (Continuity and derivability) Consider the following function, for $n \in \mathbb{N} \setminus \{0\}$

$$f(x) = x^n \sin \frac{1}{|x|},\ x \neq 0. \tag{13}$$

1. Justify that f can be extended by continuity on 0 to all $\mathbb{R}$.
2. What are the values of n such that f is a function of class $\mathcal{C}^1$

Exercice 8. (Continuity and derivability) Consider the following function, for $n \in \mathbb{N}$

$$f(x) = \frac{\sin(2n+1)x}{\sin x}. \tag{14}$$

1. Determine the domain of definition of f.
2. Prove that $f(-x) = f(x)$ for all x in the domain of definition of f.
3. Is f periodic ?
4. Can f be extended to $\mathbb{R}$?

Exercice 9. (Derivative) Let f be a given function two times differentiable such that for some $k > 0$

$$f(y) - f(x) = (y-x)f'(x) + \frac{(y-x)^2}{2} f''(\frac{x+ky}{1+k}),\ \forall x, y \in \mathbb{R}. \tag{15}$$

1. Prove that $f^{(4)} \equiv 0$.
2. Deduce that f is a polynome of degree three.

Exercice 10. (Relations using Derivatives)

1. Prove that

$$(\arccos(x) + \arcsin(x))' = 0,\ \forall x \in]-1, 1[. \tag{16}$$

2. Deduce that

$$\arccos(x) + \arcsin(x) = \frac{\pi}{2},\ \forall\, x \in [-1, 1]. \tag{17}$$

Exercice 11. (Relations) Prove the following rules

1. $$\cos(\arcsin(x)) = \sqrt{1 - x^2},\ \forall\, x \in [-1, 1]. \tag{18}$$

2. $$\tan(\arcsin(x)) = \frac{x}{\sqrt{1 - x^2}},\ \forall\, x \in]-1, 1[. \tag{19}$$

3. $$\sin(\arccos(x)) = \sqrt{1 - x^2},\ \forall\, x \in [-1, 1]. \tag{20}$$

4. $$\tan(\arccos(x)) = \frac{\sqrt{1 - x^2}}{x},\ \forall\, x \in [-1, 1] \setminus \{0\}. \tag{21}$$

5. $$\cos(\arctan(x)) = \frac{x}{\sqrt{1 + x^2}},\ \forall\, x \in \mathbb{R}. \tag{22}$$

6. $$\sin(\arctan(x)) = \frac{x}{\sqrt{1 + x^2}},\ \forall\, x \in \mathbb{R}. \tag{23}$$