A symmetric finite volume scheme for anisotropic heterogeneous second-order elliptic problems

Leo Agelas* and Daniele A. Di Pietro†

* Institut Français du Pétrole 1 & 4, avenue de Bois-Préau 92852 Rueil-Malmaison Cedex - France (leo.agelas@ifp.fr)

† Institut Français du Pétrole 1 & 4, avenue de Bois-Préau 92852 Rueil-Malmaison Cedex - France (daniele-antonio.di-pietro@ifp.fr)

Abstract

In this paper, we assess a new family of finite volume discretization schemes on benchmark test cases. These are based on the discrete variational formulation framework developped in [EGH 08], [EH 07], [EGH 07]. The use of a subgrid for each cell of the mesh enables us to obtain fluxes only between cells sharing an edge as opposed to the cell centered finite volume scheme [EGH 07] for which fluxes are also defined between cells sharing only a vertex. The resulting finite volume schemes are cell centered, symmetric and coercive on general polygonal and polyhedral meshes and anisotropic heterogeneous media and can be proved to be convergent even for L^∞ diffusion coefficients under usual shape regularity assumptions. Using L type interpolation from [AEMN 07], [AAV 07] for the intermediate subgrid unknowns enable us to take into account large jumps of the diffusion coefficients.

Introduction

We consider the following problem: find an approximation of $\tilde{u}$, weak solution to the equation:

$$\begin{cases} -\mathrm{div}\Lambda(x)\nabla\tilde{u} = f, \text{ in } \Omega \\ \tilde{u} = 0, \quad \tilde{u} \in \partial\Omega. \end{cases} \qquad [1]$$

where Ω is an open bounded connected polygonal subset of $\mathbb{R}^d, d \in \mathbb{N}^\star$, $f \in L^2(\Omega)$, Λ is a measurable function from Ω to $\mathcal{M}_d(\mathbb{R})$ such that for a.e. $x \in \Omega$, $\Lambda(x)$ is symmetric and the set of its eigenvalues is included in $[\alpha(x), \beta(x)]$ with $\alpha, \beta \in L^\infty(\Omega)$.

A function $\tilde{u} \in H_0^1(\Omega)$ is said to be a weak solution of (1) if

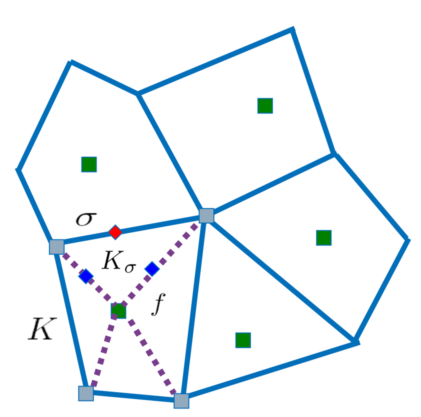
$$\begin{cases} \tilde{u} \in H_0^1(\Omega), \\ \displaystyle\int_\Omega \Lambda(x)\nabla\tilde{u}(x)\cdot\nabla v(x)\mathrm{d}x = \int_\Omega f(x)v(x)\mathrm{d}x, \quad \forall v \in H_0^1(\Omega). \end{cases} \qquad [2]$$

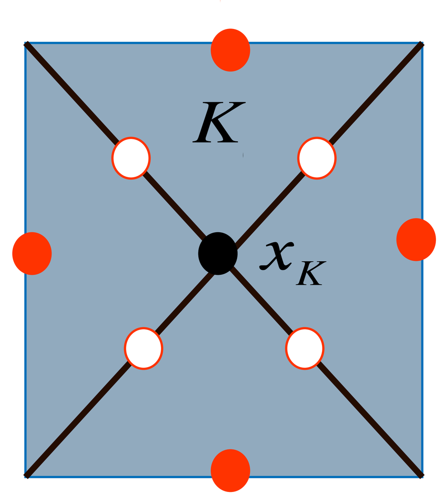
Finite volume discretization

The finite volume discretization of the domain Ω is given by $\mathcal{D} = (\mathcal{K}, \mathcal{E}, \mathcal{P})$.

- $\mathcal{K}$ the set of cells
- $\mathcal{E}$ the set of faces
- $\mathcal{P}$ the set of centers of the cells
- $\mathcal{E}_\kappa$ the set of the faces of the cell κ
- $\mathcal{E}_{\text{ext}}$ the set of boundaries faces

The size of the mesh is defined by $h_\mathcal{K} = \sup_{\kappa\in\mathcal{K}} \text{diameter}(\kappa)$. For each cell κ, we define subcells by the set of pyramids $\{\kappa_\sigma\}_{\sigma\in\mathcal{E}_\kappa}$ joining the face σ to the cell center x_κ.
Let $\mathcal{E}_{\kappa_\sigma}$ be the set of faces of the subcell κ_σ. Let S be the set of all subcells.





Discretes spaces and operators

Define the following discrete spaces:

- $H_\mathcal{K}$, the space of piecewise constant functions on each cell $\kappa \in \mathcal{K}$ identified to $\mathbb{R}^\mathcal{K}$.
- $\mathcal{H}_S$, the set of piecewise constant functions on each subcell $\kappa_\sigma \in S$.
- $H_{\mathcal{K},\mathcal{E}} = H_\mathcal{K} \times \{(v_\sigma)_{\sigma\in\mathcal{E}};\ v_\sigma \in \mathbb{R}$ for all $\sigma \in \mathcal{E}\}$ equipped with the seminorm $\|\cdot\|_{H_{\mathcal{K},\mathcal{E}}}$

We shall also need the following operators:

- $P_\mathcal{K}$ is defined for all $v_{\mathcal{K},\mathcal{E}} \in H_{\mathcal{K},\mathcal{E}}$ by $(P_\mathcal{K} v_{\mathcal{K},\mathcal{E}})(x) = v_\kappa$ for a.e. $x \in \kappa$, for all $\kappa \in \mathcal{K}$.
- $P_{\mathcal{K},\mathcal{E}}$ is the projection of the set of continuous functions which vanish on $\partial\Omega$ to $H_{\mathcal{K},\mathcal{E}}$.

A symmetric, coercive, convergent cell centered finite volume scheme on general polygonal and polyhedral meshes

The idea of the proof can be summarized as follows:
- find a discrete gradient reconstruction $\nabla_{\mathcal{K},\mathcal{E}} : \mathcal{H}_{\mathcal{K},\mathcal{E}} \to \mathcal{H}_S^d$ which strongly approximates the gradient of smooth functions in $[L^2(\Omega)]^d$;
- substitute the discrete gradient in the variational formulation [2];
- obtain [2] as the limit of the discrete variational formulation for $h_\mathcal{D} \to 0$.

The discrete variational formulation is defined as follows: find $u \in \mathcal{H}_{\mathcal{K},\mathcal{E}}$

$$\int_\Omega \nabla_{\mathcal{K},\mathcal{E}} u(x)\cdot\Lambda(x)\nabla_{\mathcal{K},\mathcal{E}} v(x)\mathrm{d}x = \int_\Omega f(x)P_\mathcal{K} v(x)\mathrm{d}x \quad \forall v \in \mathcal{H}_{\mathcal{K},\mathcal{E}}. \qquad [3]$$

where

- $\nabla_{\mathcal{K},\mathcal{E}}$ is such that for each $v \in \mathcal{H}_{\mathcal{K},\mathcal{E}}$ and for each subcell $\kappa_\sigma \in S$

$$(\nabla_{\mathcal{K},\mathcal{E}} v)_{\kappa_\sigma} = \frac{1}{|\kappa_\sigma|}\Big[|\sigma|(v_\sigma - v_\kappa)\mathbf{n}_{\kappa,\sigma} + \sum_{e\in\mathcal{E}_{\kappa_\sigma}} |e|\Big(v_e - v_\kappa\Big)\mathbf{n}_{\kappa_\sigma,e}\Big].$$

where $v_e = \Pi_e(v_\mathcal{K}, v_{\mathcal{E}_{\text{ext}}})$ and Π_e such that $|\Pi_e((\varphi(x_\kappa), \kappa \in \mathcal{K}), (\varphi(x_\sigma), \sigma \in \mathcal{E}_{\text{ext}})) - \varphi(x_e)| \le C(\varphi)\ h_\mathcal{K}\ \epsilon(h_\mathcal{K})$ for all $\varphi \in C_c^\infty$.

The above formulation (3) is equivalent to a hybrid finite volume scheme where the fluxes are such that

$$a_\mathcal{D}(u_{\mathcal{K},\mathcal{E}}, v_{\mathcal{K},\mathcal{E}}) = \sum_{\kappa\in\mathcal{K}}\sum_{\sigma\in\mathcal{E}_\kappa} F_{\kappa,\sigma}(u_\mathcal{K}, u_\mathcal{E})(v_\kappa - v_\sigma), \quad \forall v_{\mathcal{K},\mathcal{E}} \in H^0_{\mathcal{K},\mathcal{E}}. \qquad [4]$$

Convergence of the scheme

Under mild regularity assumptions on the mesh, the following inequalities hold for all $u \in \mathcal{H}_{\mathcal{K},\mathcal{E}}$ (see e.g. [EGH 00]):
- for all $q \in [2, 2d/(d-2)]$, $\|P_\mathcal{K} u\|_{L^q(\Omega)} \le q\,\sqrt{d}\,C_{\text{sob}}\,\|u\|_{H_{\mathcal{K},\mathcal{E}}}$
- $\|P_\mathcal{K} u(\cdot+\xi) - P_\mathcal{K} u\|_{L^1(\mathbb{R}^d)} \le |\xi|\ \|u\|_{H_{\mathcal{K},\mathcal{E}}}(d\,|\Omega|)^{1/2}$, $\forall \xi \in \mathbb{R}^d$,
- $\|\nabla_{\mathcal{K},\mathcal{E}} u\|_{L^2(\Omega)} \le \sqrt{d}\|u\|_{H_{\mathcal{K},\mathcal{E}}}$

Let $a_\mathcal{D}$ be the bilinear form defined as follows :

$$a_\mathcal{D}(u_{\mathcal{K},\mathcal{E}}, v_{\mathcal{K},\mathcal{E}}) = \int_\Omega \nabla_{\mathcal{K},\mathcal{E}} u(x)\cdot\Lambda(x)\nabla_{\mathcal{K},\mathcal{E}} v(x)\mathrm{d}x \quad \forall (u_{\mathcal{K},\mathcal{E}}, v_{\mathcal{K},\mathcal{E}}) \in \mathcal{H}_{\mathcal{K},\mathcal{E}} \times \mathcal{H}_{\mathcal{K},\mathcal{E}}. \qquad [5]$$

Provided $a_\mathcal{D}$ is coercive (i.e., $\exists\alpha > 0$, $\forall u_{\mathcal{K},\mathcal{E}} \in \mathcal{H}_{\mathcal{K},\mathcal{E}}$, $a_\mathcal{D}(u_{\mathcal{K},\mathcal{E}}, u_{\mathcal{K},\mathcal{E}}) \ge \alpha\|u_{\mathcal{K},\mathcal{E}}\|^2_{\mathcal{K},\mathcal{E}}$), $\|u_{\mathcal{K},\mathcal{E}}\|_{\mathcal{K},\mathcal{E}} \le 2\,\sqrt{d}\,C_{\text{sob}}\|f\|_{L^2(\Omega)}$.

Rellich theorem states that
- $\exists \overline{u} \in H_0^1(\Omega)$, $u_{\mathcal{K},\mathcal{E}} \to \overline{u}$ in $L^2(\Omega)$ as $h_\mathcal{K} \to 0$
- $\nabla_{\mathcal{K},\mathcal{E}} u$ converges weakly to $\nabla\overline{u}$ in $L^2(\Omega)$ as $h_\mathcal{K} \to 0$.

The convergence of the method can be proved taking $v = P_{\mathcal{K},\mathcal{E}}\varphi$, $\varphi \in C_c^\infty(\Omega)$, as a test function in (3) and letting $h_\mathcal{K} \to 0$.

To ensure coercivity, we add penalty terms to the bilinear form $a_\mathcal{D}$, thus obtaining, for all $(u\mathcal{K},\mathcal{E}, v\mathcal{K},\mathcal{E}) \in \mathcal{H}_{\mathcal{K},\mathcal{E}} \times \mathcal{H}_{\mathcal{K},\mathcal{E}}$,

$$\tilde{a}_\mathcal{D}(u_{\mathcal{K},\mathcal{E}}, v_{\mathcal{K},\mathcal{E}}) = a_\mathcal{D}(u_{\mathcal{K},\mathcal{E}}, v_{\mathcal{K},\mathcal{E}}) + \sum_{\kappa\in\mathcal{K}}\sum_{\sigma\in\mathcal{E}_\kappa}\Big[\alpha_{\kappa,\sigma}\sum_{s\in\mathcal{E}_{\kappa_\sigma}\cup\{\sigma\}}\frac{|s|}{d_{\kappa_\sigma,s}}R_{\kappa_\sigma,s}(u_{\mathcal{K},\mathcal{E}})R_{\kappa_\sigma,s}(v_{\mathcal{K},\mathcal{E}})\Big],$$

where $\alpha_{\kappa,\sigma}$ is a positive real and the residuals $R_{\kappa_\sigma,s}$ are defined as follows:

$$R_{\kappa_\sigma,s}(v_{\mathcal{K},\mathcal{E}}) = v_s - v_\kappa - (x_\sigma - x_\kappa)^t(\nabla v_{\mathcal{K},\mathcal{E}})_{\kappa_\sigma}$$

The convergence can be proved using a similar argument.

Numerical tests

• **Test 5 Heterogeneous rotating anisotropy,** min $= 0$, max $= 1$ **, uniform rectangular mesh,** `mesh2`

i	nunkw	nnmat	sumflux	erl2	ratioerl2	ergrad	ratioergrad
2_1	16	180	−4.19e−01	1.07e+01	–	2.28e+00	–
2_2	64	1012	−1.27e−01	1.50e+00	2.83e+00	1.46e+00	6.42e−01
2_3	256	4692	−1.05e−01	2.20e−01	2.77e+00	7.15e−01	1.03e+00
2_4	1024	20116	−7.03e−02	3.99e−02	2.47e+00	3.20e−01	1.16e+00
2_5	4096	83220	−3.55e−02	8.69e−03	2.20e+00	1.06e−01	1.59e+00
2_6	16384	338452	−1.33e−02	2.05e−03	2.09e+00	2.99e−02	1.83e+00

ocvl2=2.09, **ocvgradl2**=1.83

i	umin	umax
2_1	−1.92e+01	5.38e+00
2_2	−5.28e+00	1.34e+00
2_3	−1.39e+00	1.03e+00
2_4	−3.57e−01	1.00e+00
2_5	−9.06e−02	1.00e+00
2_6	−2.28e−02	1.00e+00

• **Test 6 Oblique drain,** min $= -1.2$, max $= 0$, **Coarse** `mesh6` **and Fine** `mesh7` **oblique meshes**

i	nunkw	nnmat	sumflux	erl2	ergrad
C	210	3748	−4.48e−14	8.18e−16	8.93e−15
F	230	3976	2.10e−12	3.41e−11	3.65e−09

i	umin	umax
C	−1.15e+00	−5.43e−02
F	−1.15e+00	−5.43e−02

References

[AAV 07] Aavatsmark I., Eigestad G.T., Heimsund B.O., Mallison B.T. and Nordbotten J.M. A new Finite Volume Approach to Efficient Discretization on Challenging Grids, *Proc. SPE 106435, Houston, 2007.*

[AEMN 07] Aavatsmark I., Eigestad G.T., Mallison B.T. and Nordbotten J.M. A compact multipoint flux approximation method with improved robustness, *Numerical Methods for Partial Differential Equations, 2007.*

[EGH 00] Eymard R., Gallouët T. and Herbin R. The Finite Volume Method, *Handbook of Numerical Analysis, P.G. Ciarlet, J.L. Lions editors, Elsevier,* **7**, *2000.*

[EGH 07] Eymard R., Gallouët T. and Herbin R., A new finite volume scheme for anisotropic diffusion problems on general grids: convergence analysis, *Comptes rendus Mathmatiques de l'Acadmie des Sciences*, 344,6, 2007, p. 403-406.

[EGH 08] Eymard R., Gallouët T., Herbin R., Discretization schemes for heterogeneous and anisotropic diffusion problems on general nonconforming meshes, *arXiv : 0804.1430.*

[EH 07] Eymard R. and Herbin R., A new colocated finite volume scheme for the incompressible Navier-Stokes equations on general non matching grids, *Comptes rendus Mathmatiques de l'Acadmie des Sciences*, 344(10), p. 659-662, 2007.