A Gradient Reconstruction Formula for Finite Volume Schemes and Discrete Duality

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Contents

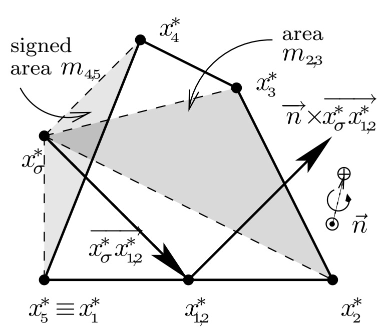
- A 2D affine interpolation formula. Reconstruction of discrete gradient on a 2D "median dual mesh".
- Associated discrete duality and discrete entropy dissipation formulae.
- Application in 3D (we refer to [H2] for a numerical study; see [P] for a different 3D scheme) : an analogue of the "DDFV" schemes in the spirit of Hermeline and Domelevo, Omnès. Discrete duality and entropy dissipation formulae.
- Usefulness of the discrete duality formulae.

Discrete duality formulae like formula (3) are a crucial tool of the "discrete calculus" for finite volume schemes. They allow in particular to discretize coercive and monotone diffusion operators with the help of coercive and monotone finite volume schemes. Further, (3) ensures that the variational character of a diffusion operator is preserved at the discrete level. If $\vec{\mathfrak{a}}(\cdot)$ is the gradient of a convex functional $\Phi(\cdot)$, so that the diffusion operator $-\mathrm{div}\,\vec{\mathfrak{a}}(\nabla w)$ derives from the functional $w\mapsto\int_\Omega\Phi(\nabla w)$, then the discrete diffusion operator $-\mathrm{div}^{\mathfrak{T}}\vec{\mathfrak{a}}(\nabla^{\mathfrak{T}}w^{\overline{\mathfrak{T}}})$ derives from the discrete functional $w^{\mathfrak{T}}\mapsto\sum_{D\in\mathfrak{D}}Vol(D)\,\Phi(\nabla^{\mathfrak{T}}w^{\overline{\mathfrak{T}}})$ (with the notation below). This allows us to calculate discrete solutions by minimization algorithms such as the Polak-Ribière method. Discrete entropy dissipation formulae like (4) require strong constraints on the geometry of the mesh; but they appear as necessary in order to work with the "nonlinear" notions of entropy/renormalized solutions.

- Application to a doubly nonlinear convection-diffusion problem : DDFV scheme in 2D and 3D, convergence proof.

Gradient interpolation from the vertices of a 2D polygon

Polygon $\sigma\subset\Pi$, oriented by $\vec n\perp\Pi$



Let Π be a plane in $\mathbb{R}^3$ with a unit normal vector $\vec n$, and $\sigma\subset\Pi$ be a polygon[1] with l vertices $x_1^*,\dots,x_l^*$ numbered in the counter-clockwise sense with respect to the orientation of Π induced by $\vec n$. Let x_{l+1}^* stand for x_1^*. Introduce the barycentre (i.e., the middlepoint) $x_{i,i+1}^*$ of $[x_i^*,x_{i+1}^*]$. Let $x_\sigma^*\in\Pi$. Introduce the (signed) area of the triangle $x_i^*x_\sigma^*x_{i+1}^*$:

$$m_{i,i+1}=0.5\,\langle\vec n,\overrightarrow{x_\sigma^*x_{i,i+1}^*},\overrightarrow{x_i^*x_{i+1}^*}\rangle\qquad(1)$$

Denote the area of σ by m; we have $m=\sum_{i=1}^l m_{i,i+1}$.

Lemma 1 *For all $\vec r\parallel\Pi$,* $\quad\vec r=\frac{1}{m}\sum_{i=1}^l(\vec r\cdot\overrightarrow{x_i^*x_{i+1}^*})\,[\vec n\times\overrightarrow{x_\sigma^*x_{i,i+1}^*}]$.

The proof just combines the formulae of [ED,Lemma 6.1],[ABH,Lemma 2.4].

Corollary 2 *Take $(w_i^*)_{i=1}^l\subset\mathbb{R}$, $w_{l+1}^*:=w_1^*$. Consider the expression*

$$\Big(\sum_{i=1}^l m_{i,i+1}\Big)^{-1}\sum_{i=1}^l(w_{i+1}^*-w_i^*)\,[\vec n\times\overrightarrow{x_\sigma^*x_{i,i+1}^*}].\qquad(2)$$

If w_i^* are the values of an affine function w at the vertices x_i^* of the polygon σ, expression (2) gives the projection of ∇w on the plane Π.

Remark 3 *We guess that the affine interpolation formula (2) is well known. Unless $l=3$, formula (2) is one among infinitely many linear forms in $(w_i^*)_{i=1}^l$ which share the consistency property of Corollary 2. Our choice of (2) is motivated by the calculation that leads to the discrete duality property (3). If $l=3$, then (2) is equivalent to any of the known formulae for three-point affine interpolation.*

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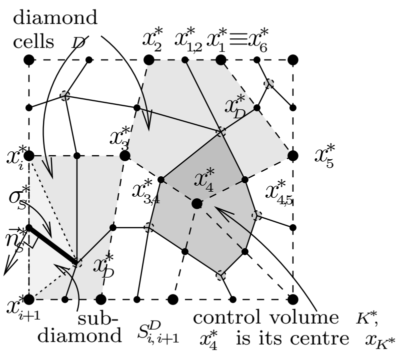
[1] In what follows, we identify $\mathbb{R}^2$ with $\mathbb{R}^2\times\{0\}\subset\mathbb{R}^3$, set $\vec n=\overrightarrow{(0,0,1)}$ and use the 3D formalism : $\|\vec a\|$ denotes the euclidean norm of $\vec a\in\mathbb{R}^3$; by $\vec a\cdot\vec b$, $\vec a\times\vec b$ and $\langle\vec a,\vec b,\vec c\rangle$ we denote the scalar, vector and mixed products, respectively, for $\vec a,\vec b,\vec c\in\mathbb{R}^3$.

2D mesh, discrete gradient and divergence operators. Discrete duality formulae.



- We consider $\mathfrak{T}$ a specific couple $(\mathfrak{D},\overline{\mathfrak{M}^*})$ of meshes. We take $\mathfrak{D}$ a partition (e.g., a triangulation) of Ω; each element of $\mathfrak{D}$ is denoted by D and called a *diamond cell*. Each $D\in\mathfrak{D}$ is supplied with a centre x^*_D; for the sake of simplicity, one may assume that $x^*_D\in D$ and D is convex.
- For each $D\in\mathfrak{D}$, we fix a counter-clockwise numbering of its vertices by $x^*_1,\dots,x^*_l$ $(l\ge 3)$, letting $l{+}1:=1$. We set $x^*_{i,i+1}=0.5\,(x^*_i+x^*_{i+1})$ (the middlepoint of $[x^*_i,x^*_{i+1}]$).
- A generic vertex of $\mathfrak{D}$ is denoted by x_{K^*}. Each x_{K^*} is the centre of a *control volume* K^*. The mesh $\overline{\mathfrak{M}^*}=\mathfrak{M}^*\cup\partial\mathfrak{M}^*$ is the *median dual mesh* of $\mathfrak{D}$. If $x_{K^*}\in\partial\Omega$, we write $K^*\in\partial\mathfrak{M}^*$; if $x_{K^*}\in\Omega$, we write $K^*\in\mathfrak{M}^*$. In the case each D admits a circumcircle and x^*_D is its centre, $\mathfrak{M}^*$ is just the Voronoï dual mesh of $\mathfrak{D}$.

- Each diamond $D\in\mathfrak{D}$ is a polygon with vertices $x^*_1=x_{K^*_1}$, ..., $x^*_l=x_{K^*_l}$; it is split into l *subdiamonds* $S^D_{i,i+1}$ which are the triangles with vertices x^*_D, x^*_i, x^*_{i+1}. For $K^*\in\mathfrak{M}^*$, $\nu^*(K^*)$ is the set of all subdiamonds having x_{K^*} for a vertex. The set of all subdiamonds is denoted by $\mathfrak{S}$. In a subdiamond $S=S^D_{i,i+1}$, we denote by σ^*_S the part of $\partial K^*_i\cap\partial K^*_{i+1}$ included into S; we denote its length by m^*_S. We have $\sigma^*_S\equiv[x^*_D,x^*_{i,i+1}]$; denote by $\vec n^*_S$ its unit normal vector such that $\vec n^*_S=\vec n\times\overrightarrow{x^*_Dx^*_{i,i+1}}/m^*_S$ (if $m^*_S\equiv\|\overrightarrow{x^*_Dx^*_{i,i+1}}\|=0$, $\vec n^*_S$ is arbitrary).
- For $K^*\in\mathfrak{M}^*$ and $S\in\nu^*(K^*)$, set $\epsilon^{K^*}_S:=0$ if $K^*=K^*_i$, and $\epsilon^{K^*}_S:=1$ if $K^*=K^*_{i+1}$.

> Diamonds, respectively subdiamonds, serve to define the discrete gradient, respectively the discrete divergence operators between the spaces of discrete functions and discrete fields defined below.

- A *discrete function on* Ω is a set $w^{\mathfrak{T}}=\big(w_{K^*}\big)_{K^*\in\mathfrak{M}^*}$ of real values. The set of all such functions is denoted by $\mathbb{R}^{\mathfrak{T}}$. A discrete function $w^{\mathfrak{T}}$ is identified with $x\in\Omega\mapsto\sum_{K^*\in\mathfrak{M}^*}w_{K^*}1\!\!1_{K^*}(x)$. Similarly, a *discrete function on* $\overline{\Omega}$, $w^{\overline{\mathfrak{T}}}\in\mathbb{R}^{\overline{\mathfrak{T}}}$, is a set $\big(w_{K^*}\big)_{K^*\in\overline{\mathfrak{M}^*}}$. If the components of w_{K^*} with $K^*\in\partial\mathfrak{M}^*$ are zero, we write $w^{\overline{\mathfrak{T}}}\in\mathbb{R}^{\overline{\mathfrak{T}}}_0$.
- A *discrete field on* Ω is a set $\vec{\mathcal{F}}^{\mathfrak{T}}=\big(\vec{\mathcal{F}}_D\big)_{D\in\mathfrak{D}}$ in $\mathbb{R}^2$. If a subdiamond S is included into D, we set $\vec{\mathcal{F}}_S:=\vec{\mathcal{F}}_D$. The set of all discrete fields is denoted by $(\mathbb{R}^2)^{\mathfrak{D}}$, and identify $\vec{\mathcal{F}}^{\mathfrak{T}}$ with $x\in\Omega\mapsto\sum_{D\in\mathfrak{D}}\vec{\mathcal{F}}_D1\!\!1_D(x)$.

> On $\mathbb{R}^{\mathfrak{T}}$, resp. on $(\mathbb{R}^2)^{\mathfrak{D}}$, define the scalar products
>
> $$[\![w^{\mathfrak{T}},v^{\mathfrak{T}}]\!]=\sum_{K^*\in\mathfrak{M}^*}Vol(K^*)\,w_{K^*}v_{K^*},\qquad\text{respectively}\quad\big\{\vec{\mathcal{F}}^{\mathfrak{T}},\vec{\mathcal{G}}^{\mathfrak{T}}\big\}=\sum_{D\in\mathfrak{D}}Vol(D)\,\vec{\mathcal{F}}_D\cdot\vec{\mathcal{G}}_D$$

- Define the *discrete gradient* operator $\nabla^{\mathfrak{T}}$ and the *discrete divergence* operator $\operatorname{div}^{\mathfrak{T}}$:

> $$\nabla^{\mathfrak{T}}:\ \begin{array}{l}\mathbb{R}^{\overline{\mathfrak{T}}}\longrightarrow(\mathbb{R}^2)^{\mathfrak{D}}\\ w^{\overline{\mathfrak{T}}}\mapsto\big(\nabla_Dw^{\overline{\mathfrak{T}}}\big)_{D\in\mathfrak{D}},\end{array}$$
> where the value $\nabla_Dw^{\overline{\mathfrak{T}}}$ is reconstructed by formulae (2),(1) from the values $u^*_1=w_{K^*_1},\dots,u^*_l=w_{K^*_l}$ of $w^{\overline{\mathfrak{T}}}$ at the vertices $x^*_1=x_{K^*_1},\dots,x^*_l=x_{K^*_l}$ of $\sigma=D$, with $x^*_\sigma=x^*_D$.

> $$\operatorname{div}^{\mathfrak{T}}:\ \begin{array}{l}(\mathbb{R}^2)^{\mathfrak{D}}\longrightarrow\mathbb{R}^{\mathfrak{T}}\\ \vec{\mathcal{F}}^{\mathfrak{T}}\mapsto(v_{K^*})_{K^*\in\mathfrak{M}^*},\end{array}$$
> where the entries v_{K^*} of the discrete function $\operatorname{div}^{\mathfrak{T}}\vec{\mathcal{F}}^{\mathfrak{T}}$ are given by
> $$\frac{1}{Vol(K^*)}\sum_{S\in\nu^*(K^*)}m^*_S\vec{\mathcal{F}}_S\cdot(-1)^{\epsilon^{K^*}_S}\vec n^*_S\equiv\frac{1}{Vol(K^*)}\sum_{S\in\nu^*(K^*)}(-1)^{\epsilon^{K^*}_S}\langle\,\vec{\mathcal{F}}_S,\vec n,\overrightarrow{x^*_Dx^*_{i,i+1}}\,\rangle.$$

Here we mean that each S in $\nu^*(K^*)$ is of the form $S^D_{i,i+1}$; the notation $\epsilon^{K^*}_S$, $x^*_D,x^*_{i,i+1}$ under the sign "$\sum$" refers to $S^D_{i,i+1}$. This formula corresponds to the standard finite volume (i.e. based upon Stokes' formula) discretization procedure on $\mathfrak{M}^*$. The value $Vol(K^*)\,v_{K^*}$ is the flux of the vector field $\vec{\mathcal{F}}^{\mathfrak{T}}$ through the boundary ∂K^*, thus it represents $\int_{K^*}\operatorname{div}\vec{\mathcal{F}}^{\mathfrak{T}}$. Indeed, thanks to the constraint $x^*_D\in D$, whenever x_{K^*} is a vertex of $D\supset S$, the vector $(-1)^{\epsilon^{K^*}_S}\vec n^*_S$ is the normal vector to $\sigma_S\subset\partial K^*$ exterior to K^*. The formulae for $\nabla^{\mathfrak{T}},\operatorname{div}^{\mathfrak{T}}$ work also when $x^*_D\notin D$; one just need to consider subdiamonds of signed area, as in (1). This allows to use e.g. the mesh $\mathfrak{T}=(\mathfrak{D},\overline{\mathfrak{M}^*})$ consisting of a Delaunay triangulation $\mathfrak{D}$ with its Voronoï dual mesh $\overline{\mathfrak{M}^*}$.

> **Proposition 4** *$(d=2)$ The discrete divergence and gradient operators* $\operatorname{div}^{\mathfrak{T}}$, $\nabla^{\mathfrak{T}}$ *defined above are linked by the following duality property :*
>
> $$\forall\,w^{\overline{\mathfrak{T}}}\in\mathbb{R}^{\overline{\mathfrak{T}}}_0\quad\forall\,\vec{\mathcal{F}}^{\mathfrak{T}}\in(\mathbb{R}^d)^{\mathfrak{D}}\qquad\Big[\!\!\Big[-\operatorname{div}^{\mathfrak{T}}[\vec{\mathcal{F}}^{\mathfrak{T}}],\,w^{\mathfrak{T}}\Big]\!\!\Big]=\Big\{\vec{\mathcal{F}}^{\mathfrak{T}},\,\nabla^{\mathfrak{T}}w^{\overline{\mathfrak{T}}}\Big\}.\qquad(3)$$

While property (3) is well suited for the study of diffusion problems by variational techniques, in the entropy or renormalized solutions setting it is not enough. Indeed, if we work with "nonlinear" test functions of the unknown solution (e.g., $\operatorname{sign}(u^{\mathfrak{T}} - k)$, or the "truncations" $T_k(u^{\mathfrak{T}})$), we need the "entropy dissipation" inequality (4) stated below. We are able to show it in the case $\overline{\mathfrak{M}^*}$ is "orthogonal". More exactly, we ask that each diamond cell D admits a circumcircle, and $x_D^* \in D$ is its centre (thus $\overline{\mathfrak{M}^*}$ is the Voronoï dual mesh of $\mathfrak{D}$).

For $A : \mathbb{R} \mapsto \mathbb{R}$ and a discrete function $u^{\overline{\mathfrak{T}}}$ with entries u_{K^*}, denote by $A(u^{\overline{\mathfrak{T}}})$ the discrete function with the entries $A(u_{K^*})$. For $\psi \in L^1(\Omega)$, denote by $\psi^{\overline{\mathfrak{T}}}$ the discrete function on $\overline{\Omega}$ with the entries $\psi_{K^*} = \frac{1}{Vol(K^*)} \int_{K^*} \psi$, $K^* \in \overline{\mathfrak{M}^*}$.

> Proposition 5 *(d=2) Let $\overline{\mathfrak{M}^*}$ be an orthogonal mesh in the above sense.*
> *Let $u^{\overline{\mathfrak{T}}} \in \mathbb{R}_0^{\overline{\mathfrak{T}}}$ be a discrete function, and $\mathcal{G}^{\mathfrak{T}} \in (\mathbb{R}^d)^{\mathfrak{D}}$ be a discrete field. Let $\psi \in \mathcal{D}(\overline{\Omega})$, $\psi \geq 0$.*
>
> *Let $S' : \mathbb{R} \mapsto \mathbb{R}$ be a bounded non-decreasing function, and $S(r) = \int_0^r S'(s)\,ds$.*
> *Let $A : \mathbb{R} \to \mathbb{R}$ be a non-decreasing continuous function; set $A_{S'}(r) = \int_0^r S'(s)\,dA(s)$.*
> *(e.g. in the case $A = Id$ and $S(r) = \operatorname{sign}(r-k)$, the function $A_{S'}(r) = |r-k|$ is the Kruzhkov's entropy).*
>
> *Assume that either $S'(0) = 0$, or $\psi \in \mathcal{D}(\Omega)$ and $\max_{K^* \in \partial\mathfrak{M}^*} \operatorname{diam}(K^*)$ is small enough. Then*
>
> $$\left[\!\left[-\operatorname{div}^{\mathfrak{T}} \left[|\mathcal{G}^{\mathfrak{T}}|\, \nabla^{\mathfrak{T}} A(u^{\overline{\mathfrak{T}}}) \right],\, S'(u^{\mathfrak{T}})\psi^{\mathfrak{T}} \right]\!\right] \geq \left\{\!\!\left\{ |\mathcal{G}^{\mathfrak{T}}|\, \nabla^{\mathfrak{T}} A_{S'}(u^{\overline{\mathfrak{T}}}),\, \nabla^{\mathfrak{T}} \psi^{\overline{\mathfrak{T}}} \right\}\!\!\right\}. \qquad (4)$$

Application ([ABK]) : $\left[\!\left[-\operatorname{div}^{\mathfrak{T}} \left[\left| \nabla^{\mathfrak{T}} u^{\overline{\mathfrak{T}}} \right|^{(p-2)} \nabla^{\mathfrak{T}} u^{\overline{\mathfrak{T}}} \right],\, \operatorname{sign}^+(u^{\mathfrak{T}} - k)\, \psi^{\mathfrak{T}} \right]\!\right] \geq \left\{\!\!\left\{ \left| \nabla^{\mathfrak{T}} u^{\overline{\mathfrak{T}}} \right|^{(p-2)} \nabla^{\mathfrak{T}} \left(u^{\overline{\mathfrak{T}}} - k \right)^+,\, \nabla^{\mathfrak{T}} \psi^{\overline{\mathfrak{T}}} \right\}\!\!\right\}.$

"Double" 3D finite volume scheme with the discrete duality

- A "double" finite volume mesh of a bounded polyhedre $\Omega \subset \mathbb{R}^3$ is a triple $\mathfrak{T} = \left(\overline{\mathfrak{M}}, \overline{\mathfrak{M}^*}, \mathfrak{D} \right)$. Here $\overline{\mathfrak{M}} = \mathfrak{M} \cup \partial\mathfrak{M}$, $\mathfrak{M}$ is a partition of Ω in tetrahedra K, called *primal volumes.* We call $\partial\mathfrak{M}$ the set of all faces of volumes that are included in $\partial\Omega$. These faces are considered as *boundary volumes.* For $K \in \partial\mathfrak{M}$, we choose a centre $x_K \in K$. The set of vertices of all volumes $K \in \overline{\mathfrak{M}}$ is denoted by $(x_{K^*})_{K^* \in \overline{\mathfrak{M}^*}}$; these are the centres of *dual volumes.*

- If $K_\odot, K_\oplus \in \overline{\mathfrak{M}}$ have a common face, we denote it by $K_\odot|K_\oplus$. Each face is supplied with a centre $x_{K_\odot|K_\oplus}$, assumed, for the sake of simplicity, to belong to $K_\odot|K_\oplus$. If x_{K^*}, x_{L^*} are neighbour vertices of $K_\odot|K_\oplus$, then $x_{K^*|L^*}$ is the middlepoint of $[x_{K^*}, x_{L^*}]$; $K^*|L^*$ is the common boundary of the dual volumes with centres x_{K^*}, x_{L^*}.

- Now, $\overline{\mathfrak{M}^*}$ is the dual mesh of $\overline{\mathfrak{M}}$ such that the dual volume K^* with centre x_{K^*} has its vertices in the set $(x_K)_K \cup (x_{K|L})_{K|L} \cup (x_{K^*|L^*})_{K^*|L^*}$ (see [H2]). We write $x_{K^*} \in \partial\Omega$ if $K^* \in \partial\mathfrak{M}^*$, and $K^* \in \mathfrak{M}^*$ otherwise.

- A couple of neighbours $\{K_\odot, K_\oplus\}$ define an (oriented) *diamond* $D = D^{K_\odot|K_\oplus}$. This is the convex hull of $x_{K_\odot}$, $x_{K_\oplus}$ and $K_\odot|K_\oplus$. The set $\mathfrak{D}$ of all diamonds is a partition of Ω. In an oriented diamond $D^{K_\odot|K_\oplus}$, $\vec{e}_{K_\odot,K_\oplus}$ is the unit vector pointing from $x_\odot$ to $x_\oplus$; $\vec{n}_{K_\odot|K_\oplus}$ is the unit normal vector to $K_\odot|K_\oplus$ such that $\vec{n}_{K_\odot|K_\oplus} \cdot \vec{e}_{K_\odot,K_\oplus} > 0$. This induces an orientation in the triangle $K_\odot|K_\oplus$ (see Fig.3). We denote by Proj_D, Proj_D^* the orthogonal projectors of $\mathbb{R}^3$ on the line $< \vec{e}_{K_\odot,K_\oplus} >$, resp. on the plane containing $K_\odot|K_\oplus$.

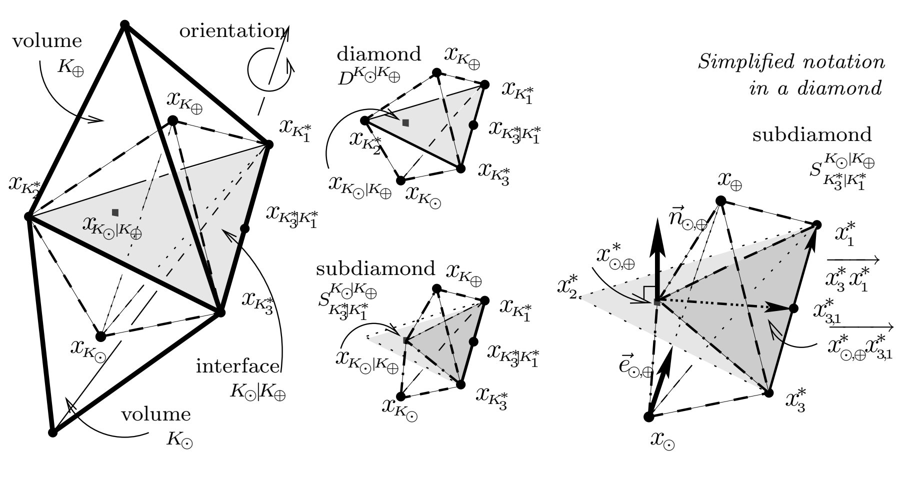


Figure 3: 3D neighbour volumes, diamond, subdiamond. Zoom on a subdiamond.

- As in 2D, each diamond $D^{K_\odot|K_\oplus}$ is split into $l=3$ subdiamonds; a generic subdiamond $S=S^{K_\odot|K_\oplus}_{K_i^*|K_{i+1}^*}$ is the convex hull of $x_{K_\odot}, x_{K_\oplus}, x_{K_\odot|K_\oplus}$ and of the neighbour vertices $x_{K_i^*}, x_{K_{i+1}^*}$ of $K_\odot|K_\oplus$. Whenever a diamond $D^{K_\odot|K_\oplus}$ is fixed, we use a simplified notation, as shown in Fig.3. Notations $\mathcal{V}(K)$, resp. $\mathcal{V}^*(K^*)$ stand for the sets of all subdiamonds intersecting K, resp., K^*. Further, for $S=S^{K_\odot|K_\oplus}_{K_i^*|K_{i+1}^*}$ we define

$$\epsilon_S^K := \begin{cases} 0, \text{ if } K=K_\odot \\ 1, \text{ if } K=K_\oplus \end{cases}, \qquad \epsilon_S^{K^*} := \begin{cases} 0, \text{ if } K^*=K_i^* \\ 1, \text{ if } K^*=K_{i+1}^* \end{cases}.$$

We define similarly the spaces $\mathbb{R}^{\mathfrak{T}}$, and $(\mathbb{R}^3)^{\mathfrak{D}}$; the appropriate scalar product on $\mathbb{R}^{\mathfrak{T}}$ is now

$$[\![w^{\mathfrak{T}}, v^{\mathfrak{T}}]\!] := \frac{1}{3}\sum_{K\in\mathfrak{M}} Vol(K)\, w_K v_K + \frac{2}{3}\sum_{K^*\in\mathfrak{M}^*} Vol(K^*)\, w_{K^*} v_{K^*}$$

The discrete gradient $\nabla^{\mathfrak{T}} w^{\overline{\mathfrak{T}}} \in (\mathbb{R}^3)^{\mathfrak{D}}$ is defined by its entries: for $D=D^{K_\odot|K_\oplus}$,

$$\nabla_D w^{\overline{\mathfrak{T}}} \text{ is s.t. } \begin{cases} \mathrm{Proj}_D(\nabla_D w^{\overline{\mathfrak{T}}}) = \frac{w_\oplus - w_\odot}{d_{\odot,\oplus}}\vec{e}_{\odot,\oplus}, \text{ with } w_\odot = w_{K_\odot},\ w_\oplus = w_{K_\oplus}; \\ \mathrm{Proj}^*_D(\nabla_D w^{\overline{\mathfrak{T}}}) \quad \text{is the vector defined by formulae (2), (1)} \\ \qquad\qquad \text{with } w_i^* = w_{K_i^*},\ \vec{n} = \vec{n}_{\odot,\oplus},\ x_\sigma^* = x^*_{\odot\oplus}. \end{cases} \tag{5}$$

In a sense, the primal (resp., dual) mesh yields $\frac{1}{3}$ (resp., $\frac{2}{3}$) of the components of $\nabla_D w^{\overline{\mathfrak{T}}}$ (like in $[\![\cdot,\cdot]\!]$!).

The discrete divergence $\mathrm{div}^{\mathfrak{T}} \vec{\mathcal{F}}^{\mathfrak{T}} \in \mathbb{R}^{\mathfrak{T}}$ is defined by its entries as follows:

$$\begin{aligned} v_K &= \frac{1}{2Vol(K)} \sum_{S\in\mathcal{V}(K)} (-1)^{\epsilon_S^K} \langle \vec{\mathcal{F}}_S, \overrightarrow{x^*_{\odot\oplus}x^*_{i,i+1}}, \overrightarrow{x_i^* x_{i+1}^*} \rangle, \\ v_{K^*} &= \frac{1}{2Vol(K^*)} \sum_{S\in\mathcal{V}^*(K^*)} (-1)^{\epsilon_S^{K^*}} \langle \vec{\mathcal{F}}_S, \overrightarrow{x_\odot x_\oplus}, \overrightarrow{x^*_{\odot\oplus}x^*_{i,i+1}} \rangle. \end{aligned} \tag{6}$$

As in 2D, (6) comes by calculation from the finite volume (Stokes'-formula based) discretization on $\mathfrak{M}$, $\mathfrak{M}^*$.

Proposition 6 *(3D analogue of Propositions 4,5)*
(i) With the above definitions, the discrete duality property (3) holds.
(ii) Further, assume that x_K is the centre of the circumscribed ball of K and $x_{K_\odot|K_\oplus}$ is the one of the circumscribed circle of $K_\odot|K_\oplus$ (in which case $\overline{\mathfrak{M}^*}$ is the Voronoï mesh dual to $\mathfrak{M}$). Let $A, S, \mathcal{G}^{\mathfrak{T}}, \psi, u^{\overline{\mathfrak{T}}}$ be as in Proposition 5. Then the discrete entropy dissipation inequality (4) holds.

A DDFV scheme for approximation of entropy solutions of a nonlinear convection-diffusion problem

In [ABK], we treat the "doubly nonlinear" problem

$$\begin{cases} \partial_t u + \mathrm{div}\, \vec{\mathfrak{f}}(u) - \mathrm{div}\, \vec{\mathfrak{a}}(\nabla w) = f,\ w = A(u) \quad \text{in } Q=(0,T)\times\Omega, \\ u|_{t=0} = u_0 \text{ in } \Omega, \quad u=0 \text{ on } \Sigma = (0,T)\times\partial\Omega, \end{cases} \tag{7}$$

with bounded data u_0, f. The function $\vec{\mathfrak{a}}:\mathbb{R}^N\to\mathbb{R}^N$, taken under the form $\vec{\mathfrak{a}}(\xi) = k(|\xi|)\xi$, is assumed to be of the Leray-Lions kind (i.e., $\mathrm{div}\,\vec{\mathfrak{a}}(\nabla\cdot)$ acts from $W_0^{1,p}$ into $W^{-1,p'}$). The nonlinearity $A(\cdot)$ is assumed continuous non-decreasing (it thus can degenerate, so that (7) is of hyperbolic-parabolic type). The convection flux function $\vec{\mathfrak{f}}:\mathbb{R}\to\mathbb{R}^3$ is assumed merely continuous.

We combine a finite volume discretization in space (2D, as in [ABH] / 3D, as described above) on the orthogonal "double" mesh with the time-implicit discretization in time with step Δt. The diffusion term at the time $n\Delta t$ is discretized under the form $-\mathrm{div}^{\mathfrak{T}}\vec{\mathfrak{a}}(\nabla^{\mathfrak{T}} w^{n,\overline{\mathfrak{T}}})$, with the notation above; here $u^{n,\overline{\mathfrak{T}}} \in \mathbb{R}_0^{\overline{\mathfrak{T}}}$ is the unknown solution at time $n\Delta t$, and $w^{n,\overline{\mathfrak{T}}} = A(u^{n,\overline{\mathfrak{T}}})$. A term penalizing the differences $(w_K - w_{K^*})$ for $K\cap K^*\neq\emptyset$ is added in order to enforce the convergence of the functions $w^{n,\mathfrak{M}} = \sum_{K\in\mathfrak{M}} A(u_K^n)1\!\!1_K$, $w^{n,\mathfrak{M}^*} = \sum_{K^*\in\mathfrak{M}^*} A(u^n_{K^*})1\!\!1_{K^*}$ to the same limit. The convection term is discretized in the usual way (see [EGHM] and references therein). We adapt the associated "weak BV" techniques to the non-Lipschitz case, and obtain an analogue of the entropy dissipation property of Proposition 5 for the convection terms. The scheme converges to the entropy solution of (7) in the sense of J. Carrillo (see [EGHM]). Formulae (3),(4) are essential; other tools come from [EGHM, DO, ABH].

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