University of Annaba–"Ecole Preparatoire aux Sciences et Techniques"
Second year undergraduation **2010–2011**

Functions of Several Variables

Derivatives and differentiability

Exercise 1. Compute the partial derivatives of the following functions on their domaines of definition:

1.
$$f(x,y) = (x^2 + y^2)\exp(-xy), \tag{1}$$

2.
$$f(x,y) = \frac{xy}{x^2+y^2}. \tag{2}$$

Exercise 2. Consider the function f defined by

$$f(x,y) = \begin{cases} \frac{x^3+y^3}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases} \tag{3}$$

1. Is f continuous on $(0,0)$?
2. Compute the partial derivatives of f. Is f of class $\mathcal{C}^1$?

Exercise 3. Consider the function f defined by

$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases} \tag{4}$$

1. Is f continuous?
2. Does the directional derivative $D_v f(0,0)$ exist for all $v = (a,b) \in \mathbb{R}^2 \setminus \{(0,0)\}$. We deduce then that f is not differentiable in $(0,0)$.
3. Compute the partial derivatives of f in $(0,0)$.
4. Use the previous question combined with the definition of differentiability to prove that f is not differentiable in $(0,0)$.

Exercise 4. Consider the function f defined by

$$f(x,y) = \begin{cases} \frac{x^2 y}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases} \tag{5}$$

1. Show that f is continuous everywhere?
2. Show that the directional derivative $D_v f(0,0)$ exist for all $v = (a,b) \in \mathbb{R}^2 \setminus \{(0,0)\}$ but f is not differentiable in $(0,0)$.

Exercise 5. Compute the second partial derivatives of the following functions and decide if the Schwartz Theorem can be applied on the their domains of definition:

1.
$$f(,y) = y \ln x, \tag{6}$$

2.
$$f(x,y) = \frac{1}{\sqrt{x^2+y^2+z^2}}, \tag{7}$$

3.
$$f(x,y) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(x^2+y^2)\right). \tag{8}$$

Exercise 6. Compute the directional derivative of $f(x,y) = xy^2$ with respect to the vector $v = (1,-2)$ on the point $M = (2,1)$.

Exercise 7. Let us consider

$$f(x,y) = y^3 \ln(x). \tag{9}$$

Compute the partial derivatives $\frac{\partial^4 f}{\partial x \partial y^3}, \frac{\partial^4 f}{\partial x^2 \partial y^2}, \frac{\partial^4 f}{\partial y \partial x \partial y^2}$.

Exercise 8. Let us consider

$$f(x,y) = \begin{cases} \frac{4xy(x^2-y^2)}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases} \tag{10}$$

1. Compute the partial derivatives $\frac{\partial f}{\partial x}(x_0,y_0)$, $\frac{\partial f}{\partial y}(x_0,y_0)$ for $(x_0,y_0) \neq (0,0)$.
2. Compute the partial derivatives $\frac{\partial f}{\partial x}(0,0)$, $\frac{\partial f}{\partial y}(0,0)$, $\frac{\partial^2 f}{\partial x \partial y}(0,0)$, $\frac{\partial^2 f}{\partial y \partial x}(0,0)$.

Exercise 9. Compute the differential and the pertial derivatives of f when they exist in each case of the following:

1.
$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases} \tag{11}$$

2.
$$f(x,y) = \begin{cases} (x^2+y^2)^x, & (x,y) \neq (0,0) \\ 1, & (x,y) = (0,0). \end{cases} \tag{12}$$