Benchmark on Discretization Schemes for Anisotropic Diffusion Problems on General Grids

Overview of the results - Part I

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The participating schemes and teams

Cell centred schemes

- CMPFA, by S. Mundal, D. A. Di Pietro and I. Aavatsmark.
- FVHYB, by L. Agelas and D. A. Di Pietro.
- FVSYM, by C. Le Potier.
- SUSHI-P or SUSHI-NP (full barycentric version), by R. Eymard, T. Gallouët and R. Herbin.

Control volume finite element schemes

- CVFE, by M. Afif and B. Amaziane.

Discontinous Galerkin schemes

- DG-C, by A. Dedner and R. Klöfkorn.
- DG-W, by D. A. Di Pietro and A. Ern.

Discrete duality finite volume schemes

- DDFV-BHU, by F. Boyer and F. Hubert.
- DDFV-HER, by F. Hermeline.
- DDFV-MNI, by I. Moukouop Nguena and A. Njifenjou.
- DDFV-OMN, by P. Omnes.

Finite element schemes

- FEP1, FEP2, FEQ1, FEQ2, by G. Ansanay-Alex, B. Piar and D. Vola.

Lattice Boltzmann schemes

- LATTB, by F. Dubois, P. Lallemand and M. M. Tekitek.

Mixed or hybrid methods

- MFD-BLS, by K. Lipnikov.
- MFD-FHE, by B. Flemisch and R. Helmig.
- MFD-MAN, by G. Manzini.
- MFD-MAR, by S. Marnach.
- MFE, programmed by the benchmark organizers for lack of a submission.
- MFV, by C. Chainais-Hillairet, J. Droniou and R. Eymard.
- SUSHI-P or SUSHI-NP (full hybrid version),by R. Eymard, T. Gallouët and R. Herbin.

Nonlinear schemes

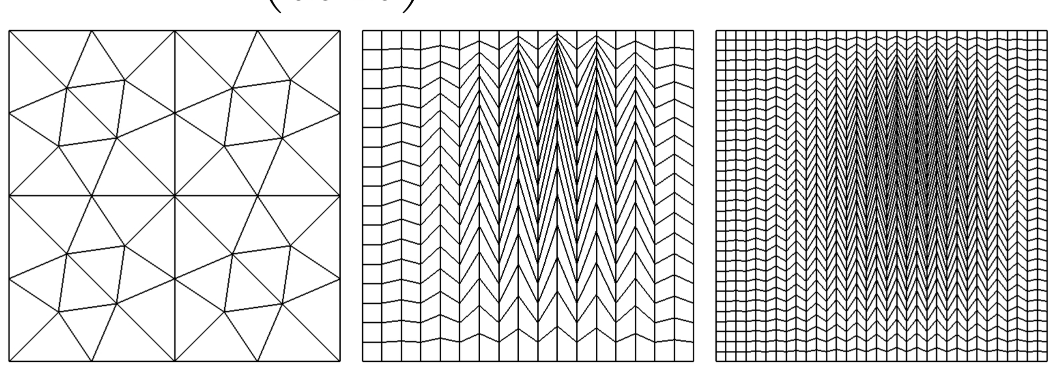
- FVPMMD, by C. Le Potier.
- NMFV, by D. Svyatskiy.

Test 1.1 - mesh1_1

$$-\mathrm{div}(\mathbf{K}\nabla u) = f \text{ in } \Omega$$

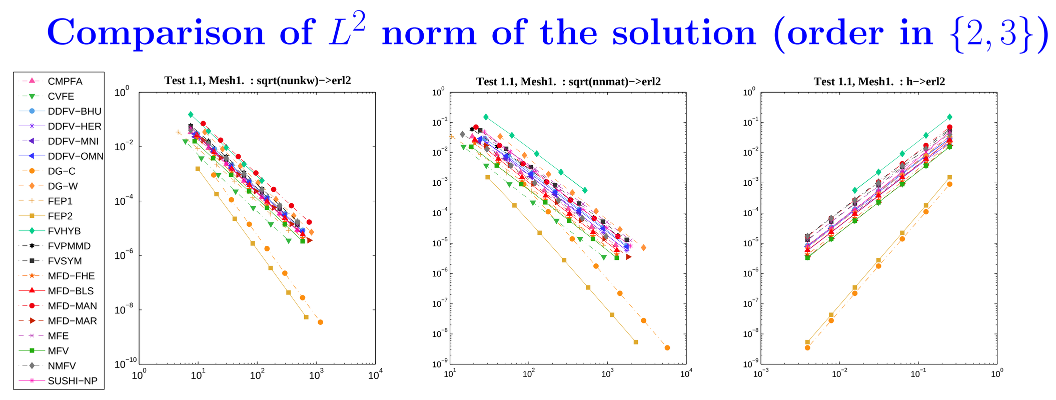
Non homogeneous Dirichlet boundary conditions

with $\mathbf{K} = \begin{pmatrix} 1.5 & 0.5 \\ 0.5 & 1.5 \end{pmatrix}$ and $u(x,y) = 16x(1-x)y(1-y)$.

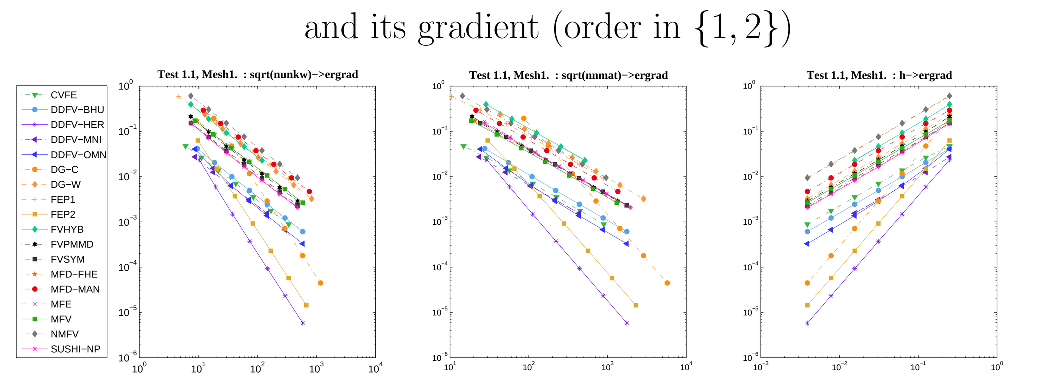


Mesh1 Mesh4_1 Mesh4_2

Comparison of L^2 norm of the solution (order in $\{2,3\}$)



and its gradient (order in $\{1,2\}$)



Minimum and maximum of the approximate solutions

	mesh 4_1		mesh 4_2	
	umin	umax	umin	umax
CMPFA	9.95E-03	1.00E+00	2.73E-03	9.99E-01
CVFE	0.00E+00	8.43E-01	0.00E+00	9.14E-01
DDFV-BHU	1.33E-02	9.96E-01	3.63E-03	9.99E-01
DDFV-HER	0.00E+00	**1.03E+00**	0.00E+00	**1.01E+00**
DDFV-MNI	**-3.09E-01**	**1.03E+00**	0.00E+00	1.00E+00
DDFV-OMN	1.34E-02	**1.03E+00**	3.65E-03	**1.01E+00**
DG-C	**-2.33E-03**	9.96E-01	**-3.24E-04**	9.99E-01
DG-W	**-7.90E-05**	9.22E-01	**-8.18E-06**	9.66E-01
FEQ1	0.00E+00	8.61E-01	0.00E+00	9.37E-01
FEQ2	0.00E+00	9.99E-01	0.00E+00	1.00E+00
FVHYB	2.14E-03	9.84E-01	7.16E-04	9.93E-01
FVSYM	7.34E-03	9.59E-01	2.33E-03	9.89E-01
MFD-BLS	8.54E-03	9.55E-01	2.44E-03	9.87E-01
MFD-FHE	9.73E-03	9.45E-01	2.90E-03	9.83E-01
MFD-MAN	6.64E-03	9.71E-01	1.50E-03	9.93E-01
MFD-MAR	8.82E-03	9.60E-01	2.47E-03	9.88E-01
MFV	1.08E-02	9.42E-01	3.34E-03	9.82E-01
NMFV	1.30E-02	**1.11E+00**	3.61E-03	**1.04E+00**
SUSHI-NP	7.64E-03	8.88E-01	2.33E-03	9.61E-01

- Maximum principle violated by DDFV and DG method.

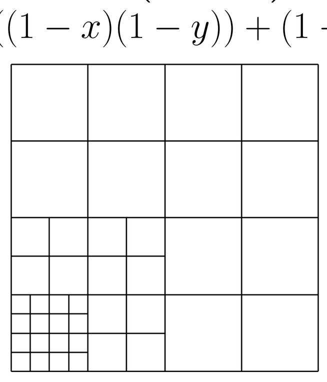
Results for Test 1.2 : Mild Anisotropy

$$-\mathrm{div}(\mathbf{K}\nabla u) = f \text{ in } \Omega$$

Non homogeneous Dirichlet boundary conditions

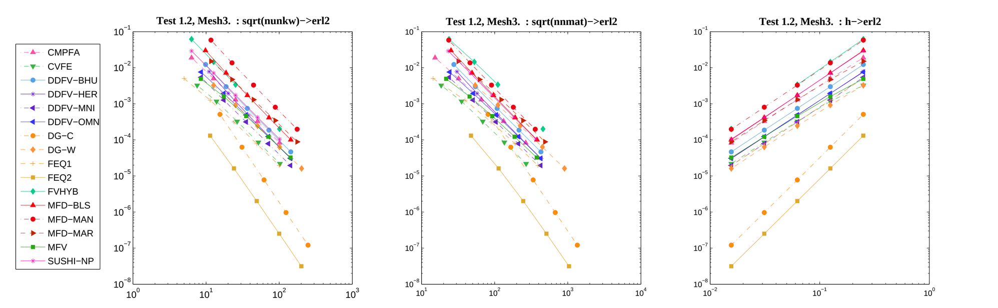
with $\mathbf{K} = \begin{pmatrix} 1.5 & 0.5 \\ 0.5 & 1.5 \end{pmatrix}$ and

$$u(x,y) = \sin\left((1-x)(1-y)\right) + (1-x)^3(1-y)^2.$$

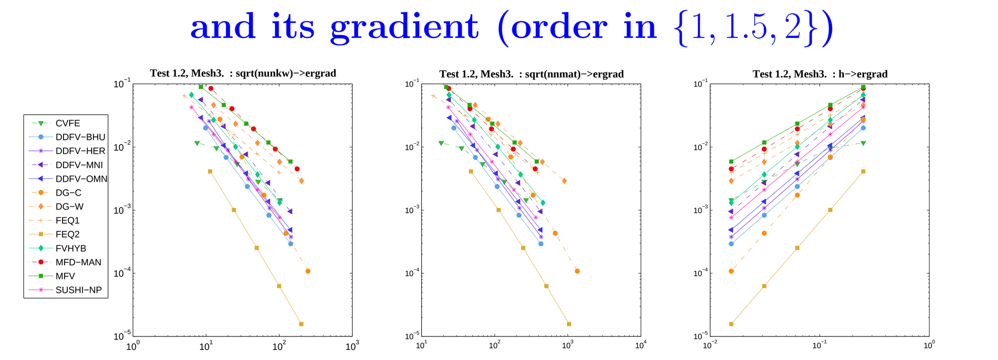


Mesh3

Comparison of L^2 norm of the solution (order in $\{2,3\}$)



and its gradient (order in $\{1, 1.5, 2\}$)



Results for Test 2 : Numerical Locking

$$-\mathrm{div}(\mathbf{K}\nabla u) = f \text{ in } \Omega$$

Neumann BC

$\int_\Omega u\, dx = 0$

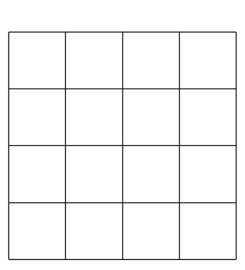
with $\mathbf{K} = \begin{pmatrix} 1 & 0 \\ 0 & 10^5 \end{pmatrix}$ and

$$u(x,y) = \sin(2\pi x)e^{-2\pi 10^{-2.5} y}.$$

	min(umin), i	max(umax), i	ocvl2	ocvgrad	erflm
CMPFA	**-1.10E+00, 1**	1.04E+00, 3	**1.09**	/	6.36E+02
CVFE	**-1.01E-00, 2**	**1.01E-01, 2**	2.00	1.00	1.55E+01
DDFV-BHU	**-9.27E-01, 1**	1.17E+00, 1	1.76	1.21	7.51E+00
DDFV-HER	**-4.20E-01, 2**	9.12E+00, 4	/	/	7.16E-03
DDFV-OMN	**-8.24E-01, 1**	7.76E-01, 1	2.00	1.00	2.11E+00
DG-W	**-1.18E-01, 1**	1.18E-01, 1	2.00	1.00	1.69E+01
FEP1	**-9.48E-03, 1**	9.75E-03, 1	2.00	1.01	/
FEP2	**-9.56E-01, 1**	9.56E-01, 1	**2.97**	**2.00**	/
FVSYM	**-1.76E+00, 2**	**1.80E+00, 2**	2.38	1.47	7.29E+00
MFD-BLS	**-6.50E+00, 2**	**5.75E+00, 2**	2.54	/	3.59E+01
MFD-FHE	**-6.50E+00, 2**	**5.75E+00, 2**	2.54	1.51	3.59E+01
MFD-MAN	**-6.62E+00, 2**	**5.50E+00, 2**	2.49	1.50	3.58E+01
MFD-MAR	**-6.50E+00, 2**	**5.75E+00, 2**	2.53	/	3.59E+01
MFE	**-6.50E+00, 2**	**5.75E+00, 2**	2.53	1.47	3.59E+01
MFV	**-6.50E+00, 2**	**5.75E+00, 2**	2.41	1.51	3.58E+01
SUSHI-P	**-6.50E+00, 2**	**5.75E+00, 2**	2.53	1.47	3.59E+01
SUSHI-NP	-1.93E-02, 4	1.89E-02, 4	**0.37**	1.99	/

- On such meshes, the DDFV schemes all seem to coincide.Same is true with MFD-BLS, MFD-FHE, MFD-MAN, MFD-MAR, MFE, SUSHI-P(Hybrid version).
- Order of convergence vary more than on other tests.

Results for Test 3 : Oblique flow



$$-\mathrm{div}(\mathbf{K}\nabla u) = 0 \text{ in } \Omega$$

Non homogeneous Dirichlet boundary conditions

with $\mathbf{K} = R_\theta \begin{pmatrix} 1 & 0 \\ 0 & 10^{-3} \end{pmatrix} R_\theta^{-1}$, $\theta = 40$ degrees

$$\bar{u}(x,y) = \begin{cases} 1 & \text{on } (0,.2)\times\{0.\}\cup\{0.\}\times(0,.2) \\ 0 & \text{on } (.8,1.)\times\{1.\}\cup\{1.\}\times(.8,1.) \\ \frac{1}{2} & \text{on } ((.3,1.)\times\{0\}\cup\{0\}\times(.3,1.) \\ \frac{1}{2} & \text{on } (0.,.7)\times\{1.\}\cup\{1.\}\times(0.,0.7) \end{cases}$$

The values of umin, umax

	umin_i	umax_i	i
CMPFA	6.90E-02	9.31E-01	1
	9.83E-04	9.99E-01	7
CVFE	0.00E+00	1.00E+00	1
	0.00E+00	1.00E+00	7
DDFV-BHU	**-4.72E-03**	1.00E+00	1
	-5.31E-04	1.00E+00	7
DDFV-HER	**-4.72E-03**	1.00E+00	1
	-5.96E-08	1.00E+00	7
DDFV-MNI	**-4.73E-03**	1.00E+00	1
	-1.07E-03	1.00E+00	5
DDFV-OMN	1.04E-01	8.96E-01	1
	1.01E-03	9.99E-01	7
DG-C	**-9.35E-02**	**1.07E+00**	1
	-1.32E-03	1.00E+00	7
DG-W	**-4.11E-02**	**1.04E+00**	1
	-3.71E-03	1.00E+00	7
FEQ1	0.00E+00	1.00E+00	1
	0.00E+00	1.00E+00	7
FEQ2	0.00E+00	1.00E+00	1
	0.00E+00	1.00E+00	7
FVHYB	**-1.75E-01**	**1.17E+00**	1
	-1.00E-03	1.00E+00	6

	umin_i	umax_i	i
FVSYM	6.85E-02	9.32E-01	1
	4.92E-04	9.99E-01	8
LATTB	1.14E-01	8.86E-01	1
	9.36E-04	9.99E-01	7
MFD-BLS	6.09E-02	9.39E-01	1
	1.29E-03	9.99E-01	7
MFD-FHE	7.06E-02	/	1
	1.00E-03	9.99E-01	7
MFD-MAN	7.56E-02	9.24E-01	1
	8.01E-04	9.99E-01	8
MFD-MAR	6.09E-02	9.39E-01	1
	1.00E-03	9.99E-01	8
MFE	3.12E-02	9.69E-01	1
	5.08E-04	9.99E-01	8
MFV	1.22E-02	8.78E-01	1
	7.92E-04	9.99E-01	7
NMFV	1.11e-01	8.88e-01	1
	1.28E-03	9.99E-01	7
SUSHI-NP	6.03E-02	9.40E-01	1
	8.52E-04	9.99E-01	7

- Maximum principle violated by DDFV, DG methods and FVHYB scheme.

The energies

	ener1	eren_i	i
CVFE	2.24E-01	8.42E-02	1
	2.42E-01	3.33E-03	7
DDFV-BHU	2.14E-01	9.60E-02	1
	2.42E-01	7.11E-06	7
DDFV-HER	2.14E-01	9.46E-02	1
	2.42E-01	1.91E-05	7
DDFV-MNI	2.14E-01	9.61E-02	1
	2.42E-01	1.86E-04	5
DDFV-OMN	1.81E-01	3.68E-03	1
	2.42E-01	1.77E-06	7
DG-C	**5.04E-01**	9.88E-02	1
	2.42E-02	2.48E-05	7
DG-W	**1.90E-01**	5.67E-01	1
	2.44E-01	2.85E-05	7
FEQ1	2.21E-01	3.67E-01	1
	2.44E-01	3.17E-02	7
FEQ2	2.64E-01	3.41E-01	1
	2.42E-01	0.00E+00	7
FVHYB	2.13E-01	2.55E-01	1
	2.42E-01	8.19E-03	6

	ener1	eren_i	i
FVSYM	2.20E-01	0.00E+00	1
	2.42E-01	0.00E+00	8
LATTB	2.42E-01	1.64E-02	1
	2.42E-01	3.00E-04	7
MFD-BLS	2.38E-01	4.44E-15	1
	2.42E-01	6.74E-13	7
MFD-FHE	2.19E-01	2.09E-01	1
	2.42E-01	1.05E-04	7
MFD-MAN	**1.91E-01**	1.87E-14	1
	2.42E-01	3.70E-14	8
MFD-MAR	2.38E-01	9.85E-13	1
	2.42E-01	1.97E-10	8
MFE	**1.25E-01**	2.46E-02	1
	2.41E-01	2.91E-03	8
MFV	**4.85E-01**	8.23E-07	1
	2.42E-01	9.74E-06	7
NMFV	2.33e-01	1.45e-01	1
	2.45E-01	1.94E-02	7
SUSHI-NP	2.25E-01	3.01E-01	1
	2.43E-01	1.28E-02	7

- Most schemes are accurate on the energy on the coarse grids.