Convergence of the Finite Volume MPFA O Scheme for Heterogeneous Anisotropic Diffusion Problems on General Meshes

L. Agelas* and R. Masson†

* Institut Français du Pétrole 1 & 4, avenue de Bois-Préau 92852 Rueil-Malmaison Cedex - France (leo.agelas@ifp.fr)

† Institut Français du Pétrole 1 & 4, avenue de Bois-Préau 92852 Rueil-Malmaison Cedex - France (roland.masson@ifp.fr)

Abstract

The Multipoint Flux Approximation (MPFA) O method is a cell centered finite volume discretization of second order elliptic operators. In the oil industry, it has been widely used for the discretization of diffusion fluxes in multiphase Darcy problems (see, e.g.,[2, 4, 5]). Up to now, the analysis has hinged on strong assumptions on the mesh, and discontinuous diffusion coefficients haven't been accounted for. In [1] we have stated the equivalence of the O method with a discrete variational formulation which fits in the framework proposed in [6, 7]. Convergence results have been obtained under mild assumptions on the mesh and for diffusion coefficients in $[L^\infty(\Omega)]^{d\times d}$. In particular, a sufficient condition ensuring coercivity has been proposed.

Introduction

We consider the following problem: find an approximation of $\tilde u$, weak solution to the equation:

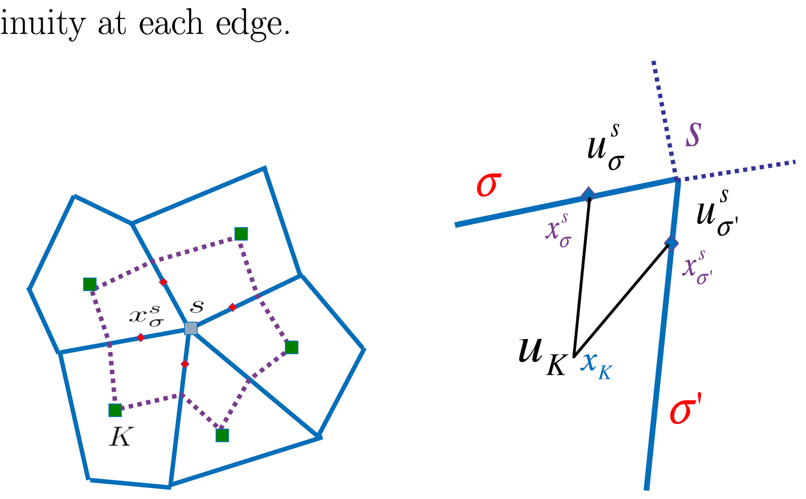
$$\begin{cases} -\mathrm{div}(\Lambda(x)\nabla \tilde u) = f, & \text{in } \Omega, \\ \tilde u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is an open bounded connected polygonal subset of $\mathbb{R}^d, d \in \mathbb{N}^\star$, $f \in L^2(\Omega)$ and Λ is a measurable function from Ω to $\mathcal{M}_d(\mathbb{R})$ s.t. for a.e. $x \in \Omega$, $\Lambda(x)$ is symmetric and the set of its eigenvalues is included in $[\alpha(x), \beta(x)]$ with $\alpha, \beta \in L^\infty(\Omega)$.
A function $\tilde u \in H^1_0(\Omega)$ is said to be a weak solution of (1) if

$$\int_\Omega \Lambda(x)\nabla \tilde u(x)\cdot\nabla v(x)\mathrm{d}x = \int_\Omega f(x)v(x)\mathrm{d}x, \quad \forall v \in H^1_0(\Omega). \tag{2}$$

Description of the Finite Volume MPFA O Scheme in 2D

The aim of the Finite Volume MPFA O Scheme is to compute fluxes at half edges around each vertex s. To compute fluxes:
- define a interaction zone in "O" (a set of subcells) around each vertex s;
- in each subcell of the interaction zone, assume that the potential u is linear;
- enforce potential continuity at points x^s_σ;
- enforce flux continuity at each edge.



Space discretization

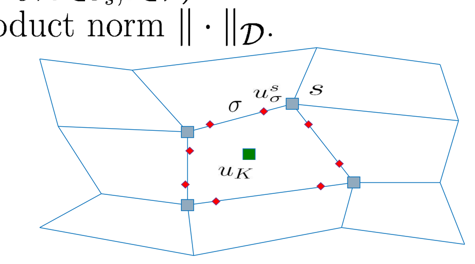
The finite volume discretization of the domain Ω is given by $\mathcal{D} = (\mathcal{M}, \mathcal{P}, \mathcal{V}, \mathcal{E})$. Define
- $\mathcal{M}$, the set of cells,
- $\mathcal{P}$, the set of centers,
- $\mathcal{V}$, the set of vertices,
- $\mathcal{E}$, the set of faces,
- $\mathcal{E}_s$, the faces sharing the vertex s,
- $\mathcal{V}_K$, the set of the vertices of the cell K,
- $\mathcal{V}_\sigma$, the set of the vertices of the face σ,
- $\mathcal{E}_K$, the set of the faces of the cell K.

Let $\mathcal{S}$ be the set of subcells and $h_\mathcal{D}$ the size of discretization.

Discrete spaces and operators

Define the following discrete spaces:
- $\mathcal{H}_\mathcal{M}$, the set of piecewise constant functions on each cell $K \in \mathcal{M}$;
- $\mathcal{H}_S$, the set of piecewise constant functions on each subcell $K_s \in S$;
- $\mathcal{H}_\mathcal{D}$, the set $\big((u_K)_{K\in\mathcal{M}}, (u^s_\sigma)_{\sigma\in\mathcal{E}_s, s\in\mathcal{V}}\big) \in \mathbb{R}^{\mathrm{card}(\mathcal{M})\times\mathrm{card}(\mathcal{E}_s)}$ s.t. $u^s_\sigma = 0$ for all $\sigma \in \mathcal{E}_{\mathrm{ext}}$.

Equip $\mathcal{H}_\mathcal{D}$ with the inner product norm $\|\cdot\|_\mathcal{D}$.



We shall also need the following operators:
- $P_\mathcal{M}$ is defined for all $u \in \mathcal{H}_\mathcal{D}$ by $(P_\mathcal{M} u)(x) = u_K$ for a.e. $x \in K$, for all $K \in \mathcal{M}$;
- $P_\mathcal{D}$ is the projection of the set of compactly supported continuous functions onto $\mathcal{H}_\mathcal{D}$.

MPFA Oscheme and its variational formulation

The idea of the proof can be summarized as follows:
- find a discrete gradient reconstruction $\nabla_\mathcal{D} : \mathcal{H}_\mathcal{D} \to \mathcal{H}^d_S$ which strongly approximates the gradient of smooth functions in $[L^2(\Omega)]^d$;
- find a second discrete gradient reconstruction $\widetilde\nabla_\mathcal{D} : \mathcal{H}_\mathcal{D} \to \mathcal{H}^d_S$ to weakly approximate gradients in $[L^2(\Omega)]^d$;
- substitute the discrete gradients in the variational formulation (2);
- obtain (2) as the limit of the discrete variational formulation for $h_\mathcal{D} \to 0$.

The discrete variational formulation is defined as follows: find $u \in \mathcal{H}_\mathcal{D}$

$$\int_\Omega \nabla_\mathcal{D} u(x)\cdot\Lambda(x)\widetilde\nabla_\mathcal{D} v(x)\mathrm{d}x = \int_\Omega f(x)P_\mathcal{M} v(x)\mathrm{d}x \quad \forall v \in \mathcal{H}_\mathcal{D}, \tag{3}$$

where
- $\nabla_\mathcal{D}$ is s.t., for each piecewise linear function u on S which vanishes on $\partial\Omega$, $\nabla_\mathcal{D}(P_\mathcal{D} u) = \nabla u$;
- $\widetilde\nabla_\mathcal{D}$ is defined as follows: for all $u \in \mathcal{H}_\mathcal{D}$ and for a.e. $x \in K_s$,

$$(\widetilde\nabla_\mathcal{D} u)(x) = \frac{1}{|K_s|}\sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_s} \mathrm{m}^s_\sigma(u^s_\sigma - u_K)\mathbf{n}_{K,\sigma}.$$

Then, the variational formulation (3) is equivalent to the Finite Volume MPFA O Scheme

Convergence of the scheme

Under mild regularity assumptions on the mesh, the following inequalities hold for all $u \in \mathcal{H}_\mathcal{D}$ (see e.g. [3]):
- for all $q \in [2, 2d/(d-2)]$, $\|P_\mathcal{M} u\|_{L^q(\Omega)} \le q\sqrt{d}\, C_{\mathrm{sob}}\|u\|_\mathcal{D}$;
- $\|P_\mathcal{M} u(\cdot+\xi) - P_\mathcal{M} u\|_{L^1(\mathbb{R}^d)} \le |\xi|\,\|u\|_\mathcal{D}(d\,|\Omega|)^{1/2}$, $\forall \xi \in \mathbb{R}^d$;
- $\|\widetilde\nabla_\mathcal{D} u\|_{L^2(\Omega)} \le \sqrt{d}\|u\|_\mathcal{D}$.

Let $a_\mathcal{D}$ be the bilinear form defined as follows:

$$a_\mathcal{D}(u,v) = \int_\Omega \nabla_\mathcal{D} u(x)\cdot\Lambda(x)\widetilde\nabla_\mathcal{D} v(x)\mathrm{d}x \quad \forall (u,v) \in \mathcal{H}_\mathcal{D}\times\mathcal{H}_\mathcal{D}. \tag{4}$$

Provided $a_\mathcal{D}$ is coercive (i.e., $\exists\alpha > 0, \forall u \in \mathcal{H}_\mathcal{D}, a_\mathcal{D}(u,u) \ge \alpha\|u\|^2_\mathcal{D}$), $\|u\|_\mathcal{D} \le 2\sqrt{d}\, C_{\mathrm{sob}}\|f\|_{L^2(\Omega)}$. Rellich theorem states that
- $\exists \overline u \in H^1_0(\Omega)$, $u \to \overline u$ in $L^2(\Omega)$ as $h_\mathcal{D} \to 0$;
- $\widetilde\nabla_\mathcal{D} u$ converges weakly to $\nabla\overline u$ as $h_\mathcal{D} \to 0$.

Using the above results together with (3), we can prove that $\nabla_\mathcal{D} u \to \nabla\overline u$ strongly in $[L^2(\Omega)]^{d\times d}$ as $h_\mathcal{D} \to 0$.The convergence of the method can be proved taking $v = P_\mathcal{D}\varphi$, $\varphi \in C^\infty_c(\Omega)$, as a test function in (3) and letting $h_\mathcal{D} \to 0$.

To improve coercivity on general meshes, we can add penalty terms to the bilinear form $a_\mathcal{D}$, thus obtaining, for all $(u,v) \in \mathcal{H}_\mathcal{D}\times\mathcal{H}_\mathcal{D}$,

$$\tilde a_\mathcal{D}(u,v) = a_\mathcal{D}(u,v) + \sum_{K\in\mathcal{T}}\sum_{s\in\mathcal{V}_K}\left(\alpha^s_K \sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_s} \frac{|K_s|}{d(x_K,\sigma)^2} R^s_{K,\sigma}(u)R^s_{K,\sigma}(v)\right). \tag{5}$$

For all $u \in \mathcal{H}_\mathcal{D}$, $\sigma \in \mathcal{E}_K\cap\mathcal{E}_s$, $s \in \mathcal{V}_K$, $K \in \mathcal{T}$, the residual terms $R^s_{K,\sigma}$ are defined by

$$R^s_{K,\sigma}(u) = u^s_\sigma - u_K - (\nabla_\mathcal{D} u)_{K_s}\cdot(x^s_\sigma - x_K).$$

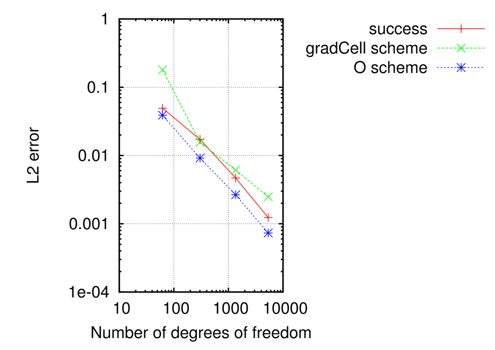
The convergence can be proved using a similar argument.

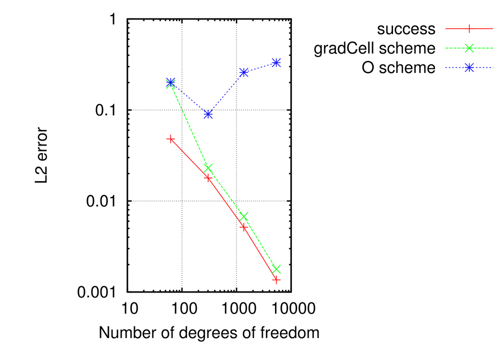
Numerical tests

Consider the following exact solution to (2):

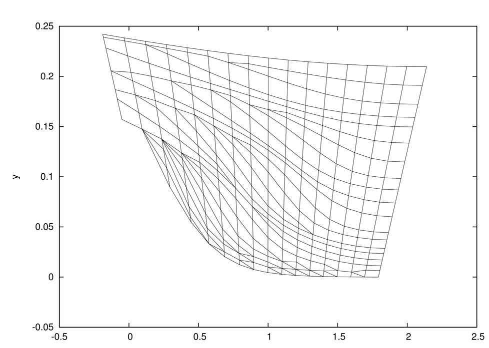
$$u = \sin(\pi x)\sin(\pi y), \quad K = \begin{bmatrix}\delta & 0\\ 0 & 1\end{bmatrix}. \tag{6}$$

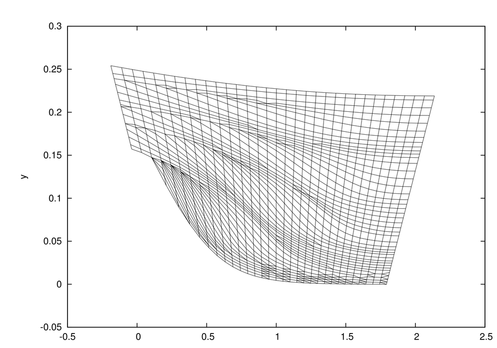
Two convergence tests with $\delta = 1$ and $\delta = 0.1$ were performed.





The tests have been run on 2d basin-like meshes.





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