Benchmark of anisotropic problems
The DDFV and m-DDFV schemes

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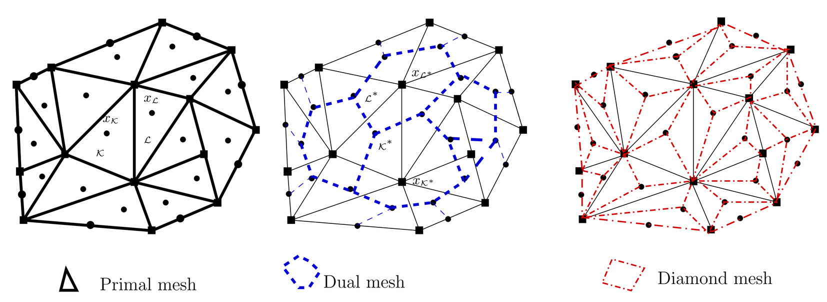
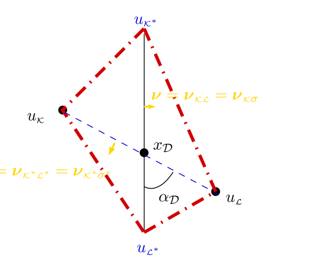
Description of the scheme

- The diffusion problem $-\mathrm{div}(\mathbf{K}(x)\nabla u) = f$.
- The DDFV Meshes



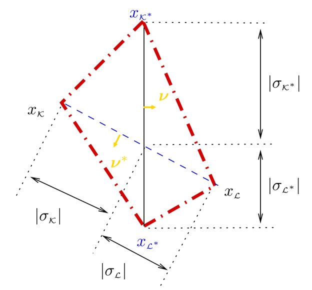
➤ Note that the vertices of the dual mesh corresponds to the center of the primal cells.

- The unknowns : $(u_\mathcal{K})_{\mathcal{K}\in\mathfrak{M}}$ and $(u_{\mathcal{K}^*})_{\mathcal{K}^*\in\mathfrak{M}}$
- A discrete gradient constant on the diamond cell $\mathcal{D}$



$$\nabla_\mathcal{D}^\mathcal{T} u^\mathcal{T} = \frac{1}{\sin\alpha_\mathcal{D}}\left(\frac{u_\mathcal{L}-u_\mathcal{K}}{d_{\mathcal{K}\mathcal{L}}}\boldsymbol{\nu}_{\mathcal{K}\mathcal{L}} + \frac{u_{\mathcal{L}^*}-u_{\mathcal{K}^*}}{d_{\mathcal{K}^*\mathcal{L}^*}}\boldsymbol{\nu}_{\mathcal{K}^*\mathcal{L}^*}\right)$$

- The scheme



$$-\sum_{\sigma\in\partial\mathcal{K}} |\sigma|\mathbf{K}_\mathcal{D}\nabla_\mathcal{D}^\mathcal{T} u^\mathcal{T}\cdot\boldsymbol{\nu}_{\mathcal{K}\sigma} = \int_\mathcal{K} f(z)\,dz, \forall\mathcal{K}\in\mathfrak{M}$$

$$-\sum_{\sigma^*\in\partial\mathcal{K}^*} |\sigma^*|\mathbf{K}_\mathcal{D}\nabla_\mathcal{D}^\mathcal{T} u^\mathcal{T}\cdot\boldsymbol{\nu}_{\mathcal{K}^*\sigma^*} = \int_{\mathcal{K}^*} f(z)\,dz, \forall\mathcal{K}^*\in\mathfrak{M}^*$$

▶ If $\mathbf{K}$ is smooth **(DDFV scheme)**

$$\mathbf{K}_\mathcal{D} = \mathbf{K}(x_\mathcal{D})$$

▶ If $\mathbf{K}$ is discont. and Cst. on the primal cells **(m-DDFV scheme)**

$$(\mathbf{K}_\mathcal{D}\boldsymbol{\nu},\boldsymbol{\nu}) = \frac{(|\sigma_\mathcal{K}|+|\sigma_\mathcal{L}|)(\mathbf{K}_\mathcal{K}\boldsymbol{\nu},\boldsymbol{\nu})(\mathbf{K}_\mathcal{L}\boldsymbol{\nu},\boldsymbol{\nu})}{|\sigma_\mathcal{L}|(\mathbf{K}_\mathcal{K}\boldsymbol{\nu},\boldsymbol{\nu})+|\sigma_\mathcal{K}|(\mathbf{K}_\mathcal{L}\boldsymbol{\nu},\boldsymbol{\nu})}$$

$$(\mathbf{K}_\mathcal{D}\boldsymbol{\nu}^*,\boldsymbol{\nu}^*) = \frac{|\sigma_\mathcal{L}|(\mathbf{K}_\mathcal{L}\boldsymbol{\nu}^*,\boldsymbol{\nu}^*)+|\sigma_\mathcal{K}|(\mathbf{K}_\mathcal{K}\boldsymbol{\nu}^*,\boldsymbol{\nu}^*)}{|\sigma_\mathcal{K}|+|\sigma_\mathcal{L}|} - \frac{|\sigma_\mathcal{K}||\sigma_\mathcal{L}|}{|\sigma_\mathcal{K}|+|\sigma_\mathcal{L}|}\frac{((\mathbf{K}_\mathcal{K}\boldsymbol{\nu},\boldsymbol{\nu}^*)-(\mathbf{K}_\mathcal{L}\boldsymbol{\nu},\boldsymbol{\nu}^*))^2}{|\sigma_\mathcal{L}|(\mathbf{K}_\mathcal{K}\boldsymbol{\nu},\boldsymbol{\nu})+|\sigma_\mathcal{K}|(\mathbf{K}_\mathcal{L}\boldsymbol{\nu},\boldsymbol{\nu})}$$

$$(\mathbf{K}_\mathcal{D}\boldsymbol{\nu},\boldsymbol{\nu}^*) = \frac{|\sigma_\mathcal{L}|(\mathbf{K}_\mathcal{L}\boldsymbol{\nu},\boldsymbol{\nu}^*)(\mathbf{K}_\mathcal{K}\boldsymbol{\nu},\boldsymbol{\nu})+|\sigma_\mathcal{K}|(\mathbf{K}_\mathcal{K}\boldsymbol{\nu},\boldsymbol{\nu}^*)(\mathbf{K}_\mathcal{L}\boldsymbol{\nu},\boldsymbol{\nu})}{|\sigma_\mathcal{L}|(\mathbf{K}_\mathcal{K}\boldsymbol{\nu},\boldsymbol{\nu})+|\sigma_\mathcal{K}|(\mathbf{K}_\mathcal{L}\boldsymbol{\nu},\boldsymbol{\nu})}$$

Results for Test 1.1

umin= 0.0, umax=1.0.

- Triangular mesh mesh1 ⇝ **ocvl2**= 1.99, **ocvgradl2**= 1.00.

$x_\mathcal{K}$ = **circumcenter**

i	erflx0	erflx1	erfly0	erfly1	erflm	uminv	umaxv	uminc	umaxc
1	2.40E-03	2.40E-03	2.40E-03	2.40E-03	2.90E-01	0.00	**1.03**	8.43E-02	9.38E-01
2	6.02E-04	6.02E-04	6.02E-04	6.02E-04	1.98E-01	0.00	**1.01**	2.31E-02	9.84E-01
3	1.51E-04	1.51E-04	1.51E-04	1.51E-04	1.07E-01	0.00	1.00	6.03E-03	9.96E-01
4	3.76E-05	3.76E-05	3.76E-05	3.76E-05	5.58E-02	0.00	1.00	1.54E-03	9.99E-01
5	9.41E-06	9.41E-06	9.41E-06	9.41E-06	2.84E-02	0.00	1.00	3.88E-04	9.99E-01
6	2.35E-06	2.35E-06	2.35E-06	2.35E-06	1.44E-02	0.00	1.00	9.76E-05	9.99E-01
7	5.88E-07	5.88E-07	5.88E-07	5.88E-07	7.21E-03	0.00	1.00	2.44E-05	1.00E+00

$x_\mathcal{K}$ = **barycenter**

i	erflx0	erflx1	erfly0	erfly1	erflm	uminv	umaxv	uminc	umaxc
1	2.47E-03	2.47E-03	2.47E-03	2.47E-03	2.80E-01	0.00	**1.02**	4.01E-02	9.47E-01
2	6.17E-04	6.17E-04	6.17E-04	6.17E-04	1.43E-01	0.00	1.00	1.08E-02	9.86E-01
3	1.54E-04	1.54E-04	1.54E-04	1.54E-04	7.62E-02	0.00	1.00	2.79E-03	9.96E-01
4	3.85E-05	3.85E-05	3.85E-05	3.85E-05	3.89E-02	0.00	1.00	7.10E-04	9.99E-01
5	9.64E-06	9.64E-06	9.64E-06	9.64E-06	1.95E-02	0.00	1.00	1.79E-04	9.99E-01
6	2.41E-06	2.41E-06	2.41E-06	2.41E-06	9.80E-03	0.00	1.00	4.49E-05	9.99E-01
7	6.02E-07	6.02E-07	6.02E-07	6.02E-07	4.91E-03	0.00	1.00	1.12E-05	1.00E+00

- Distorted quadrangular mesh mesh4_1_i ⇝ **ocvl2**= 2.03, **ocvgradl2**= 1.51.

i	nunkw	nnmat	sumflux	erl2	ergrad	ratiol2	ratiograd
1	613	5233	3.55E-14	2.14E-02	5.16E-02		
6	21013	187473	2.42E-13	5.91E-04	3.53E-03	2.03	1.51

grid	erflx0	erflx1	erfly0	erfly1	erflm	uminv	umaxv	uminc	umaxc
1	1.35E-02	8.99E-03	1.08E-02	1.72E-02	7.38E-01	0.000	**1.014**	1.337E-02	0.996
6	3.93E-04	2.76E-04	2.79E-04	4.94E-04	1.35E-01	0.000	1.001	3.836E-04	0.999

- Distorted quadrangular mesh mesh4_2_i ⇝ **ocvl2**= 2.03, **ocvgradl2**= 1.50.

grid	nunkw	nnmat	sumflux	erl2	ergrad	ratiol2	ratiograd
1	2245	19665	4.97E-14	5.70E-03	2.03E-02		
6	78805	706065	-4.80E-13	1.50E-04	1.38E-03	2.03	1.51

grid	erflx0	erflx1	erfly0	erfly1	erflm	uminv	umaxv	uminc	umaxc
1	3.71E-03	2.33E-03	3.35E-03	4.17E-03	3.13E-01	0.000	**1.003**	3.633E-03	0.999
6	1.07E-04	6.81E-05	8.97E-05	1.13E-04	5.33E-02	0.000	1.000	1.020E-04	1.000

⇒ **Maximum principle is satisfied on the centers and violated on the vertices of the mesh**

Results for Test 1.2

umin= 0.0, umax=$1+\sin(1)$, $x_\mathcal{K}$ = circumcenter.

- Triangular mesh mesh1 ⇝ **ocvl2**= 2.00, **ocvgradl2**= 1.00.

i	nunkw	nnmat	sumflux	erl2	ergrad	ratiol2	ratiograd
1	93	765	4.44E-15	7.92E-03	2.15E-02		
2	353	3041	3.11E-15	1.96E-03	1.01E-02	2.10	1.14
3	1377	12129	- 2.04E-14	4.90E-04	4.95E-03	2.04	1.04
4	5441	48449	-4.44E-16	1.23E-04	2.47E-03	2.01	1.01
5	21633	193665	4.40E-14	3.08E-05	1.24E-03	2.01	1.00
6	86273	774401	-5.29E-13	7.70E-06	6.20E-04	2.00	1.00
7	344577	3097089	-6.13E-12	1.93E-06	3.10E-04	2.00	1.00

i	erflx0	erflx1	erfly0	erfly1	erflm	uminv	umaxv	uminc	umaxc
1	3.46E-04	4.31E-03	3.28E-03	1.19E-02	1.67E-01	0.00	1.84	2.93E-03	1.46E+00
2	1.12E-04	1.19E-03	9.94E-04	3.48E-03	9.37E-02	0.00	1.84	7.25E-04	1.64E+00
3	2.86E-05	3.13E-04	2.96E-04	1.00E-03	4.94E-02	0.00	1.84	1.80E-04	1.74E+00
4	6.46E-06	8.07E-05	8.67E-05	2.83E-04	2.54E-02	0.00	1.84	4.51E-05	1.79E+00
5	1.33E-06	2.05E-05	2.49E-05	7.89E-05	1.32E-02	0.00	1.84	1.12E-05	1.81E+00
6	2.44E-07	5.16E-06	7.05E-06	2.18E-05	6.77E-03	0.00	1.84	2.82E-06	1.82E+00
7	3.73E-08	1.30E-06	1.97E-06	5.96E-06	3.43E-03	0.00	1.84	7.05E-07	1.83E+00

- Locally refined mesh mesh3 ⇝ **ocvl2**= 2.00, **ocvgradl2**= 1.50.

i	nunkw	nnmat	sumflux	erl2	ergrad	ratiol2	ratiograd
1	97	765	4.44E-16	1.23E-02	2.01E-02		
2	353	2973	1.11E-14	3.00E-03	6.84E-03	2.18	1.67
3	1345	11709	8.88E-15	7.47E-04	2.37E-03	2.08	1.59
4	5249	46461	2.26E-14	1.87E-04	8.31E-04	2.04	1.54
5	20737	185085	-2.22E-15	4.67E-05	2.93E-04	2.02	1.52

i	erflx0	erflx1	erfly0	erfly1	erflm	uminv	umaxv	uminc	umaxc
1	4.91E-03	9.65E-03	1.66E-03	2.57E-03	9.65E-02	0.00	1.84	1.52E-02	1.65E+00
2	5.92E-04	2.15E-03	6.46E-04	3.64E-03	5.32E-02	0.00	1.84	3.88E-03	1.74E+00
3	2.61E-05	5.33E-04	2.02E-04	1.72E-03	2.96E-02	0.00	1.84	9.74E-04	1.79E+00
4	5.16E-05	1.35E-04	5.95E-05	6.39E-04	1.59E-02	0.00	1.84	2.44E-04	1.81E+00
5	2.44E-05	3.40E-05	1.71E-05	2.13E-04	8.96E-03	0.00	1.84	6.10E-05	1.83E+00

⇒ **No problem with Maximum Principle in this case.**

⇒ **Same results in the case $x_\mathcal{K}$ = barycenter.**

⇒ **DDFV schemes give a very accurate approximation of the gradient of the solution.**

Results for Test 2: Numerical locking

Triangular mesh mesh1. umin= -1, umax= 1.

- $\delta = 10^5$ ⇝ **ocvl2**= 1.76, **ocvgradl2**= 1.21.

i	nunkw	nnmat	sumflux	erl2	ergrad	ratiol2	ratiograd
1	93	877	4.15E-10	3.28E-01	4.26E-01		
2	353	3265	-3.47E-10	1.12E-01	1.63E-01	1.61	1.44
3	1377	12577	1.74E-09	3.41E-02	6.33E-02	1.75	1.39
4	5441	49345	-2.29E-09	9.12E-03	2.40E-02	1.92	1.41
5	21633	195457	-2.23E-09	2.26E-03	9.08E-03	2.02	1.41
6	86273	777985	-3.73E-08	5.81E-04	3.65E-03	1.96	1.32
7	344577	3104257	6.50E-08	1.71E-04	1.57E-03	1.77	1.22

i	erflx0	erflx1	fluy0	fluy1	erflm	uminv	umaxv	uminc	umaxc
1	3.98E-10	1.57E-05	1.30E-10	-2.52E-01	5.60E+02	-0.991	1.035	-0.927	1.176
2	9.95E-11	2.36E-07	-2.54E-11	-3.79E-03	1.39E+02	-0.977	1.051	-0.969	1.060
3	2.49E-11	1.15E-08	7.76E-11	1.85E-04	4.35E+01	-0.990	1.017	-0.988	1.018
4	6.22E-12	7.43E-09	1.87E-10	-1.19E-04	1.89E+01	-0.997	1.004	-0.996	1.005
5	1.55E-12	9.39E-09	-2.34E-10	-1.51E-04	1.23E+01	-1.000	1.001	-0.999	1.001
6	3.89E-13	5.38E-09	1.32E-10	-8.65E-05	1.10E+01	-1.000	1.000	-0.999	1.000
7	9.71E-14	1.92E-09	-1.07E-09	-3.09E-05	7.51E+00	-1.000	1.000	-0.999	1.000

- $\delta = 10^6$ ⇝ **ocvl2**= 2.11, **ocvgradl2**= 1.35.

i	nunkw	nnmat	sumflux	erl2	ergrad	ratiol2	ratiograd
1	93	877	-5.62E-09	3.32E-01	4.30E-01		
2	353	3265	6.97E-09	1.15E-01	1.66E-01	1.59	1.43
3	1377	12577	1.47E-10	3.54E-02	6.43E-02	1.73	1.39
4	5441	49345	-9.84E-09	9.73E-03	2.42E-02	1.88	1.42
5	21633	195457	-2.52E-07	2.48E-03	8.96E-03	1.98	1.44
6	86273	777985	-3.20E-07	5.88E-04	3.33E-03	2.08	1.43
7	344577	3104257	2.47E-06	1.36E-04	1.30E-03	2.12	1.36

i	erflx0	erflx1	fluy0	fluy1	erflm	uminv	umaxv	uminc	umaxc
1	4.03E-12	1.86E-06	-3.82E-10	-2.96E-01	1.77E+03	-0.997	1.040	-0.917	1.194
2	1.01E-12	1.19E-07	-1.97E-09	-1.89E-02	4.42E+02	-0.979	1.053	-0.967	1.064
3	2.52E-13	9.98E-09	2.76E-10	-1.59E-03	1.38E+02	-0.990	1.018	-0.987	1.019
4	6.30E-14	1.25E-09	2.87E-10	-2.00E-04	6.00E+01	-0.997	1.005	-0.996	1.005
5	1.58E-14	3.65E-10	-4.08E-09	-5.86E-05	3.00E+01	-0.999	1.001	-0.999	1.001
6	3.94E-15	2.09E-10	1.71E-09	-3.37E-05	1.51E+01	-1.000	1.000	-0.999	1.000
7	9.85E-16	1.22E-10	-5.09E-09	-1.70E-05	1.38E+01	-1.000	1.000	-0.999	1.000

⇒ **Maximum principle is violated on both centers and vertices**

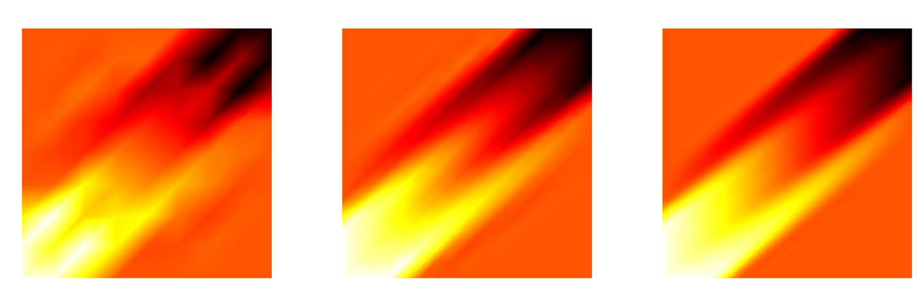
Results for Test 3: Oblique flow

- Uniform rectangular mesh mesh2. umin= 0.0, umax= 1.0.

i	flux0	flux1	fluy0	fluy1	ener1	ener2	eren
1	-1.82E-01	1.82E-01	- 1.20E-01	1.20E-01	2.14E-01	1.93E-01	9.60E-02
2	- 1.91E-01	1.91E-01	- 1.03E-01	1.03E-01	2.46E-01	2.29E-01	6.84E-02
3	-1.94E-01	1.94E-01	-9.83E-02	9.83E-02	2.40E-01	2.39E-01	4.50E-03
4	-1.94E-01	1.94E-01	-9.79E-02	9.79E-02	2.42E-01	2.41E-01	2.12E-03
5	-1.93E-01	1.93E-01	-9.82E-02	9.82E-02	2.42E-01	2.42E-01	1.85E-04
ref	-1.93E-01	1.93E-01	-9.86E-02	9.86E-02	2.42E-01	2.42E-01	7.11E-06

- Solution on mesh2_i

i=2 i=3 i=4



⇒ **Problem with Maximum Principle.**

⇒ **Good agreement with the expected results after the third refinement.**

Results for Test 4: Vertical fault

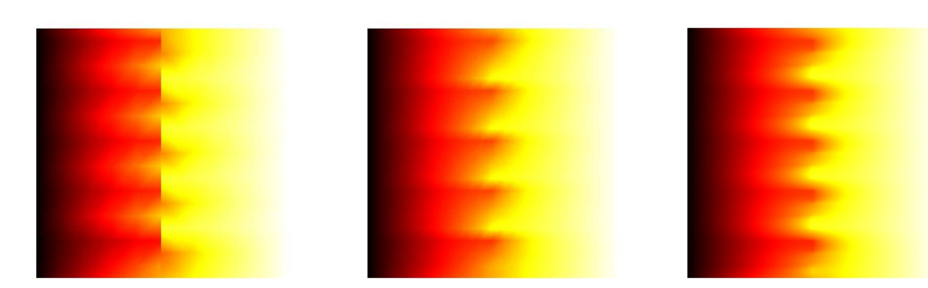
- Non conforming rectangular mesh mesh5. umin= 0.0, umax= 1.0.

	nunkw	nnmat	sumflux	uminv	umaxv	uminc	umaxc
DDFV mesh5	241	1993	3.26E-14	-1.000E-08	1.000	4.245E-02	0.962
DDFV reg	841	7237	-1.05E-13	0.000	1.000	2.088E-02	0.981
DDFV ref	205441	1843837	-7.50E-12	0.000	1.000	1.317E-03	0.998
m-DDFV mesh5	241	1993	5.29E-14	-1.000E-08	1.000	4.038E-02	0.961
m-DDFV reg	841	7237	2.27E-13	0.000	1.000	1.986E-02	0.982
m-DDFV ref	205441	1843837	1.66E-11	0.000	1.000	1.315E-03	0.998

	flux0	flux1	fluy0	fluy1	ener1	ener2	eren
DDFV mesh5	-41.2	43.7	-2.54	6.74E-03	39.4	42.1	6.39E-02
DDFV reg	-41.7	43.9	-2.20	6.98E-04	42.2	42.6	1.09E-02
DDFV ref	-42.1	44.4	-2.32	7.84E-04	43.2	43.2	1.35E-04
m-DDFV mesh5	-40.0	41.8	-1.81	9.08E-04	42.18	40.64	3.653E-02
m-DDFV reg	-39.9	42.0	-2.68	8.01E-04	43.23	41.04	5.076E-02
m-DDFV ref	-42.1	44.4	-2.33	7.97E-04	43.23	43.18	1.279E-03

- m-DDFV solutions for the vertical fault on the meshes:

mesh5 mesh5_reg mesh5_ref



⇒ **Good behaviour of the m-DDFV scheme even on coarse meshes**

Results for Test 5: Heterogeneous rotating anisotropy

- Uniform rectangular mesh mesh2. umin= 0.0, umax= 1.0.

⇝ **ocvl2**= 2.00, **ocvgradl2**= 1.95.

i	nunkw	nnmat	sumflux	erl2	ergrad	ratiol2	ratiograd
1	41	293	-8.88E-16	2.12E-02	2.57E-02		
2	145	1165	-1.78E-15	4.45E-03	7.34E-03	2.47	1.98
3	545	4637	-7.11E-15	1.01E-03	1.95E-03	2.24	2.00
4	2113	18493	-3.55E-15	2.45E-04	5.09E-04	2.09	1.98
5	8321	73853	5.33E-15	6.06E-05	1.32E-04	2.04	1.97
6	33025	295165	-5.68E-14	1.51E-05	3.42E-05	2.01	1.96
7	131585	1180157	6.22E-14	3.78E-06	8.83E-06	2.01	1.96

i	erflx0	erflx1	erfly0	erfly1	erflm	uminv	umaxv	uminc	umaxc
1	1.93E-02	1.05E-03	1.93E-02	1.05E-03	1.38E-01	0.000	0.997	1.360E-01	0.870
2	6.41E-03	6.94E-04	6.41E-03	6.94E-04	5.29E-02	0.000	1.000	3.475E-02	0.965
3	1.98E-03	2.09E-04	1.98E-03	2.09E-04	2.79E-02	0.000	1.000	8.735E-03	0.991
4	5.88E-04	5.51E-05	5.88E-04	5.51E-05	1.41E-02	0.000	1.000	2.187E-03	0.997
5	1.70E-04	1.40E-05	1.70E-04	1.40E-05	7.10E-03	0.000	1.000	5.468E-04	0.999
6	4.84E-05	3.54E-06	4.84E-05	3.54E-06	3.55E-03	0.000	1.000	1.367E-04	0.999
7	1.35E-05	8.87E-07	1.35E-05	8.87E-07	1.78E-03	0.000	1.000	3.418E-05	1.000

⇒ **No problem of Maximum Principle.**

⇒ **Second order of convergence of the gradient in L^2 norm.**

Results for Test 6 and Test 7

- Test 6 Oblique drain, umin= -1.2, umax= 0, coarse (C) and fine (F) oblique meshes, mesh6 and mesh7

	nunkw	nnmat	sumflux	erl2	ergrad
DDFV - C	452	3808	1.87E-14	1.17E-03	1.32E-02
DDFV - F	512	4348	-9.06E-14	1.23E-03	1.33E-02
m-DDFV - C	452	3808	-4.92E-14	5.45E-16	3.88E-15
m-DDFV - F	512	4348	- 3.71E-14	1.19E-15	6.77E-15

	erflx0	erflx1	erfly0	erfly1	erflm	uminv	umaxv	uminc	umaxc
DDFV - C	1.23E-03	1.23E-03	4.27E-06	4.27E-06	1.47E-01	-1.200	0.000	-1.146	-5.385E-02
DDFV - F	1.15E-03	1.15E-03	1.44E-05	1.44E-05	1.54E-01	-1.200	0.000	-1.146	-5.385E-02
m-DDFV - C	1.64E-15	6.72E-15	1.25E-15	1.39E-15	8.53E-13	-1.200	0.000	-1.146	-5.385E-02
m-DDFV - F	1.19E-15	5.08E-15	1.39E-16	1.11E-15	2.25E-12	-1.200	0.000	-1.146	-5.385E-02

- Test 7 Oblique barrier, min $= -5.575$, max $= 0.575$, coarse oblique mesh mesh6

	nunkw	nnmat	sumflux	erl2	ergrad
DDFV	452	3808	- 1.24E-14	1.87E-02	4.31E-02
m-DDFV	452	3808	-2.03E-14	1.38E-14	5.86E-14

	erflx0	erflx1	erfly0	erfly1	erflm	uminv	umaxv	uminc	umaxc
DDFV	1.63E-01	1.63E-01	1.74E-01	1.74E-01	9.35E-01	-5.575	0.575	-5.538	0.537
m-DDFV	5.69E-15	1.94E-15	5.33E-15	2.41E-14	3.55E-13	-5.575	0.575	-5.537	0.536

⇒ **The m-DDFV scheme gives the exact solution in both test 6 and 7.**

Results for Test 8 and Test 9

- Test 8 Perturbed parallelograms mesh mesh8. umin= 0.0, umax= 1.0.

nunkw	nnmat	sumflux	uminv	umaxv	uminc	umaxc
265	2197	1.39E-09	-7.755E-05	4.568E-02	-1.257E-03	8.220E-02

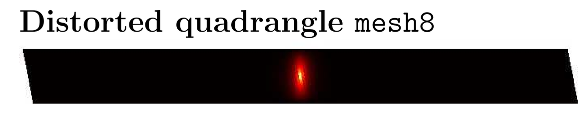
flux0	flux1	fluy0	fluy1
-5.814E-10	-3.357E-10	4.972E-01	5.028E-01

- Test 9 Anisotropy with wells. Square uniform grid mesh9. umin= 0, umax= 1.0.

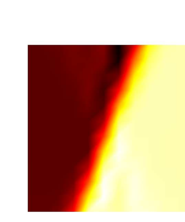
nunkw	nnmat	sumflux	uminv	umaxv	uminc	umaxc
265	2509	1.02E-15	-1.585E-01	1.083E+00	-1.389E-01	1.138E+00

- Solutions of Test 8 (left), Test 9 (right)

Distorted quadrangle mesh8



Regular fine quadrangle mesh 201×201



⇒ **Positivity and Maximum Principle violated.**