Finite Volumes for Complex Applications V

A Finite Volume Stability Result for the Convection Operator in Compressible Flows ... and some Finite Element Applications

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Introduction

- **Problem addressed**

Address an asymptotic model for **low Mach number flows**:

$$\left| \begin{array}{l} \dfrac{\partial \rho \boldsymbol{u}}{\partial t} + \nabla \cdot (\rho\, \boldsymbol{u} \otimes \boldsymbol{u}) + \boldsymbol{\nabla} p - \nabla \cdot \tau(\boldsymbol{u}) = \boldsymbol{f} \\ \dfrac{\partial \rho}{\partial t} + \nabla \cdot (\rho\, \boldsymbol{u}) = 0 \end{array} \right. \tag{1}$$

with velocity $\boldsymbol{u}$, pressure p, density ρ, shear stress tensor $\tau(\boldsymbol{u})$, forcing term $\boldsymbol{f}$. ρ does not depend on p.

The problem is solved by a **pressure correction scheme**, and discretized by a **non-conforming low-order finite element** discretization.

- **Difficulties we must tackle**
  - Build a **stable approximation** of the **convection operator**.
  - Deal with **open boundary conditions**.

1. A finite volume result

- **Continuous stabilty : kinetic equality**

Let ρ and $\boldsymbol{u}$ be such that the mass balance holds in Ω:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0, \tag{2}$$

and z be a smooth scalar function defined over Ω. Then the following stability identity holds:

$$\int_\Omega \left[\frac{\partial \rho z}{\partial t} + \nabla \cdot (z\, \rho \boldsymbol{u}) \right] z = \frac{1}{2} \frac{d}{dt} \int_\Omega \rho z^2 + \frac{1}{2} \int_{\partial\Omega} z^2\, \rho \boldsymbol{u} \cdot \boldsymbol{n} \tag{3}$$

Taking $z = \boldsymbol{u}$ in (3) yields the kinetic energy conservation theorem:

$$\frac{1}{2} \frac{d}{dt} \int_\Omega \rho u^2 \, d\Omega = \int_\Omega \boldsymbol{f} \boldsymbol{u} \, d\Omega + \int_\Omega \nabla \cdot \sigma(\boldsymbol{u})\, \boldsymbol{u} \, d\Omega \tag{4}$$

with $\sigma = -p\ \mathrm{Id} + 2\mu \nabla^s \boldsymbol{u}$.

- **Finite-volume counterpart of** (2)**:**

$$\forall K \in \mathcal{M}, \qquad \frac{|K|}{\delta t} (\rho_K - \rho_K^*) + \sum_{\sigma \in \mathcal{E}(K)} F_{\sigma,K} = 0, \tag{5}$$

with $\sigma = K|L$ the common face of K and L, $\mathcal{E}(K)$ the set of all edges or faces of K, $\bar\Omega = \bigcup_{K \in \mathcal{M}} \bar K$, $\mathcal{M}$ a finite volume admissible mesh, and $F_{\sigma,K} = -F_{\sigma,L}$.

- **Theorem : Stability of the convection operator** [4]

$$\sum_{\mathbf{K} \in \mathcal{M}} \mathbf{z_K} \left[\frac{|\mathbf{K}|}{\delta \mathbf{t}} (\rho_\mathbf{K}\, \mathbf{z_K} - \rho^*_\mathbf{K}\, \mathbf{z}^*_\mathbf{K}) + \sum_{\sigma \in \mathcal{E}(\mathbf{K})} \mathbf{F}_{\sigma,\mathbf{K}}\, \mathbf{z}_\sigma \right] \geq$$
$$\frac{\mathbf{1}}{\mathbf{2}} \sum_{\mathbf{K} \in \mathcal{M}} \frac{|\mathbf{K}|}{\delta \mathbf{t}} \left[\rho_\mathbf{K}\, \mathbf{z}_\mathbf{K}^\mathbf{2} - \rho^*_\mathbf{K}\, {\mathbf{z}^*_\mathbf{K}}^\mathbf{2} \right] + \frac{\mathbf{1}}{\mathbf{2}} \sum_{\substack{\sigma \in \mathcal{E}_{\text{ext}} \\ (\sigma \in \mathcal{E}(\mathbf{K}))}} \mathbf{F}_{\sigma,\mathbf{K}}\, \mathbf{z}_\sigma^\mathbf{2}$$

holds for both centered and upwind approximations of z_σ.

2. Stable approximation of the convection operator for low order finite elements

Non-conforming low-order finite elements:

- On **quadrangles**: Rannacher-Turek Q_1^{NC}/P_0 (named **RT**)
- On **triangles**, Crouzeix-Raviart P_1^{NC}/P_0 (named **CR**)

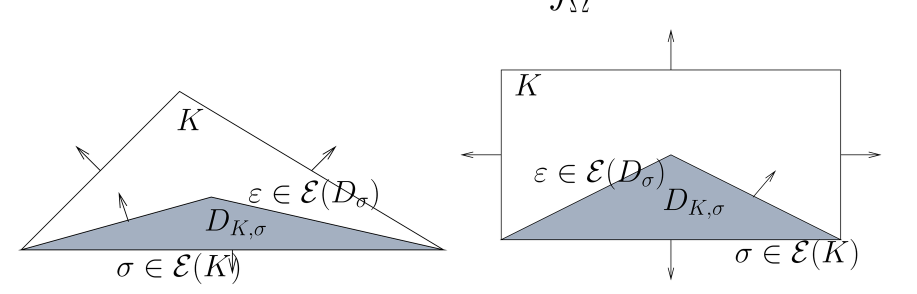
Standard Galerkin is unstable !

- **Build a finite volume approximation of the convective operator such that the FV stability theorem applies**

1. Approximate *unsteady term* in a FV manner
   - CR element: naturally diagonal mass matrix in 2D

$$\frac{1}{\Delta t} \int_\Omega (\rho^n \boldsymbol{u}^{n+1} - \rho^{n-1} \boldsymbol{u}^n)\, \boldsymbol{v} \approx \frac{1}{\Delta t} \sum_{\sigma \in \mathcal{E}(K)} |D_\sigma| (\rho^n_\sigma \boldsymbol{u}^{n+1}_\sigma - \rho^{n-1}_\sigma \boldsymbol{u}^n_\sigma)$$

with $|D_\sigma| = |D_{K,\sigma}| + |D_{L,\sigma}| = \int_K \varphi_\sigma + \int_L \varphi_\sigma$ (see figure), and φ_σ the basis function associated to the node on σ.
   - RT element: mass lumping : $|D_\sigma| = \int_\Omega \varphi_\sigma$



Cones, or "half-diamonds" for CR (left) and RT (right) elements

2. Approximate *convection term* in a FV manner
   Build each $(\rho \boldsymbol{u})_\varepsilon$ such that mass balance on $K \rightsquigarrow$ (5) on D_σ
   - CR element: natural finite element evaluation of the mass fluxes

$$(\rho \boldsymbol{u})_\varepsilon(x) = \sum_{\sigma \in \partial K} \varphi_\sigma(x_\varepsilon) (\rho \boldsymbol{u})_\sigma$$

with x_ε the center of ε
   - RT element: specific interpolation [2]

$$(\rho \boldsymbol{u})_\varepsilon \cdot \boldsymbol{n}_\varepsilon = \sum_{\sigma \in \partial K} \alpha_\sigma(\boldsymbol{x} \cdot \boldsymbol{n}_\varepsilon)((\rho \boldsymbol{u})_\sigma \cdot \boldsymbol{n}_\sigma)$$

- **Discrete momentum balance equation:**

For $1 \leq i \leq d$,

$$\sum_{\sigma \in \mathcal{E}(\mathbf{K})} \frac{\mathbf{1}}{\mathbf{\Delta t}} |\mathbf{D}_\sigma| (\rho^\mathbf{n}_\sigma \mathbf{u}^{\mathbf{n+1}}_{\sigma,\mathbf{i}} - \rho^{\mathbf{n-1}}_\sigma \mathbf{u}^\mathbf{n}_{\sigma,\mathbf{i}})$$
$$+ \sum_{\sigma \in \mathcal{E}(\mathbf{K})} \sum_{\varepsilon = \mathbf{D}_\sigma | \mathbf{D}_{\sigma'}} |\varepsilon| (\rho \mathbf{u})_{|\varepsilon} \cdot \mathbf{n}_\varepsilon \frac{\mathbf{u}^{\mathbf{n+1}}_{\sigma,\mathbf{i}} + \mathbf{u}^{\mathbf{n+1}}_{\sigma',\mathbf{i}}}{\mathbf{2}}$$
$$+ \sum_\mathbf{K} \int_\mathbf{K} \tau(\mathbf{u}^{\mathbf{n+1}}) : \nabla(\varphi_\sigma \mathbf{e_i})) - \sum_\mathbf{K} \int_\mathbf{K} \mathbf{p^n} \nabla \cdot (\varphi_\sigma \mathbf{e_i})$$
$$= \int_\mathbf{\Omega} \mathbf{f}^{\mathbf{n+1}} \cdot (\varphi_\sigma \mathbf{e_i})$$

3. A pressure correction scheme for low Mach number flows with open boundaries

- **Projection method**

1. Velocity prediction step:
   Find $\tilde{\boldsymbol{u}}^{n+1} \in \boldsymbol{W}_h$ such that, $\forall \boldsymbol{v} \in \boldsymbol{W}_h$:

$$\frac{1}{\delta t} (\rho^n \tilde{\boldsymbol{u}}^{n+1} - \rho^{n-1} \boldsymbol{u}^n, \boldsymbol{v})_h + (\nabla \cdot_h \tilde{\boldsymbol{u}}^{n+1} \otimes \rho^n \boldsymbol{u}^n, \boldsymbol{v})_h$$
$$+ a(\tilde{\boldsymbol{u}}^{n+1}, \boldsymbol{v}) + b(p^n, \boldsymbol{v}) = (\boldsymbol{f}^{n+1}, \boldsymbol{v})_h$$

with the discrete divergence operator:

$$(\nabla \cdot_h \rho^n \tilde{\boldsymbol{u}}^{n+1} \otimes \boldsymbol{u}^n)_\sigma = \frac{1}{|D_\sigma|} \sum_{\substack{\varepsilon \in \mathcal{E}(D_\sigma) \\ \varepsilon \in \mathcal{E} \setminus \mathcal{E}_{\text{ext,D}}}} |\varepsilon|\ (\rho^n \boldsymbol{u}^n)_\varepsilon\ (\tilde{\boldsymbol{u}}^{n+1})_\varepsilon$$

2. Standard algebraic projection step

- **What could be a suitable condition for the inflow boundaries where the velocity is not prescribed?**

The boundary condition must not lead to an unstable problem.
$\rightsquigarrow$ following energy estimate:

$$\underbrace{\frac{1}{2} (\rho^n \tilde{\boldsymbol{u}}^{n+1}, \tilde{\boldsymbol{u}}^{n+1})_h}_{\text{discrete kinetic energy at } t=t^{n+1}} + \delta t\, T^{n+1}_{\text{visc}} + \delta t\, T^{n+1}_{\text{pres}} \leq$$
$$\underbrace{\frac{1}{2} (\rho^{n-1} \boldsymbol{u}^n, \boldsymbol{u}^n)_h}_{\text{discrete kinetic energy at } t=t^n} + \delta t\, T^{n+1}_{\mathrm{D}} + \delta t\, T^{n+1}_{\partial\Omega}$$

By the stability theorem, we get for $T^{n+1}_{\partial\Omega}$:

$$T^{n+1}_{\partial\Omega} \leq \sum_{\sigma \in \mathcal{E}_{\text{ext}} \setminus \mathcal{E}_{\text{ext,D}}} -\frac{1}{2} |\sigma|\ (\rho^n \boldsymbol{u}^n)_\sigma\ |(\tilde{\boldsymbol{u}}^{n+1})_\sigma|^2$$
$$+ \int_\sigma \Big(\tau(\tilde{\boldsymbol{u}}^{n+1})\, \boldsymbol{n}_\sigma - p^n\, \boldsymbol{n}_\sigma \Big) \cdot \tilde{\boldsymbol{u}}^{n+1}$$

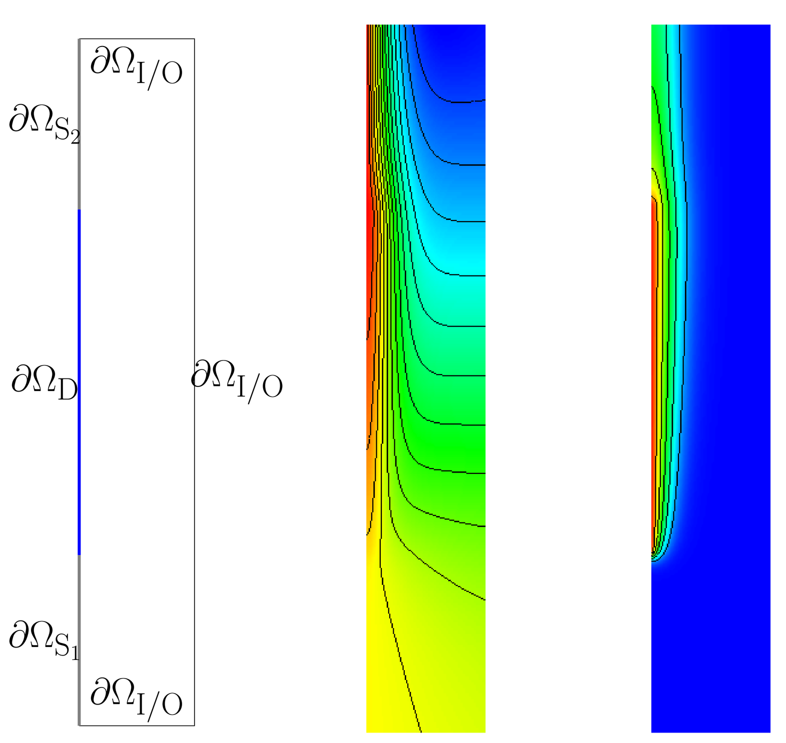
Taking the following artificial boundary condition on edges where the velocity is not prescribed

$$-\frac{\mathbf{1}}{\mathbf{2}} \rho\, \mathbf{u} \cdot \mathbf{n}_\sigma\ \mathbf{u} + \tau(\mathbf{u})\, \mathbf{n}_\sigma - \mathbf{p}\, \mathbf{n}_\sigma = \mathbf{f}_{\partial\Omega} \tag{6}$$

enables to control $T^{n+1}_{\partial\Omega}$.

- **In practice,** on σ:
  - do not compute the integrals involving the stress tensor and the pressure
  - divide the convection term by 2
  - add the integral of $\boldsymbol{f}_{\partial\Omega} \cdot \boldsymbol{v}$ over σ

4. Numerical test : a natural convection flow with open boundaries



(left) comput. domain, (center) streamlines and (right) isovalues of the temperature

Equations:

- low Mach number flow (1) (FE method with convection on dual mesh)
- linear convection-diffusion equation for T (usual FV method)

Flow simulation properties:

- Ideal gas law, $R = 287$, $p_0 = 101325\, Pa$, $\rho = \rho(p_0, T)$
- $\nu = 1.68\, 10^{-5}\, Pa.s$, $c_p = R\gamma/(\gamma - 1)$ with $\gamma = 1.4$, $\mathbf{Pr} = 0.7$, $\mathbf{Ra} = 10^6$

Boundary conditions:

- $\partial\Omega_{\mathrm{D}}$: $\boldsymbol{u} = 0$, $T = 900°\, C$, $\quad \partial\Omega_{\mathrm{S_1}}$ and $\partial\Omega_{\mathrm{S_2}}$: $\boldsymbol{u} \cdot \boldsymbol{n} = 0$, $\boldsymbol{\nabla} T \cdot \boldsymbol{n} = 0$
- $\partial\Omega_{\mathrm{I/O}}$ outflow: $\tau(\boldsymbol{u})\boldsymbol{n} - p\, \boldsymbol{n} = 0$, $\boldsymbol{\nabla} T \cdot \boldsymbol{n} = 0$
- $\partial\Omega_{\mathrm{I/O}}$ inflow: **artificial boundary condition** (6) with $\boldsymbol{f}_{\partial\Omega} = 0$, $T = 300°\, C$.

Results:

- The flow enters the domain on almost the whole boundary.
- For computations with $\mathbf{Ra} = 10^7$ and 10^8, no instability was seen.
- Computations with a larger domain $\rightsquigarrow$ remarkable agreement with initial results.

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