Some highlights on the limit in two dimensional space

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Aim of these notes

The aim of these notes is to provide some highlights on the relation between the limit of a real function defined on $\mathbb{R}^2$ at some point (x_0, y_0), the limit along every line through (x_0, y_0), and the known limit using the polar coordinates. We will see that the criterion of the limit of a real function defined on $\mathbb{R}^2$ at some point (x_0, y_0) is equivalent to the known criterion on the limit using the polar coordinates but with a uniform convergence with respect to the angle θ. However, the criterion of the limit of a real function defined on $\mathbb{R}^2$ at some point (x_0, y_0) implies (but not equivalent) to the limit along every line through (x_0, y_0).

1 Introduction

Throughout these notes, we assume that f is a function defined on some ball (for the sake of simplicity and in order to use simply polar coordinates, we choose the Euclidean norm) $\mathcal{B}((x_0, y_0), r) \subset \mathbb{R}^2$ except perhaps on (x_0, y_0) (in fact the definition of the limit at a point requires the definition of the function under consideration on some neighborhood of the point). Using the translation $(x, y) \to (x - x_0, y - y_0)$, any computation concerning the limit at (x_0, y_0) can be considered as limit at $(0, 0)$. For this reason, we can assume without loss of generality that $(x_0, y_0) = (0, 0)$.
The following definition of the limit can be used when it is needed:

Definition 1 (Definition of the limit) *The limit* $\lim\limits_{(x,y)\to(0,0)} f(x, y)$ *is said to be equal to* l*, for some* $l \in \mathbb{R}$ *(the limit* l *can also be considered as infinity), if for all* $\varepsilon > 0$*, there exists some* $\eta > 0$ *(the real number* $\eta > 0$ *can be chosen sufficiently small in a such way that* $\eta < r$ *and consequently the function* f *is defined for all* (x, y) *such that* $\sqrt{x^2 + y^2} \le \eta$*), such that for all* (x, y) *satisfying* $\sqrt{x^2 + y^2} \le \eta$ *we have*

$$| f(x, y) - l| < \varepsilon. \tag{1}$$

2 Some known rules to test the limit

Let $y = y(x)$ an arbitrary curve such that $(x, y(x)) \subset \mathcal{B}(0, r)$ (recall that f is a function defined on the ball $\mathcal{B}(0, r)$). Assume that $\lim\limits_{(x,y)\to(0,0)} f(x, y) = l$ and $\lim\limits_{x\to 0} y(x) = 0$. Therefore, thanks to the definition of the limit $\lim\limits_{x\to 0} f(x, y(x)) = l$.
One of the particular and easy cases is take $y = kx$, where $k \in \mathbb{R}$ is arbitrary. If we find that $\lim\limits_{x\to 0} f(x, kx)$ does not exist or depending on k, then $\lim\limits_{(x,y)\to(0,0)} f(x, y)$ does not exist.
The $\lim_{x\to 0} f(x, kx)$ called the limit along the line $y = kx$.
Two important notes should be mentioned:

1. If $\lim\limits_{x\to 0} f(x, kx)$ does not exist for some k or depending on k, then $\lim\limits_{(x,y)\to(0,0)} f(x, y)$ does not exists.
2. The fact that $\lim\limits_{x\to 0} f(x, kx) = l$, for all k, and for some given l does not imply that $\lim\limits_{(x,y)\to(0,0)} f(x, y) = l$ as it is shown in the following example:

$$f(x,y) = \frac{xy^2}{x^2+y^4}. \tag{2}$$

The function f is defined everywhere in $\mathbb{R}^2 \setminus \{(0,0)\}$.

We remark that

$$f(x,kx) = \frac{xk^2x^2}{x^2+k^4x^4} = \frac{k^2x^3}{x^2(1+k^4x^2)} = \frac{k^2x}{1+k^4x^2} \to 0, \text{ as } x \to 0. \tag{3}$$

On the other hand

$$f(x^2,x) = \frac{x^2x^2}{x^4+x^4} = \frac{1}{2} \not\to 0, \text{ as } x \to 0. \tag{4}$$

The two facts (3) and (4) implies that $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist even the limit along every line $y = kx$ is 0.

3 The use of polar coordinates to compute the limit

The following theorem is useful:

Theorem 3.1 (The use of polar coordinates to compute the limit) *The following statements are equivalent:*

A. $\lim_{(x,y)\to(0,0)} f(x,y) = l$

B. $\lim_{\rho\to 0} f(\rho\cos\theta, \rho\sin\theta) = l$, *uniformly in* θ.

Remark 3.1 (What does it mean the sentence "convergence uniformly in θ") *The statement* $\lim_{r\to 0} f(\rho\cos\theta, \rho\sin\theta) = l$, *uniformly in* θ *means that for all* $\varepsilon > 0$ *there exists some* $\eta > 0$ *only depending on* ε *such that for all* ρ *satisfying* $0 < \rho \le \eta$ *we have*

$$|\,f(\rho\cos\theta, \rho\sin\theta) - l| < \varepsilon, \ \forall\,\theta. \tag{5}$$

The statement $\lim_{\rho\to 0} f(\rho\cos\theta, \rho\sin\theta) = l$, *uniformly in* θ, *is replaced in some literature by the fact that* $\lim_{\rho\to 0} f(\rho\cos\theta, \rho\sin\theta) = l$ *does not depend on* θ. *However the statement* $\lim_{\rho\to 0} f(\rho\cos\theta, \rho\sin\theta) = l$, *uniformly in* θ, *is more precise since the constant of* $\eta > 0$, *such that* $0 < \rho \le \eta$ *implies inequality* (5), *should be independent of* θ.

Remark 3.2 (The convergence along every line and the convergence using polar coordinates) *As we mentioned, that the use of the limit along lines through* $(0,0)$ *serves us*

1. *to predict the a priori limit since the computation of* $f(x,kx)$ *is easy*
2. *to decide that the limit* $\lim_{(x,y)\to(0,0)} f(x,y)$ *does not exist when* $\lim_{x\to 0} f(x,kx)$ *does not exist for some* k *or it is depending on* k

However, if $\lim_{x\to 0} f(x,kx) = l$, *for some fixed* l, *does not imply that* $\lim_{(x,y)\to(0,0)} f(x,y) = l$.
Whereas, the computation of $\lim_{\rho\to 0} f(\rho\cos\theta, \rho\sin\theta)$ *serves us to decide whether there is a convergence or not.*

Proof of Theorem 3.1

1. Statement **A.** implies statement **B.** is clear when we set $(x,y) = (\rho\cos\theta, \rho\sin\theta)$ in the Definition 1 of the limit
2. Statement **B.** implies statement **A.** is clear when we set $(x,y) = (\rho\cos\theta, \rho\sin\theta)$ in (5).

The proof of Theorem 3.1 is completed. ∎

4 Some examples

1. First example: let us consider the function f given by (2). We have remarked, thanks to (3) and (4) implies that $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist even the limit along every line $y = kx$ is 0. In polar coordinates, we have

$$f(\rho\cos\theta, \rho\sin\theta) = \frac{(\rho\cos\theta)(\rho^2\sin^2\theta)}{(\rho^2\cos^2\theta) + (\rho^4\sin^4\theta)} = \frac{\rho\cos\theta\sin^2\theta}{\cos^2\theta + \rho^2\sin^4\theta}. \tag{6}$$

   We remark that $\lim_{\rho\to 0} f(\rho\cos\theta, \rho\sin\theta) = 0$ but this convergence is not uniform in in θ.

2. Second example:let us consider the function f given by

$$f(x,y) = \frac{x^4 - y^4}{x^2 + y^2}. \tag{7}$$

   The function f is defined everywhere in $\mathbb{R}^2 \setminus \{(0,0)\}$.
   In polar coordinates, we have, using $\cos^2\theta + \sin^2\theta = 1$

$$f(\rho\cos\theta, \rho\sin\theta) = \frac{\rho^4\cos^4\theta - \rho^4\sin^4\theta}{\rho^2\cos^2\theta + \rho^2\sin^2\theta} = \rho^2(\cos^4\theta - \sin^4\theta). \tag{8}$$

   Using the fact that $0 \leq \cos^4\theta \leq 1$ and $0 \leq \sin^4\theta \leq 1$, the previous expression yields

$$|\, f(\rho\cos\theta, \rho\sin\theta)| \leq 2\rho^2, \;\; \forall\,\theta \in \mathbb{R}. \tag{9}$$

   Which implies that $\lim_{\rho\to 0} f(\rho\cos\theta, \rho\sin\theta) = 0$, uniformly in θ and consequently, thanks to Theorem 3.1, $\lim_{(x,y)\to(0,0)} f(x,y) = 0$