Benchmark on discretization schemes for anisotropic diffusion problems on general grids (December 10th)

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ABSTRACT. *We present here a number of test cases and meshes which were designed to form a benchmark for finite volume schemes. We address a two-dimensional anisotropic diffusion problem, which is discretized on general, possibly nonconforming meshes. In all cases, the diffusion tensor is taken to be anisotropic, and at times heterogenous and/or discontinuous. The meshes are either triangular or quadrangular. The results which are expected from the participants to the benchmark range from the number of unknowns, the errors on the fluxes or the minimum and maximum values, to the order of convergence (when available).*

KEYWORDS: *Anisotropic medium, diffusion process, finite volume schemes, benchmark*

1. Introduction

The aim of this benchmark is to provide a number of test cases in order to compare the properties (convergence, robustness...) of existing discretization schemes for anisotropic diffusion problems using general grids.

In all test cases except test 8, the domain Ω is the unit square. The boundary of the domain is divided into $\partial\Omega = \Gamma_D \cup \Gamma_N$ where Dirichlet (resp. Neumann) boundary conditions are given on Γ_D (resp. on Γ_N).

The considered diffusion problem is formulated as:

$$\begin{cases} -\nabla \cdot (\mathbf{K}\nabla u) = f \text{ on } \Omega, \\ u = \bar{u} \text{ on } \Gamma_D, \\ \mathbf{K}\nabla u \cdot \mathbf{n} = g \text{ on } \Gamma_N, \end{cases} \tag{1}$$

where $\mathbf{K} : \Omega \to \mathbb{R}^{2\times 2}$ is the diffusion (or permeability) tensor, f the source term, $\bar{u}$ and g the Dirichlet and Neumann boundary conditions, and $\mathbf{n}$ denotes the outward unit normal vector to Γ_N.

For each test case, we propose some meshes which will be used for the comparison between the various schemes. The corresponding data files are given in the different formats which are explained in the README file of the web site. For any related question to these meshes or formats please get in touch with one of the organizers (`herbin@cmi.univ-mrs.fr`, `fhubert@cmi.univ-mrs.fr`).

The scheme which is used should be described in the introduction of the paper, along with the known (mathematically proven) results of convergence, stability, or error estimates.

In order to facilitate the programming, some FORTRAN subroutines giving the source terms, the diffusion tensors, the exact solutions and their derivatives (when available), are given in the file `sources.f90` available on this web site.

One may submit a benchmark paper even if only a partial number of test cases and meshes are performed. The file `template.tex` should be used for submission and display of the numerical results.

Please make sure to register on the web site for the benchmark if you intend to participate, since all updates will be sent to the benchmark mailing list produced by the registrations.

2. The tests

Test 1: Mild anisotropy

A homogeneous anisotrotic tensor is considered:

$$\mathbf{K} = \begin{pmatrix} 1.5 & 0.5 \\ 0.5 & 1.5 \end{pmatrix},$$

– Test 1.1: `mesh1` (triangular mesh), `mesh4` (distorted quadrangular mesh) This first solution is very regular, and is tested first on a "regular" triangular mesh and then on a distorted quadrangular mesh. On this latter mesh, we wish to see whether oscillations appear and whether the approximate solution remains within the bounds of the exact solution.

$$\begin{cases} u(x,y) = 16x(1-x)y(1-y),\ f = -\nabla \cdot (\mathbf{K}\nabla u) \\ \Gamma_D = \partial\Omega, \Gamma_N = \emptyset, \\ \bar{u} = u|_{\partial\Omega} \end{cases}$$

– Test 1.2: `mesh1` (triangular mesh), `mesh3` (locally refined nonconforming rectangular mesh) This solution increases at the origin, and therefore we use an nonconforming rectangular mesh to see how the schemes behaves.

$$\begin{cases} u(x,y) = \sin\left((1-x)(1-y)\right) + (1-x)^3(1-y)^2,\ f = -\nabla\cdot(\mathbf{K}\nabla u) \\ \Gamma_D = \partial\Omega, \Gamma_N = \emptyset, \\ \bar{u} = u|_{\partial\Omega} \end{cases}$$

Test 2: Numerical locking [BAB 92, MAN 07]

$$\mathbf{K} = \begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix},$$

– Meshes : `mesh1` (triangular mesh)

– Values of the parameter δ : $10^5, 10^6$.

$$\begin{cases} u(x,y) = \sin(2\pi x)e^{-2\pi\sqrt{1/\delta}y},\ f = -\nabla\cdot(\mathbf{K}\nabla u) \\ \Gamma_D = \emptyset, \Gamma_N = \partial\Omega, \\ g = (\mathbf{K}\nabla u\cdot\mathbf{n})|_{\partial\Omega}, \\ \displaystyle\int_\Omega u\,dx = 0. \end{cases}$$

Note that the maximum and the minimum of the solution are located on the boundary, and are more difficult to obtain with the Neumann boundary conditions imposed here. Since δ is large, the solution is almost constant in the y variable.

Test 3: Oblique flow

This test case represents a flow with boundary conditions such that the pressure driven flow 'would like' to go from vertex (0,0) to vertex (1,1), but is impeded by a heterogeneous anisotropic tensor with high permeability in a direction at 40 degrees from the horizontal and low permeability in the orthogonal direction. This test case is inspired by a talk given by I. Aavatsmark in Paris in December 2006 at GDR MOMAS. After the first publication of this benchmark on the web, I. Aavatsmark told us that in fact, there are more severe test cases for monotony, which he generously handed out to us. They are described in Tests 8 and 9 below.

$$\mathbf{K} = R_\theta \begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix} R_\theta^{-1},$$

where R_θ is the rotation of angle $\theta = 40$ degrees and $\delta = 10^{-3}$.

The shape of the solution is depicted in Figure 1; it was obtained by a computation by a “hybrid finite volume scheme" (see [EYM 07]) on a fine grid.

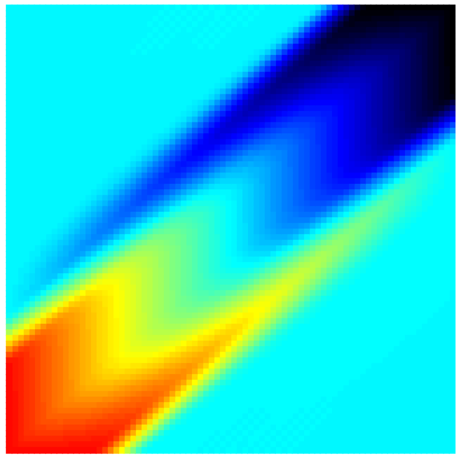

Figure 1. *Approximate solution on a fine grid for Test 3, oblique flow*

We wish to see how the schemes respect the maximum principle. Hence the results should show the maximum and miminum values of the approximate solution. Since the exact solution is not known, it is difficult to measure the precision of the scheme with respect to the values of the approximate solution; hence the outward fluxes should also be given to compare the various schemes, along with two computations of the energy given by the discrete counterpart of the formulae:

$$E_1 = \int_\Omega \mathbf{K}\nabla u \cdot \nabla u dx, \; E_2 = \int_{\partial\Omega} \mathbf{K}\nabla u \cdot \mathbf{n}\, u dx \tag{2}$$

Note that E_1 may only be computed for those methods which include a discrete gradient while E_2 can be computed with the boundary outward normal fluxes only. Even though E_1 and E_2 should converge to the same value on fine grids, there could be a noticeable difference between E_1 and E_2 on the coarsest meshes, and the authors are encouraged to comment on this difference.

– Meshes: `mesh2` (uniform rectangular mesh) and a reference mesh.

$$\begin{cases} \Gamma_D = \partial\Omega, \Gamma_N = \emptyset, \; f = 0, \\ \bar{u} \text{ is continuous and piecewise linear on } \partial\Omega \text{ and such that} \\ \bar{u}(x,y) = \begin{cases} 1 & \text{on} & ((0,.2)\times\{0.\}\cup\{0.\}\times(0,.2) \\ 0 & \text{on} & ((.8,1.)\times\{1.\}\cup\{1.\}\times(.8,1.) \\ \frac{1}{2} & \text{on} & ((.3,1.)\times\{0\}\cup\{0\}\times(.3,1.) \\ \frac{1}{2} & \text{on} & ((0.,.7)\times\{1.\}\cup\{1.\}\times(0.,0.7) \end{cases} \end{cases}$$

Test 4: Vertical fault

The medium considered here is a pile of anisotropic layers with a fault in the middle, which leads to a discontinuity of the layers at $x = .5$. Each geological layer is meshed with one layer of discretization cells only. A Dirichlet boundary condition is imposed.

The domain Ω may be decomposed as $\Omega = \Omega_1 \cup \Omega_2$, with $\Omega_2 = \Omega \setminus \Omega_1$, with $\Omega_1 = \Omega_1^\ell \cup \Omega_1^r$, and

$$\Omega_1^\ell = (0.; .5] \times \left(\bigcup_{k=0}^{4} [.05 + 2k \times .1; .05 + (2k+1) \times .1) \right),$$
$$\Omega_1^r = (.5; 1) \times \left(\bigcup_{k=0}^{4} [2k \times .1; (2k+1) \times .1) \right).$$

It is described in Figure 2 where Ω_1 is in black and Ω_2 in white.

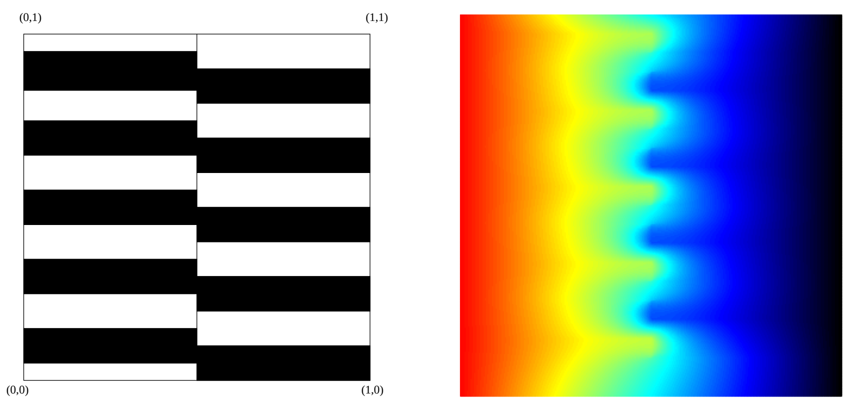

Figure 2. *The computational domain and approximate solution on a fine grid (*320×320*) for Test 4, vertical fault*

As in the case of Test 3, the exact solution is not known, and the expected results are the same as those of test 3, namely minimum and maximum values, outward fluxes and the energies E_1 and E_2.

The diffusion tensor $\mathbf{K}$ is anisotropic and heterogenous, and is given by:

$$\mathbf{K} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \text{ with } \begin{cases} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 10^2 \\ 10 \end{pmatrix} \text{ on } \Omega_1, \\ \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 10^{-2} \\ 10^{-3} \end{pmatrix} \text{ on } \Omega_2 \end{cases}$$

– Meshes: `mesh5` (nonconforming rectangular mesh) see Figure 8, the square mesh 20×20 denoted by $\texttt{mesh5}_{reg}$ and a reference mesh, for instance the square mesh 320×320 called $\texttt{mesh5}_{ref}$.

– Boundary conditions:

$$\begin{cases} \Gamma_D = \partial\Omega, \Gamma_N = \emptyset,\ f = 0, \\ \bar{u}(x,y) = 1 - x \end{cases}$$

Test 5: Heterogeneous rotating anisotropy

This test is inspired from [AND 07, LEP 05], and induces numerical locking for some schemes.

$$\mathbf{K} = \frac{1}{(x^2+y^2)} \begin{pmatrix} 10^{-3}x^2 + y^2 & (10^{-3}-1)xy \\ (10^{-3}-1)xy & x^2 + 10^{-3}y^2 \end{pmatrix},$$

$$u(x,y) = \sin \pi x \sin \pi y,\ f = -\nabla \cdot (\mathbf{K}\nabla u),$$

– Meshes : `mesh2` (uniform rectangular meshes)

– Test Boundary conditions:

$$\begin{cases} \Gamma_D = \partial\Omega, \Gamma_N = \emptyset, \\ \bar{u}(x,y) = \sin \pi x \sin \pi y \end{cases}$$

Test 6: Oblique drain

This test case represents a situation which is encountered in underground flow engineering where an oblique drain consisting in a very permeable layer concentrates most part of the flow; this drain is meshed with only one layer of discretization cells. in the case of a pressure gradient driven transport, as often described in reservoir engineering, it seems important that the discretization cells consist in only one homoneneous material: numerical experiments show that otherwise the solution may be badly approximated. Here we consider the steady case, but wish to verify that the outward fluxes are as close as possible to the exact values for the meshes considered here, both for the conforming and nonconforming meshes.

The domain Ω is composed of 3 subdomains:

$$\begin{cases} \Omega_1 = \{(x,y) \in \Omega; \phi_1(x,y) < 0\}, \\ \Omega_2 = \{(x,y) \in \Omega; \phi_1(x,y) > 0, \phi_2(x,y) < 0\}, \\ \Omega_3 = \{(x,y) \in \Omega; \phi_2(x,y) > 0\}, \end{cases}$$

with

$$\begin{cases} \phi_1(x,y) = y - \delta(x - .5) - .475, \\ \phi_2(x,y) = \phi_1(x,y) - 0.05. \end{cases}$$

We take the slope of the drain $\delta = 0.2$ and define the exact solution and the source term by:

$$u(x,y) = -x - \delta y, \text{ on } \Omega, \ f = -\nabla(\mathbf{K}\nabla u),$$

where the permeability tensor $\mathbf{K}$ is such that its principal axes are parallel and perpendicular to the drain:

$$\mathbf{K} = R_\theta \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} R_\theta^{-1},$$

with θ such that $\delta = \tan\theta$ and :

$$\begin{cases} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 10^2 \\ 10 \end{pmatrix}, \text{ on } \Omega_2, \\ \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 \\ 10^{-1} \end{pmatrix}, \text{ on } \Omega_1 \cup \Omega_3. \end{cases}$$

– Meshes : `mesh6`, `mesh7`

– Boundary conditions:

$$\begin{cases} \Gamma_D = \partial\Omega, \Gamma_N = \emptyset, \\ \bar{u}(x,y) = -x - \delta y \end{cases}$$

Test 7: Oblique barrier

This test case is similar to the Test 6, except that we now have to deal with a barrier, and the aim is that the scheme should respect this barrier as well as the outward fluxes.

We take the same geometry as test 6 above, with the slope of the drain $\delta = 0.2$.

We take the exact solution to be

$$u(x,y) = \begin{cases} -\phi_1(x,y) \text{ on } \Omega_1, \\ -\phi_1(x,y)/10^{-2} \text{ on } \Omega_2, \\ -\phi_2(x,y) - 0.05/10^{-2} \text{ on } \Omega_3, \end{cases}$$

and $f = -\nabla\cdot(\mathbf{K}\nabla u)$, where the permeability tensor K is heterogeneous and isotropic:

$$\mathbf{K} = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix},$$

with :

$$\alpha = \begin{cases} 1 \text{ on } \Omega_1, \\ 10^{-2} \text{ on } \Omega_2, \\ 1 \text{ on } \Omega_3. \end{cases}$$

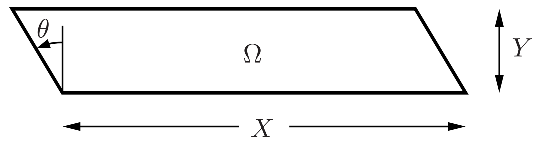

Figure 3. *Parallelogram-shaped domain Ω showing the distances X and Y and the angle θ.*

– Meshes : `mesh6`

– Boundary conditions:

$$\begin{cases} \Gamma_D = \partial\Omega, \Gamma_N = \emptyset, \\ \bar{u}(x,y) = \begin{cases} -\ \phi_1(x,y) \text{ on } \partial\Omega \cap \partial\Omega_1, \\ -\ \phi_1(x,y)/10^{-2} \text{ on } \partial\Omega \cap \partial\Omega_2, \\ -\ \phi_2(x,y) - 0.05/10^{-2} \text{ on } \partial\Omega \cap \partial\Omega_3. \end{cases} \end{cases}$$

Test 8: Perturbed parallelograms [AAV 07]

This test case was given to us by I. Aavatsmark [AAV 07], and is meant to test the schemes for the violation of the maximum principle within the domain. The domain Ω is parallelogram shaped, as shown in figure 3. The parameters shown in figure 3 are $X = 1$, $Y = 1/30$ and $\theta = 30°$. The medium is homogeneous and isotropic with $\mathbf{K} = Id$.

– Mesh: `mesh8` (perturbed parallelogram mesh) see Figure 10.

– Boundary conditions and right hand side:

$$\begin{cases} \Gamma_D = \partial\Omega, \Gamma_N = \emptyset,\ f = 0 \text{ in all cells except cell } (6,6) \text{ where } \displaystyle\int_{\text{cell}(6,6)} f(x)\,dx = 1. \\ \bar{u}(x,y) = 0 \text{ on } \partial\Omega. \end{cases}$$

Note that the solution u of this problem should be a function with a maximum in cell $(6,6)$, decreasing smoothly to zero towards the boundary. If u shows internal oscillations or if $u < 0$, Hopf's first lemma is violated.

Note that cell (i,j) is numbered by $i + 11(j-1)$ in the data `mesh8`.

Test 9: Anisotropy and wells [AAV 07]

Here, Ω is again the square unit domain $\Omega = (0,1) \times (0,1)$. The medium is homogeneous and anisotropic with

$$\mathbf{K} = M(-\theta) \begin{bmatrix} 1 & 0 \\ 0 & 10^{-3} \end{bmatrix} M(\theta), \qquad M(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \tag{3}$$

where $\theta = 67.5°$.

– Mesh: `mesh9`, the grid is a square uniform grid with 11×11 cells (Figure 11).

– Boundary conditions and right hand side:

- The source density f is zero in all cells.

- The pressure is fixed in two cells, approximating a sink and a source with fixed pressure:

$$\begin{aligned} u &= 0 \quad \text{in cell } (4,6), \\ u &= 1 \quad \text{in cell } (8,6). \end{aligned} \tag{4}$$

- Homogeneous Neumann conditions apply at the outer boundary:

$$-\mathbf{K}\nabla u \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega. \tag{5}$$

The solution u of this problem should satisfy $u \in [0,1]$. If u has extrema on the no-flow boundary with $u \notin [0,1]$, Hopf's second lemma is violated.

Note that cell (i,j) is numbered by $i + 11(j-1)$ in the data `mesh9`.

3. Expected results

When refined the meshes are numbered `i` $= 1$ to `ngrid`, from coarsest to finest. The structure of the expected results is given in the file `template.tex`. For each value of `i`, one should provide:

For all runs:

– `nunkw` number of unknowns

– `nnmat` number of nonzero terms in the matrix

– `sumflux` the discrete flux balance, that is:

`sumflux= flux0+flux1+fluy0+fluy1-sumf`,

where `flux0, flux1, fluy0, fluy1` are the outward fluxes at the boundaries $x = 0$, $x = 1$, $y = 0$, $y = 1$, for example

$$\texttt{flux0} \text{ is an approximation of } -\int_{x=0} \mathbf{K}\nabla u \cdot \mathbf{n}\, ds$$

and $\texttt{sumf} = \sum_{K\in\mathcal{T}} |K| f(x_K)$ where x_K denotes some point (which should be precised) of the control volume K.

– $\texttt{umin}$: value of the minimum of the approximate solution.

– $\texttt{umax}$: value of the maximum of the approximate solution.

When the analytical solution is known and the mesh refined:

Let us denote by u the exact solution, by $\mathcal{T}$ the mesh and by $u_{\mathcal{T}} = (u_K)_{K\in\mathcal{T}}$ the piecewise constant approximate solution.

– $\texttt{erl2}$, relative discrete L^2 norm of the error $\texttt{erl2}$ where:

$$\texttt{erl2} = \left(\frac{\displaystyle\sum_{K\in\mathcal{T}} |K| (u(x_K) - u_K)^2}{\displaystyle\sum_{K\in\mathcal{T}} |K| u(x_K)^2} \right)^{\frac{1}{2}}$$

where x_K denotes some point (which should be precised) of the control volume K, (or a variant of such a norm, to be precised).

– $\texttt{ergrad}$ relative L^2 norm of the error on the gradient, if available (give the definition of the discrete gradient)

– $\texttt{ratiol2}$: for $\texttt{i} \geq 2$,

$$\texttt{ratiol2}(\texttt{i}) = -2 \frac{\ln(\texttt{erl2}(\texttt{i})) - \ln(\texttt{erl2}(\texttt{i} - 1))}{\ln(\texttt{nunkw}(\texttt{i})) - \ln(\texttt{nunkw}(\texttt{i} - 1))}$$

– $\texttt{ratiograd}$, for $\texttt{i} \geq 2$,

$$\texttt{ratiograd}(\texttt{i}) = -2 \frac{\ln(\texttt{ergrad}(\texttt{i})) - \ln(\texttt{ergrad}(\texttt{i} - 1))}{\ln(\texttt{nunkw}(\texttt{i})) - \ln(\texttt{nunkw}(\texttt{i} - 1))}.$$

– $\texttt{erflx0}$, $\texttt{erflx1}$, $\texttt{erfly0}$, $\texttt{erfly1}$ relative error between $\texttt{flux0}$, $\texttt{flux1}$, $\texttt{fluy0}$, $\texttt{fluy1}$ and the corresponding flux of the exact solution:

$$\texttt{erflx0} = \left| \frac{\texttt{flux0} + \int_{x=0} \mathbf{K}\nabla u \cdot \mathbf{n}}{\int_{x=0} \mathbf{K}\nabla u \cdot \mathbf{n}} \right|$$

(except for the fluxes at y=0 and y=1 for case test 2 -Numerical locking- because these are zero,give the value of the approximate fluxes only in this case).

– $\texttt{erflm}$ L^∞ norm of the error on the meanvalue of the flux through the edges of the mesh, if available (give the definition of numerical flux $(\mathbf{K}\nabla u \cdot \mathbf{n})_{\mathcal{T}}$)

$$\texttt{erflm} = \max \left\{ \left| \frac{1}{|\sigma|} \int_\sigma (\mathbf{K}\nabla u \cdot \mathbf{n} - (\mathbf{K}\nabla u \cdot \mathbf{n})_{\mathcal{T}}) \right|, \sigma \text{ edges of } \mathcal{T} \right\}$$

– `ocvl2` order of convergence of the method in the L^2 norm of the solution as defined by `exrl2` with respect to the mesh size:

$$\texttt{ocvl2} = \frac{\ln(\texttt{erl2}(\mathrm{imax})) - \ln(\texttt{erl2}(\mathrm{imax} - 1))}{\ln(\texttt{h}(\mathrm{imax})) - \ln(\texttt{h}(\mathrm{imax} - 1))}$$

where `h` is the maximum of the diameter of the control volume

– `ocvgradl2` order of convergence of the method in the L^2 norm of the gradient as defined by `ergradl2` with respect to the mesh size:

$$\texttt{ocvgrad} = \frac{\ln(\texttt{ergrad}(\mathrm{imax})) - \ln(\texttt{ergrad}(\mathrm{imax} - 1))}{\ln(\texttt{h}(\mathrm{imax})) - \ln(\texttt{h}(\mathrm{imax} - 1))}$$

For tests 3, 4, 8,9, the exact solution is not known, the maximum principle and the overall precision of the scheme can be tested by computing:

– `umin`: value of the minimum of the approximate solution.

– `umax`: value of the maximum of the approximate solution.

– `flux0, flux1, fluy0, fluy1` outward normal fluxes to the boundaries $x = 0$, $x = 1$, $y = 0$ and $y = 1$ (useless in case of test 9).

The same values should be computed on a reference fine grid so as to be able to compare the results.

For tests 3, 4, since $f = 0$, we can compute :

– `ener1, ener2`: values of the discrete computations of the energies given by the formulae (2) (if a discrete gradient is available in the case of `ener1`).

– `eren`: relative error between `ener1, ener2`:

$$\texttt{eren} = \frac{|\texttt{ener1} - \texttt{ener2}|}{\max(\texttt{ener1}, \texttt{ener2})}.$$

4. The meshes

Figures of the meshes are given after the references. We provide two different formats for the meshes:

`*.typ1` and `*.typ2`, which are described in the file `README` in the directory `Meshes`.

The size steps of the meshes are given in the following table:

i	1	2	3	4	5	6	7
mesh1	2.50E-01	1.25E-01	6.25E-02	3.12E-02	1.56E-02	7.81E-03	3.91E-03
mesh2	3.54E-01	1.77E-01	8.84E-02	4.42E-02	2.21E-02	1.10E-02	5.52E-03
mesh3	3.54E-01	1.77E-01	8.84E-02	4.42E-02	2.21E-02		
mesh 4	3.29E-01	1.70E-01					
mesh 5	1.41E-01						
mesh6	1.25E-01						
mesh7	1.25E-01						
mesh8	1.24E-01						
mesh9	1.29E-01						

Please use the same format as above when entering your results in the tables of `template.tex`, that is the format: `sign integer dot integer ingeger E sign integer integer`. This format is given for instance by the FORTRAN format `ES9.2`

5. References

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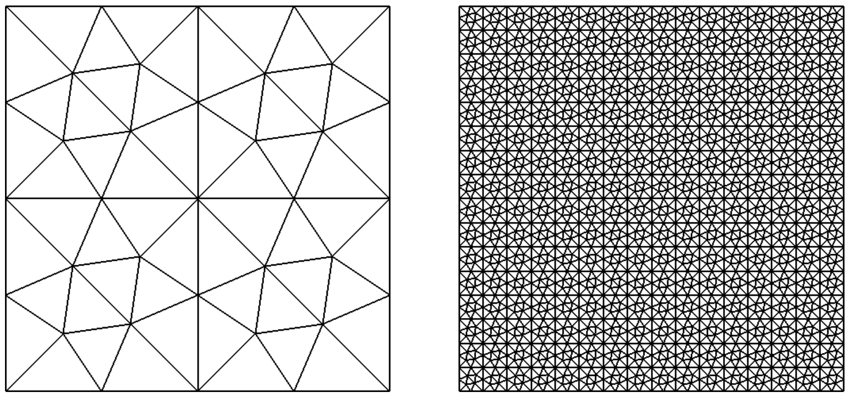

Figure 4. *Triangular mesh with acute angles: meshes* mesh1_1 *(left) and* mesh1_4 *(right)*

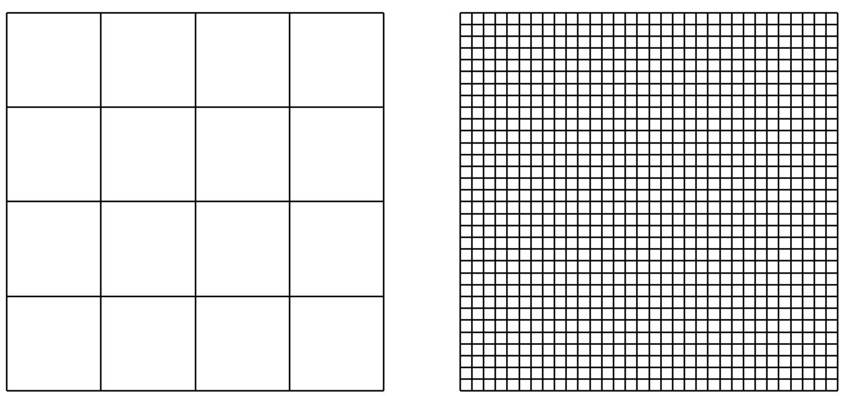

Figure 5. *Uniform rectangular mesh: meshes* mesh2_1 *(left) and* mesh2_4 *(right)*

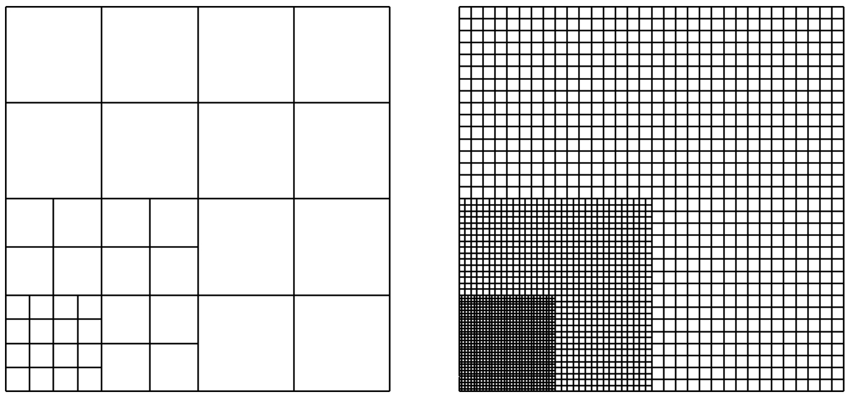

Figure 6. *Locally refined non conforming rectangular mesh: meshes* `mesh3_1` *(left) and* `mesh3_4` *(right)*

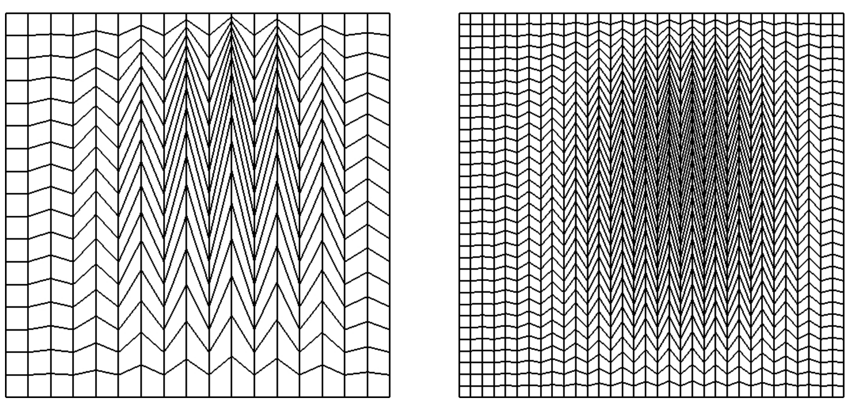

Figure 7. *Conforming distorted quadrangular mesh: meshes* `mesh4_1` *(left) and* `mesh4_2` *(right)*

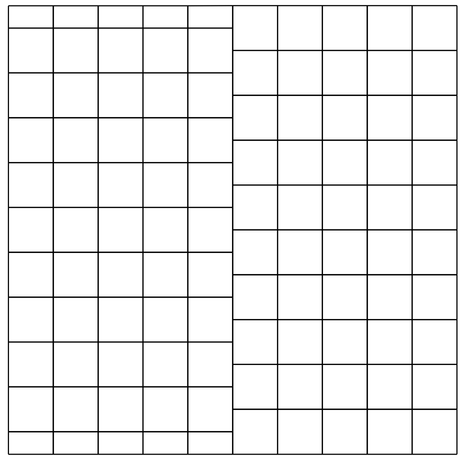

Figure 8. *Non conforming regular rectangular mesh* `mesh5`

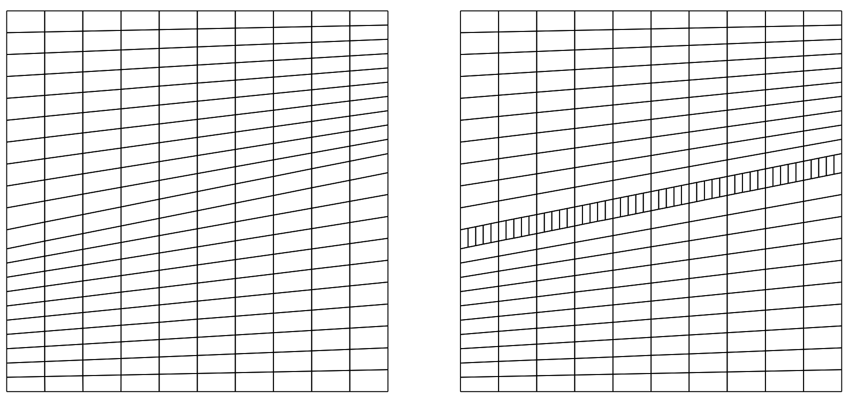

Figure 9. *Coarse oblique mesh:* `mesh6` *(Left).Fine oblique mesh for the oblique barrier and drain tests:* `mesh7` *(Right)*

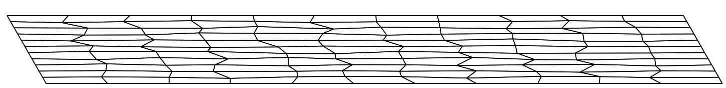

Figure 10. *Perturbed parallelogram grid* `mesh8` *with* 11×11 *cells. To visualize the grid, on this picture the height is 3 times the real height of the grid described in the test case.*

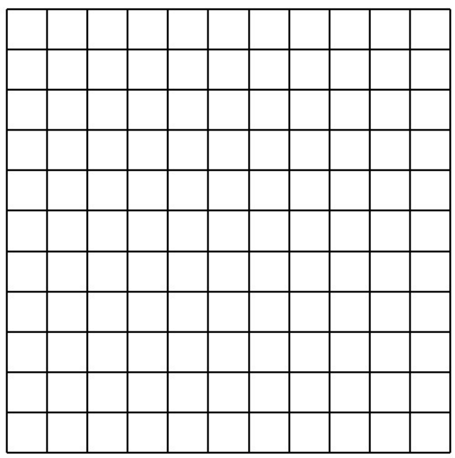

Figure 11. *square uniform grid* `mesh9` *with* 11×11 *cells.*