University of Annaba–"Ecole Preparatoire aux Sciences et Techniques"
Second year undergraduation **2010–2011**

Functions of Several Variables

Limits and Continuity

Exercise 1. Draw the open balls of center $(0,0)$ and diameter 1 in $\mathbb{R}^2$ in the three norms of $\mathbb{R}^2$.

Exercise 2. Give the domain of definition of each of the following functions: :

1. $$f(x,y) = \ln\left(\frac{x^2+y^2}{x+y}\right), \tag{1}$$
2. $$g(x,y) = \frac{2x-1}{\sqrt{1-x^2-y^2}}, \tag{2}$$
3. $$h(x,y) = \cos(z) + \frac{z+2}{\sin(2x)}. \tag{3}$$

Exercise 3. Find the domain of definition of the following functions and study the limit of f in $(0,0)$:

1. $$f(x,y) = \frac{1+x+y}{x^2-y^2}. \tag{4}$$
2. $$f(x,y) = \frac{1+x^2+y^2}{y}\sin(y). \tag{5}$$
3. $$f(x,y) = \frac{1-\cos\sqrt{xy}}{y}. \tag{6}$$

Exercise 4. Let us consider the following function :

$$f(x,y) = \begin{cases} \frac{x^2y^2}{x^2y^2+(x-y)^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases} \tag{7}$$

1. Show that
$$\lim_{y\to 0}(\lim_{x\to 0} f(x,y)) = \lim_{x\to 0}(\lim_{y\to 0} f(x,y))0. \tag{8}$$
2. Can we deduce that
$$\lim_{(x,y)\to(0,0)} f(x,y) = 0. \tag{9}$$

Exercise 5. Study the following limits:

1. $$\lim_{(x,y)\to(0,0)} \frac{3xy}{x^2y^2}, \tag{10}$$
2. $$\lim_{(x,y)\to(0,0)} \frac{3x-y^2}{x^2+y^2}, \tag{11}$$

3.

$$\lim_{(x,y)\to(0,0)} \frac{3x+2y}{x^2+y^2}, \tag{12}$$

Exercise 6. Decide if the following functions can be extended by continuity on the point $\mathcal{P}$:

1.

$$f(x,y) = \frac{x^2-2y}{x^2+y^2}, \; \mathcal{P} = (0,0), \tag{13}$$

2.

$$f(x,y) = \frac{x+5}{(x+5)^2+y^2}, \; \mathcal{P} = (-5,0), \tag{14}$$

3.

$$f(x,y) = \frac{x^7+x^4y+x^3y}{x^6+x^3y+y^2}, \; \mathcal{P} = (0,0), \tag{15}$$

Exercise 7. Consider the following function

$$f(x,y) = \begin{cases} \frac{x^2y}{x^4-2x^2y+3y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases} \tag{16}$$

1. Study the limit of f when (x,y) tends to $(0,0)$ with $y = ax$
2. Study the limit of f when (x,y) tends to $(0,0)$ with $y = x^2$
3. Show that f has no limit at the point $(0,0)$.

Exercise 8. Study the continuity of the function f given by

$$f(x,y) = \begin{cases} y\sin\frac{x}{y}, & y \neq 0 \\ 0, & y = 0. \end{cases} \tag{17}$$

1. Decide if the following function can be extended by continuity to function defined everywhere on $\mathbb{R}^2$:

$$g(x,y) = \frac{\sin xy}{y}, \quad y \neq 0 \tag{18}$$

Exercise 9. Study the continuity of the function f given by

1.

$$f(x,y) = \begin{cases} \frac{(x+y)^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases} \tag{19}$$

2.

$$f(x,y) = \begin{cases} (x+y)^2 \sin\left(\frac{1}{x^2+y^2}\right), & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases} \tag{20}$$

3.

$$f(x,y) = \begin{cases} \frac{x^3+y^3}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases} \tag{21}$$

4.

$$f(x,y) = \begin{cases} xy\ln(x^2+y^2), & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases} \tag{22}$$

Exercise 10. Study the continuity of the function f given by

$$f(x,y) = \begin{cases} \frac{|y|}{x^2}\exp\left(\frac{|y|}{x^2}\right), & x \neq 0 \\ 0, & x = 0. \end{cases} \tag{23}$$