Benchmark on Discretization Schemes
for Anisotropic Diffusion Problems on General Grids
Overview of the results - Part II

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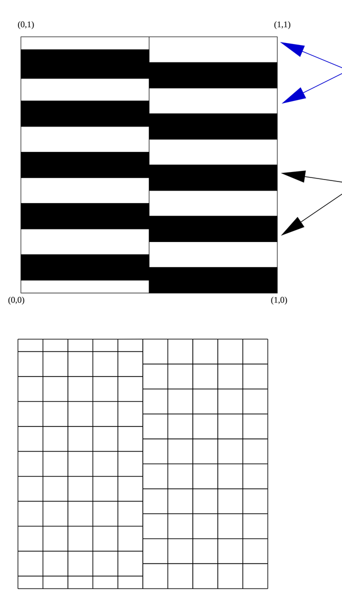
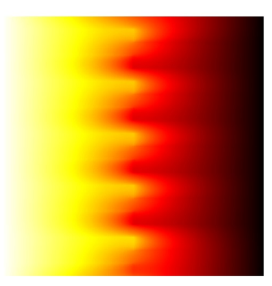
Test 4 : Vertical fault



$-\text{div}(\mathbf{K}\nabla u) = 0$ in Ω
$u = \bar{u}$ in $\partial\Omega$

$$\mathbf{K} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \text{ with } \begin{cases} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 10^2 \\ 10 \end{pmatrix} \text{ on } \Omega_1, \\ \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 10^{-2} \\ 10^{-3} \end{pmatrix} \text{ on } \Omega_2 \end{cases}$$

and $\bar{u}(x, y) = 1 - x$.



Maximum principle

- Problems only with the DG methods.

The values of the energies

	ener1 mesh5	eren mesh5	ener1 mesh5_ref	eren mesh5_ref
CVFE	45.9	1.04E-02	**43.3**	6.25E-04
DDFV-BHU	**42.1**	3.65E-02	**43.2**	1.27E-03
DDFV-HER	49.3	1.75E-01	43.8	1.64E-02
DDFV-MNI	/	/	43.8	6.23E-02
DDFV-OMN	**42.2**	3.65E-02	**43.2**	1.28E-03
DG-W	**43.5**	1.38E-02	**43.2**	7.63E-04
FEQ1	/	/	**43.3**	2.31E-03
FEQ2	/	/	**43.2**	0.00E+00
FVHYB	**41.4**	6.12E-02	/	/
MFD-BLS	33.9	7.93E-14	**43.2**	2.84E-12
MFD-FHE	/	/	**43.2**	3.53E-04
MFD-MAN	31.4	1.16E-12	**43.2**	4.71E-14
MFD-MAR	**41.1**	1.30E-13	**43.2**	2.69E-12
MFV	49.9	4.21E-05	**43.2**	1.88E-05
NMFV	/	/	**43.2**	5.92E-04
SUSHI-NP	39.1	6.67E-02	**43.1**	8.88E-04

- The methods that have trouble with PPMax are the most accurate for the energy on coarse meshes.

The fluxes

	flux0 mesh5	flux0 mesh5_ref	flux1 mesh5	flux1 mesh5_ref	fluy0 mesh5	fluy0 mesh5_ref	fluy1 mesh5
CMPFA	-45.2	**-42.1**	46.1	**44.4**	-0.95	**-2.33**	4.84E-04
CVFE	-46.6	**-42.2**	48.5	**44.5**	0.87	-2.25	8.02E-04
DDFV-BHU	-40.0	**-42.1**	41.8	**44.4**	-1.81	**-2.33**	9.08E-04
DDFV-HER	-40.0	**-42.0**	41.8	**44.3**	-1.81	**-2.35**	9.08E-04
DDFV-MNI	-43.8	-39.9	45.5	42.6	-2.8	-2.68	1.18E+00
DDFV-OMN	-40.0	**-42.1**	41.8	**44.4**	-1.81	**-2.33**	9.08E-04
DG-W	**-43.1**	**-42.1**	**45.3**	**44.5**	-2.19	**-2.32**	1.50E-03
FEQ1	/	**-42.2**	/	**44.5**	/	-2.16	/
FEQ2	/	**-42.1**	/	**44.5**	/	**-2.32**	/
FVHYB	-44.3	/	46.3	/	0.49	/	1.55E-04
MFD-BLS	-32.3	**-42.1**	36.2	**44.4**	-3.94	**-2.33**	1.22E-03
MFD-FHE	/	**-42.1**	/	**44.5**	/	-2.47	/
MFD-MAN	-29.7	**-42.1**	34.1	**44.4**	-4.37	**-2.33**	1.01E-03
MFD-MAR	-39.8	**-42.1**	42.5	**44.4**	**-2.68**	**-2.33**	9.95E-04
MFE	/	**-42.1**	/	**44.4**	/	**-2.33**	/
MFV	-44.0	**-42.1**	50.3	**44.4**	-8.03	**-2.33**	1.72E+00
NMFV	**-43.2**	**-42.1**	**44.5**	**44.4**	-1.23	**-2.33**	2.32E-04
SUSHI-NP	**-40.9**	**-42.1**	**43.1**	**44.4**	**-2.21**	**-2.33**	6.94E-04

Test 5: heterogeneous rotating anisotropy

$-\text{div}(\mathbf{K}\nabla u) = f$ in Ω
Non homogeneous Dirichlet boundary conditions

avec

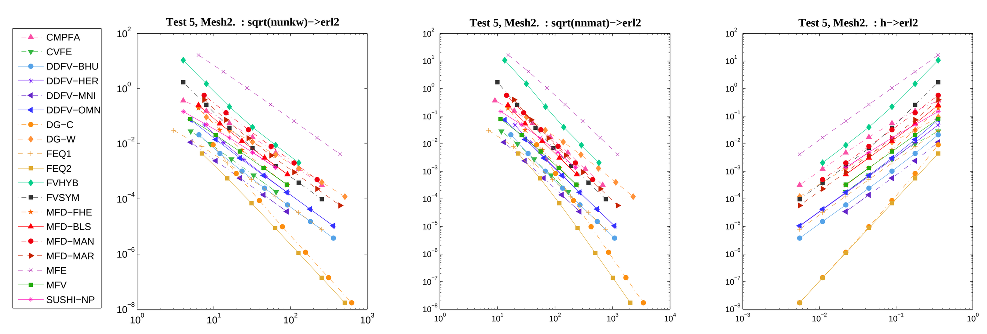
$$\mathbf{K} = \frac{1}{(x^2+y^2)} \begin{pmatrix} 10^{-3}x^2 + y^2 & (10^{-3}-1)xy \\ (10^{-3}-1)xy & x^2 + 10^{-3}y^2 \end{pmatrix}$$

and $u(x, y) = \sin \pi x \sin \pi y$.

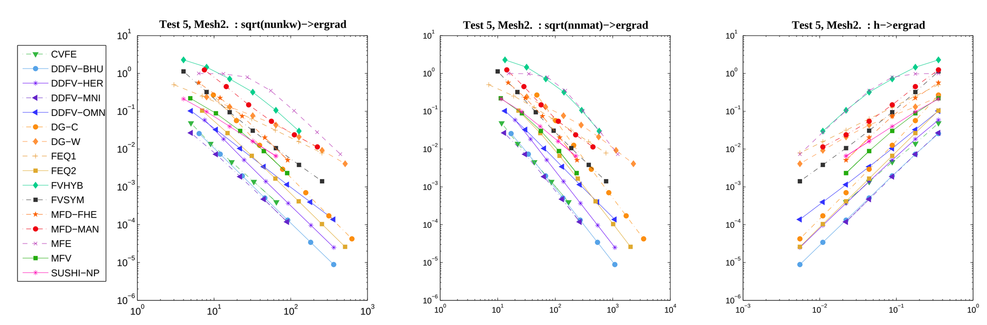
Minimum and maximum values on the coarsest mesh for the schemes which do not satisfy the maximum principle

	umin	umax
CMPFA	**-1.06E-01**	**1.09E+00**
DDFV-HER	0.00E+00	**1.01E+00**
DG-C	**-7.95E-04**	**1.02E+00**
DG-W	**-7.68E-02**	**1.06E+00**
FEQ1	0.00E+00	**1.05E+00**
FVHYB	**-1.92E+01**	**5.38E+00**
FVSYM	**-8.67E-01**	**2.57E+00**
MFE	**-1.62E+00**	**1.90E+01**

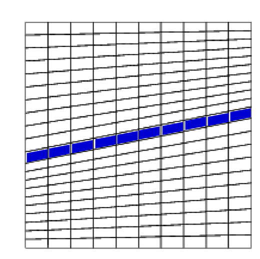
Comparison of L^2 norm of the solution (order in $\{2,3\}$)

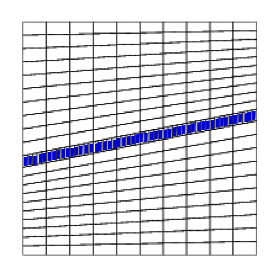


and its gradient (order in $[1,2]$)



Test 6: oblique drain





$-\text{div}(\mathbf{K}\nabla u) = f$ in Ω
Non homogeneous Dirichlet boundary conditions

with

$$\mathbf{K} = R_\theta \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} R_\theta^{-1},$$

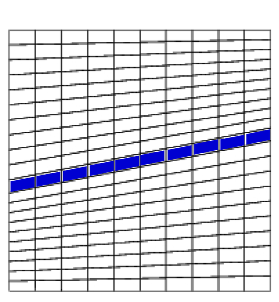
and $u(x, y) = -x - \delta y$ with $\delta = \tan\theta = 0.2$.

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 10^2 \\ 10 \end{pmatrix} \text{ on } \Omega_2$$

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 \\ 10^{-1} \end{pmatrix} \text{ on } \Omega_1 \cup \Omega_3.$$

► Order 2 schemes ⇒ exact results.

Test 7: oblique barrier



$-\text{div}(\mathbf{K}\nabla u) = f$ in Ω
Non homogeneous Dirichlet boundary conditions

with

$$\mathbf{K} = \alpha Id, \text{ for } \alpha = \begin{cases} 1 & \text{on } \Omega_1 \cup \Omega_3, \\ 10^{-2} & \text{on } \Omega_2, \end{cases}$$

and

$$u(x, y) = \begin{cases} -(y - 0.2x - 0.375) \text{ on } \Omega_1, \\ -(y - 0.2x - 0.375)/10^{-2} \text{ on } \Omega_2, \\ -(y - 0.2x - 0.425) - 0.05/10^{-2} \text{ on } \Omega_3 \end{cases}$$

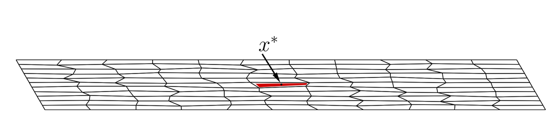
Test 8: perturbed parallelograms

$-\text{div}(\mathbf{K}\nabla u) = f$ in Ω
Homogeneous Dirichlet boundary conditions

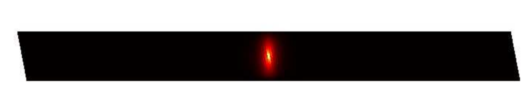
with $\mathbf{K} = Id$.
The source term f is defined by $f = \delta_{C^*}$

- C^* =cell (6, 6)
- $\int_{\text{cell}(6,6)} f(x)\,dx = 1.$

- $C^* = x^*$
- $\int_\Omega f(x)\,dx = 1.$



The distorted quadrangle mesh mesh9



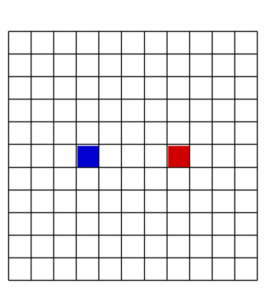
Obtained on a regular quadrangle fine grid

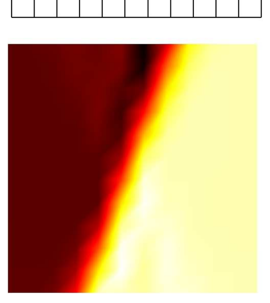
Minimum and maximum values

	umin	umax		umin	umax
Fine grid	**1.07E-24**	**4.10E-01**			
CMPFA	-2.31E-02	1.03E-01	FVHYB	-3.38E-02	1.12E-01
CVFE	-1.23E-03	4.24E-02	FVSYM	-7.21E-02	1.52E-01
DDFV-BHU	-1.25E-03	8.22E-02	FVPMMD	**1.22E-09**	**3.99E-01**
DDFV-HER	-1.61E-03	8.99E-02	MFD-BLS	**-1.03E-01**	1.85E-01
DDFV-MNI	-1.46E-03	6.69E-02	MFD-FHE	-6.54E-02	1.44E-01
DDFV-OMN	-1.77E-03	8.36E-02	MFD-MAR	-2.62E-02	9.07E-02
DG-C	-7.33E-03	1.05E-01	MFV	-8.08E-03	5.81E-02
DG-W	-9.03E-03	6.57E-02	NMFV	**3.05E-15**	9.42E-02
FEQ1	-4.17E-03	4.90E-02	SUSHI-NP	-1.19E-03	5.65E-02
FEQ2	-5.07E-03	8.04E-02	SUSHI-P	**3.26E-06**	**6.77E-03**

► Maximum principle satisfied only by the two nonlinear schemes FVPMMD and NMFV.

Test 9: anisotropy and wells





$-\text{div}(\mathbf{K}\nabla u) = 0$ in Ω
Neumann boundary conditions

with

$$\mathbf{K} = R_\theta^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 10^{-3} \end{bmatrix} R_\theta,$$

where $\theta = 67.5°$. and

$u = 0$ in cell (4, 6),
$u = 1$ in cell (8, 6).

Values of `umin` and `umax` should be `umin`=0 and `umax`=1

	umin	umax		umin	umax
CMPFA	-6.77E-01	1.68E+00			
CVFE	-1.16E-01	1.12E+00	FVPMMD	**0.00E+00**	**1.00E+00**
DDFV-BHU	-1.38E-01	1.14E+00	MFD-BLS	-4.30E-02	1.04E+00
DDFV-HER	-1.03E-01	1.10E+00	MFD-FHE	-4.21E-02	1.04E+00
DDFV-OMN	-7.07E-02	1.07E+00	MFD-MAR	-4.30E-02	1.04E+00
DG-C	**-1.02E-03**	**9.98E-01**	MFE	**0.00E+00**	**1.00E+00**
FEQ1	-2.36E-02	1.02E+00	MFV	-1.22E-01	1.07E+00
FEQ2	-5.94E-03	1.01E+00	NMFV	**1.83E-02**	**1.01E+00**
FVHYB	-3.69E-02	1.04E+00	SUSHI-NP	-1.00E+00	2.00E+00
FVSYM	-7.63E-02	1.07E+00	SUSHI-P	**0.00E+00**	**1.00E+00**

► Maximum principle satisfied only by the two nonlinear schemes FVPMMD NMFVand SUSHI-NPscheme.

Conclusions

⇝ Relative homogeneity of the results.

⇝ Higher order schemes are more precise in both L^2 and H^1 norms. DDFV methods are more precise in H^1 norm.

⇝ Cell centered schemes and mimetic schemes are generally more robust.

⇝ Nonlinear schemes seem necessary to enforce the maximum principle and/or the positiveness (or monotonicity) of the schemes.

Possible extensions

⇝ Positive schemes and schemes satisfying the maximum principle.

⇝ Coupling strongly heterogeneous problems with transport equations:

$$\begin{cases} -\text{div}(\mathbf{K}\nabla p) = 0 \\ c_t + \text{div}(f(c)\mathbf{K}\nabla p) = 0 \end{cases}$$

⇝ 3D tests on distorted nonconforming meshes.