Vector Penalty-Projection Methods
for the solution of unsteady incompressible flows

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Objectives and motivations

Work on the constraint of free divergence with penalty methods

- How to deal efficiently with the free-divergence constraint with splitting methods (prediction-correction steps) ?
- Overcome the major drawbacks of the projection methods including a scalar correction step of the Lagrange multiplier with a solution of a Poisson-type equation [Guermond et al., CMAME 06 - Jobelin et al., JCP 06]:
  - Time order of the splitting error ?
    i.e. error between the numerical solutions of the implicit (or semi-implicit) method and the fractional-step method
  - Spurious B.C. for pressure: $\nabla\phi\cdot\mathbf{n} = 0$ on Γ_D, $\phi = p^{n+1} - p^n$
    $\Rightarrow$ existence of an artificial pressure boundary layer in space
  - Open boundary conditions: $\phi = 0$ on Γ_N
    $\Rightarrow$ convergence in time and space spoiled for outflow B.C.: splitting error varying like $\mathcal{O}(\delta t^{\frac{1}{2}})$ (pressure) and no more negligible (for both velocity and pressure) with respect to the time and space discretization error
  - Pressure-correction step strongly dependent on density and viscosity for non-homogeneous flows
    $\Rightarrow$ very poor convergence for large ratios of $\varrho \sim 10^3$.

Example of flow models with the pressure field as Lagrange multiplier
$\Rightarrow$ solution of unsteady incompressible Navier-Stokes problems with the primitive variables (velocity and pressure): $\nabla\cdot\mathbf{v} = 0$
$\Rightarrow$ solution of magnetohydrodynamics (MHD) problems: $\nabla\cdot\mathbf{B} = 0$

A new family of vector penalty-projection methods: two-step penalty methods

The two-parameter family of $(\mathrm{VPP}_{r,\varepsilon})$ methods:
a family of two-step artificial compressibility methods

$\mathbf{v}^0 \in H^1(\Omega)^d$, $p^0 \in L^2_0(\Omega)$ given, for all $n \in \mathbb{N}$ s.t. $(n+1)\delta t \le T$,

Penalty-prediction step with an augmentation parameter $r \ge 0$

$$\frac{\tilde{\mathbf{v}}^{n+1} - \mathbf{v}^n}{\delta t} + (\mathbf{v}^n\cdot\nabla)\,\tilde{\mathbf{v}}^{n+1} - \frac{1}{\mathrm{Re}}\Delta\tilde{\mathbf{v}}^{n+1} - r\nabla\left(\nabla\cdot\tilde{\mathbf{v}}^{n+1}\right) + \nabla p^n = \mathbf{f}^{n+1} \quad \text{in } \Omega$$

$$\tilde{\mathbf{v}}^{n+1} = \mathbf{v}_D^{n+1} \quad \text{on } \Gamma = \partial\Omega$$

$$\tilde{p}^{n+1} = p^n - r\nabla\cdot\tilde{\mathbf{v}}^{n+1} \quad \text{in } \Omega$$

Vector penalty-projection step with a penalty parameter $0 < \varepsilon \le 1$

$$\left(\frac{\hat{\mathbf{v}}^{n+1}}{\delta t} + (\mathbf{v}^n\cdot\nabla)\,\hat{\mathbf{v}}^{n+1} - \frac{1}{\mathrm{Re}}\Delta\hat{\mathbf{v}}^{n+1}\right) - \frac{1}{\varepsilon}\nabla\left(\nabla\cdot(\hat{\mathbf{v}}^{n+1} + \tilde{\mathbf{v}}^{n+1})\right) = 0 \quad \text{in } \Omega$$

$$\hat{\mathbf{v}}^{n+1} = 0 \quad \text{on } \Gamma = \partial\Omega$$

Correction step for velocity and pressure

$$\mathbf{v}^{n+1} = \tilde{\mathbf{v}}^{n+1} + \hat{\mathbf{v}}^{n+1} \quad \text{in } \Omega$$

$$\varepsilon\left(p^{n+1} - p^n\right) + r\varepsilon\,\nabla\cdot\tilde{\mathbf{v}}^{n+1} + \nabla\cdot\mathbf{v}^{n+1} = 0 \quad \text{in } \Omega$$

$\Rightarrow$ No spurious boundary condition on pressure
$\Rightarrow$ No artificial pressure boundary layer

Generalization for outflow boundary conditions

$(\mathrm{VPP}_{r,\varepsilon})$ methods for open boundary conditions on Γ_N

For a given stress vector on a part Γ_N of $\Gamma = \partial\Omega = \Gamma_D \cup \Gamma_N$:

$$(\boldsymbol{\sigma}(\mathbf{v},p)\cdot\mathbf{n})_{|\Gamma_N} \equiv -p\,\mathbf{n} + \mu\left(\nabla\mathbf{v} + (\nabla\mathbf{v})^T\right)\cdot\mathbf{n} = \mathbf{g}$$

we get for the Dirichlet and Neumann velocity boundary conditions:

Penalty-prediction step:

$$\tilde{\mathbf{v}}^{n+1} = \mathbf{v}_D^{n+1} \quad \text{on } \Gamma_D$$

$$-p^n\,\mathbf{n} + \mu^{n+1}\left(\nabla\tilde{\mathbf{v}}^{n+1} + (\nabla\tilde{\mathbf{v}}^{n+1})^T\right)\cdot\mathbf{n} = \mathbf{g}^{n+1} \quad \text{on } \Gamma_N$$

Vector penalty-projection step:

$$\hat{\mathbf{v}}^{n+1} = 0 \quad \text{on } \Gamma_D$$

$$-(\tilde{p}^{n+1} - p^n)\,\mathbf{n} + \mu^{n+1}\left(\nabla\hat{\mathbf{v}}^{n+1} + (\nabla\hat{\mathbf{v}}^{n+1})^T\right)\cdot\mathbf{n} = 0 \quad \text{on } \Gamma_N$$

$\Rightarrow$ Original boundary conditions not spoiled through a scalar projection step with a Poisson-like pressure correction

Generalization for non-homogeneous flows

$(\mathrm{VPP}_{r,\varepsilon})$ for incompressible and variable density flows

Advection step for density:

$$\frac{\varrho^{n+1} - \varrho^n}{\delta t} + \nabla\cdot(\varrho^{n+1}\mathbf{v}^n) = 0$$

Penalty-prediction step:

$$\varrho^{n+1}\left(\frac{\tilde{\mathbf{v}}^{n+1} - \mathbf{v}^n}{\delta t} + (\mathbf{v}^n\cdot\nabla)\tilde{\mathbf{v}}^{n+1}\right) - \nabla\cdot\mu^{n+1}\left(\nabla\tilde{\mathbf{v}}^{n+1} + (\nabla\tilde{\mathbf{v}}^{n+1})^T\right) - r\nabla\left(\nabla\cdot\tilde{\mathbf{v}}^{n+1}\right) + \nabla p^n = \mathbf{f}^{n+1}$$

Vector penalty-projection step:

$$\varrho^{n+1}\left(\frac{\hat{\mathbf{v}}^{n+1}}{\delta t} + (\mathbf{v}^n\cdot\nabla)\hat{\mathbf{v}}^{n+1}\right) - \nabla\cdot\mu^{n+1}\left(\nabla\hat{\mathbf{v}}^{n+1} + (\nabla\hat{\mathbf{v}}^{n+1})^T\right) - \frac{1}{\varepsilon}\nabla\left(\nabla\cdot\hat{\mathbf{v}}^{n+1}\right) = \frac{1}{\varepsilon}\nabla\left(\nabla\cdot\tilde{\mathbf{v}}^{n+1}\right)$$

Correction step for velocity and pressure:

$$\mathbf{v}^{n+1} = \tilde{\mathbf{v}}^{n+1} + \hat{\mathbf{v}}^{n+1}$$

$$p^{n+1} = p^n - r\nabla\cdot\tilde{\mathbf{v}}^{n+1} - \frac{1}{\varepsilon}\nabla\cdot\mathbf{v}^{n+1}$$

$\Rightarrow$ Velocity correction $\hat{\mathbf{v}}$ all the more quasi-independent on the density ϱ or viscosity μ as $\varepsilon \to 0$
$\Rightarrow$ We can drop these terms in practical algorithms for ε small enough

Well-posedness, stability and convergence

Theorem (Global solvability of the $(\mathrm{VPP}_{r,\varepsilon})$ method.)
With $\mathbf{f} \in L^2(0,T;H^{-1}(\Omega)^d)$, $\mathbf{v}^0 \in H^1(\Omega)^d$ and $p^0 \in L^2_0(\Omega)$ given, both the prediction and correction steps of the $(\mathrm{VPP}_{r,\varepsilon})$ method are well-posed for all $\delta t > 0$, $r \ge 0$ and $\varepsilon > 0$, i.e. for all $n \in \mathbb{N}$ such that $(n+1)\delta t \le T$, there exists a unique solution $(\mathbf{v}^{n+1}, p^{n+1}) \in H^1(\Omega)^d \times L^2_0(\Omega)$ to the $(\mathrm{VPP}_{r,\varepsilon})$ scheme such that:

$$\frac{\mathbf{v}^{n+1} - \mathbf{v}^n}{\delta t} + (\mathbf{v}^n\cdot\nabla)\,\mathbf{v}^{n+1} - \frac{1}{\mathrm{Re}}\Delta\mathbf{v}^{n+1} + \nabla p^{n+1} = \mathbf{f}^{n+1} \quad \text{in } \Omega$$

$$(\varepsilon\delta t)\frac{p^{n+1} - p^n}{\delta t} + \nabla\cdot\mathbf{v}^{n+1} + r\varepsilon\,\nabla\cdot\tilde{\mathbf{v}}^{n+1} = 0 \quad \text{in } \Omega$$

Theorem (A priori estimates for $\mathrm{VPP}_{r,\varepsilon}$ and stability for small $r \ge 0$.)
There exists $K = K\left(\|\mathbf{f}\|_{L^2(0,T;H^{-1})}, \|\mathbf{v}_0\|_1, \|p_0\|_0\right) > 0$ and r_0 small enough satisfying the additionnal assumption:

$$(\mathcal{H}_{r,\varepsilon}) \quad 4r_0(\mathrm{Re} + \varepsilon) \le 1, \qquad 4c(\Omega)\sqrt{\mathrm{Re}}\,r_0\varepsilon \le \sqrt{\delta t}$$

where $c(\Omega)$ is the Poincaré constant, such that for all $r \le r_0$ we have:

$$(i) \quad \|\mathbf{v}^{n+1}\|_0^2 + \varepsilon\delta t\,\|p^{n+1}\|_0^2 + \sum_{k=0}^{n}\frac{\delta t}{16\mathrm{Re}}\|\nabla\mathbf{v}^{k+1}\|_0^2 + \sum_{k=0}^{n}\left(\frac{1}{4}\|\mathbf{v}^{k+1} - \mathbf{v}^k\|_0^2 + \varepsilon\delta t\,\|p^{k+1} - p^k\|_0^2\right) \le K$$

$$(ii) \quad \sum_{k=0}^{n}\delta t\,\|p^{k+1}\|_0^2 \le C$$

$$(iii) \quad \sum_{k=0}^{n}\delta t\,\|\nabla\cdot\mathbf{v}^{k+1}\|_0^2 \le C\,\varepsilon.$$

$\Rightarrow$ Convergence to the Navier-Stokes problem when $\delta t \to 0$ with compactness

Convergence analysis for small values of $r \ge 0$

First-order analysis for practical algorithms: r small enough

Theorem (Error estimates for $\mathrm{VPP}_{r,\varepsilon}$ with the Stokes problem.)
Assume $(\mathbf{v}, p)$ the solution of the Stokes Dirichlet problem smooth enough in time, well-prepared initial conditions and small enough parameters such that, $c(\Omega)$ being the Poincaré constant:

$$(\mathcal{H}_{r,\varepsilon}) \quad 4r(\mathrm{Re} + \varepsilon) \le 1, \quad 4c(\Omega)\sqrt{\mathrm{Re}}\,r\varepsilon \le \sqrt{\delta t}, \qquad 0 < \delta t \le 1$$

then there exists $C = C(\Omega, T, \mathrm{Re}, \mathbf{f}, \mathbf{v}_0, \mathbf{e}^0, \pi^0) > 0$ such that we have for all $n \in \mathbb{N}$ with $(n+1)\delta t \le T$,

$$(i) \quad \|\mathbf{e}^{n+1}\|_0^2 + \varepsilon\delta t\,\|\pi^{n+1}\|_0^2 + \sum_{k=0}^{n}\frac{\delta t}{\mathrm{Re}}\|\nabla\mathbf{e}^{k+1}\|_0^2 \le C\left(\delta t^2 + \varepsilon^2\delta t^{\frac{3}{2}}\right)$$

$$(ii) \quad \sum_{k=0}^{n}\delta t\,\|\pi^{k+1}\|_0^2 \le C\left(\delta t^2 + \varepsilon^2\delta t\right)$$

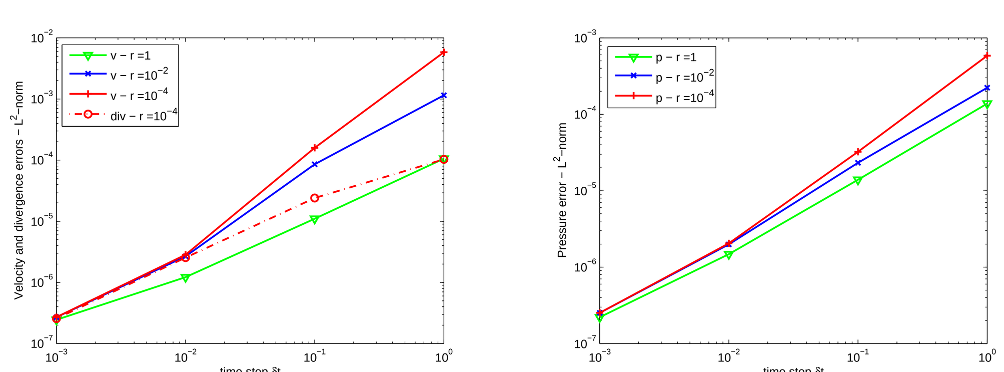
$$(iii) \quad \sum_{k=0}^{n}\delta t\,\|\nabla\cdot\mathbf{v}^{k+1}\|_0^2 = \sum_{k=0}^{n}\delta t\,\|\nabla\cdot\mathbf{e}^{k+1}\|_0^2 \le C\,(\delta t + \varepsilon)\,\varepsilon\delta t^2$$

$$(iv) \quad \|\nabla\mathbf{e}^{n+1}\|_0^2 \le C\,\mathrm{Re}^2\left(\delta t + \varepsilon^2\right).$$

$\Rightarrow$ Quasi-optimal error estimates in $\mathcal{O}(\delta t)$ for smooth solutions
$\Rightarrow$ $\|\nabla\cdot\mathbf{v}^n\|_{L^2} \approx \mathcal{O}(\varepsilon\delta t)$

Numerical results: Green-Taylor vortices

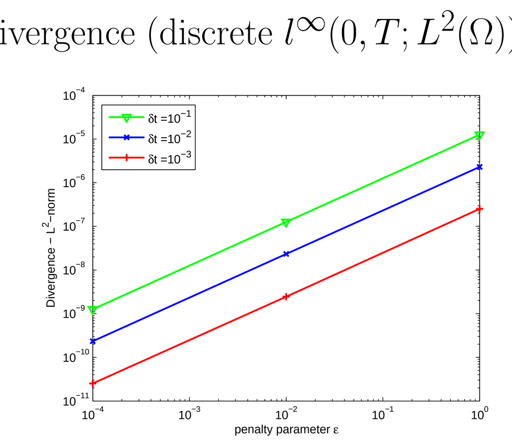
Navier-Stokes problems with Dirichlet B.C. on the MAC mesh
Velocity and pressure ($l^\infty(L^2)$ norm) errors versus time step δt



Velocity (Left) and pressure (Right) convergence in time at Re$=100$, $t=10$ - $h=1/512$, $\varepsilon=1$, $|res|_2 < 10^{-10}$

$\Rightarrow$ Time convergence in $\mathcal{O}(\delta t)$ – $\|\nabla\cdot\mathbf{v}^n\|_{L^2} = \mathcal{O}(\delta t)$ for small r
N.B.: Stagnation threshold = space discretization error in $\mathcal{O}(h^2)$

Divergence (discrete $l^\infty(0,T;L^2(\Omega))$ norm) versus penalty ε

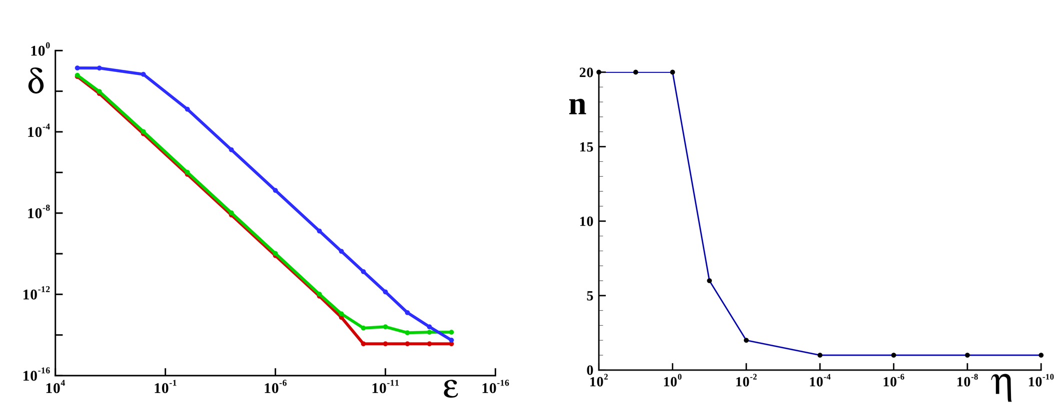


$\Rightarrow$ $\|\nabla\cdot\mathbf{v}^n\|_{L^2} = \mathcal{O}(\varepsilon\delta t)$

Numerical results: Rayleigh-Bénard natural convection in a differentially heated cavity

Convergence and cost of the penalty-correction step

divergence L^2-norm δ versus ε (Left) and number of MILU-BiCGStab solver iterations versus $\eta = \varepsilon/\delta t$ (Right)



Natural convection at $Ra = 10^5$ and $t = 2\delta t$ with $\delta t = 1$, $h = 1/256$ – $\mu = 0$ or $1.85\,10^{-5}$ (idem) and $\mu = 1.85\,10^{-1}$

$\Rightarrow$ $\|\nabla\cdot\mathbf{v}\|_{L^2} = \mathcal{O}(\varepsilon)$ until 10^{-15} (machine zero)
$\Rightarrow$ Solution all the easier as $\varepsilon \to 0$

Conclusion and perspectives

Vector penalty-projection methods
for incompressible and non-homogeneous flows
[Angot, Caltagirone and Fabrie, FVCA5 08 - CRAS 08]

- $\Rightarrow$ A general splitting approach to efficiently solve penalty problems
- The Lagrangian augmentation with $r > 0$ in the prediction step plays the role of a preconditioner
- Small values of $0 < r \le 10^{-2}$ sufficient to get a good pressure field
- Approximate projection with a vector penalty-correction step all the cheaper as $\varepsilon\delta t \to 0$
- Same convergence properties as the scalar penality-projection method
- Vector penalty-correction step all the less dependent on density or viscosity as $\varepsilon\delta t \to 0$
- L^2-norm of velocity divergence as $\mathcal{O}(\varepsilon\delta t)$ until machine precision
  $\Rightarrow$ cheap method for small values of $r \le 10^{-2}$ and $\varepsilon \le 10^{-2}$
- Other numerical experiments: second-order methods
- Other stability analysis for Navier-Stokes problems $\forall r > 0$
- Other convergence analysis: outflow B.C., variable density flows

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