# Some Error Estimates in Finite Volume Methods for Parabolic Equations


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## Problem to be approximated

Consider the following simple parabolic model

$$u_t(x,t) - \Delta u(x,t) = f(x,t),\ (x,t) \in \Omega \times (0,T), \tag{1}$$

where $\Omega$ is an open polygonal bounded subset in $\mathbb{R}^d$, with $d=2$ or $d=3$, $T>0$ , and $f$ is a given function.
Initial condition:

$$u(x,0) = u^0(x),\ x \in \Omega. \tag{2}$$

Homogeneous Dirichlet boundary conditions:

$$u(x,t) = 0,\ (x,t) \in \partial\Omega \times (0,T). \tag{3}$$

## Finite volume scheme for (1)–(3)

To define a finite volume approximation for (1)–(3), we first introduce an admissible mesh $\mathcal{T}$ of $\Omega$ in the sense of [3]

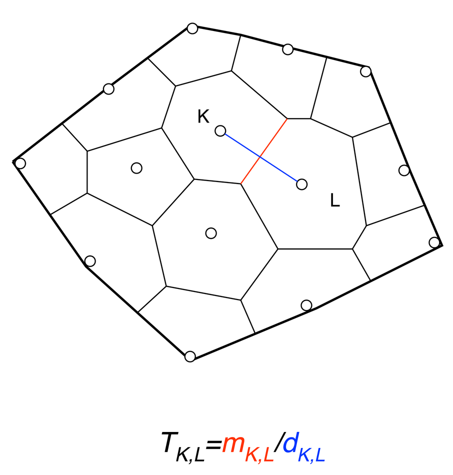


Figure: transmissivity between $K$ and $L$: $T_{K,L} = \frac{m_{K,L}}{d_{K,L}}$

Time discretization $t_n = nk$, $k$ is the time step size.
Denote by $\{u_K^n, K \in \mathcal{T}, n \in [\![0, N+1]\!]\}$ the discrete unknowns; the value $u_K^n$ is expected to approximate $u(x_K, nk)$. An implicit finite volume scheme may be given by, see [3]:

$$\mathrm{m}(K)\frac{u_K^{n+1} - u_K^n}{k} - \sum_{\sigma \in \mathcal{E}_K} \tau_\sigma (u_L^{n+1} - u_K^{n+1}) = \mathrm{m}(K) f_K^n,\ \forall K \in \mathcal{T},\ \forall n \in [\![0, N]\!]. \tag{4}$$

where we have denoted $\sigma = K|L$ if $\sigma \in \mathcal{E}_{\mathrm{int}}$, and $u_L^{n+1} = 0$ if $\sigma \in \mathcal{E}_{\mathrm{ext}} \cap \mathcal{E}_K$, and $f_K^n = \frac{1}{k\mathrm{m(K)}} \int_{nk}^{(n+1)k} \int_K f(x,t)dx\,dt$.
Approximation of (2) could be performed using one of the following choices

1.
$$u_K^0 = u^0(x_K),\ K \in \mathcal{T}. \tag{5}$$

2.
$$u_K^0 = \frac{1}{\mathrm{m(K)}} \int_K u^0(x)dx,\ K \in \mathcal{T}. \tag{6}$$

## Convergence order of volume scheme (4) and(5)/(6)

Thanks to the techniques of the proof of [3], we can prove the existence and uniqueness of a solution $(u_K)_{K\in\mathcal{T}}$ satisfying (4)and (5)/(6), and a "discrete $L^\infty(0,T;L^2(\Omega))$-estimate":

$$\sum_{K\in\mathcal{T}} \mathrm{m}(K)(u(x_K,t_n) - u_K^n)^2 \le C(h+k)^2,\ \forall n \in [\![0, N+1]\!], \tag{7}$$

where the size $h$ of the space discretization $\mathcal{T}$ is given by
$h = \sup\{\mathrm{diam}(K),\ K \in \mathcal{T}\}$

## Shortcomings of the error estimate (7)

- Estimate (7) could only allow us to approximate $\{u(x_K,t_n), K \in \mathcal{T}, n \in [\![0, N+1]\!]\}$.
- Is it possible to obtain an estimate for the Flux of $u$ over the edges of the control volumes ? Note that such an estimate is known for Elliptic Equations
- In addition to the previous item, equation (1) contains also a Time Derivative, is it possible to get an estimate for $u_t$?

## References


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## Flux approximation for the scheme (4)– (5)

In case of (4)– (5), we can prove the following error estimate

$$\sum_{n=0}^{N} \sum_{\sigma\in\mathcal{E}} \mathrm{m}(\sigma) d_\sigma \left( \frac{u_L^n - u_K^n}{d_\sigma} - \frac{1}{\mathrm{m}(\sigma)} \int_\sigma \nabla u(x,t_n) \cdot \mathbf{n}_{K,\sigma} d\gamma(x) \right)^2 \le C(h+k)^2, \tag{8}$$

where $\mathbf{n}_{K,\sigma}$ is the normal vector to $\sigma$ outward to $K$.
Therefore a Flux approximation for $u$ of order $h+k$, in an only discrete norm of $L^2(0,T;L^2(\Omega))$, could be derived from (8)

## Time derivative approximation for the scheme (4)– (5)

In case of (4)– (5), we can prove the following error estimate

$$\sum_{n=0}^{N} \sum_{K\in\mathcal{T}} k\mathrm{m}(K) \left(\frac{u(x_K,t_{n+1}) - u(x_K,t_n)}{k} - \frac{u_K^{n+1} - u_K^n}{k}\right)^2 \le C\frac{(h+k)^2}{k} \tag{9}$$

Therefore a Time derivative approximation for $u$ of only order $\frac{h+k}{\sqrt{k}}$, in an only discrete norm of $L^2(0,T;L^2(\Omega))$, could be derived from (9)

## Other direction to "improve" error estimates (8) and (9)

The idea is as follows:

1 We keep the same scheme (4),

2 instead of the approximations (5) or (6) of (2), we introduce the following discretization of (2) (which is based on a discrete projection of $u_0$):

$$-\sum_{\sigma\in\mathcal{E}_K} \tau_\sigma (u_L^0 - u_K^0) = -\int_K \Delta u^0(x)dx,\ \forall K \in \mathcal{T}. \tag{10}$$

## Flux approximation for the new scheme (4) and (10)

We prove that the following error estimate holds for the scheme (4) and (10):

$$\sum_{\sigma\in\mathcal{E}} \mathrm{m}(\sigma) d_\sigma \left( \frac{u_L^n - u_K^n}{d_\sigma} - \frac{1}{\mathrm{m}(\sigma)} \int_\sigma \nabla u(x,t_n) \cdot \mathbf{n}_{K,\sigma} d\gamma(x) \right)^2 \le C(h+k)^2. \tag{11}$$

Therefore a Flux approximation for $u$ of order $h+k$, in a discrete $L^\infty(0,T;L^2(\Omega))$ (instead of $L^2(0,T;L^2(\Omega))$ in (8)) , could be derived from (11)

## Time derivative approximation for the new scheme (4) and (10)

We prove that the following error estimate holds for the scheme (4) and (10), for all $n \in [\![0, N]\!]$:

$$\sum_{K\in\mathcal{T}} \mathrm{m}(K) \left(\frac{u(x_K,t_{n+1}) - u(x_K,t_n)}{k} - \frac{u_K^{n+1} - u_K^n}{k}\right)^2 \le C(h+k)^2 \tag{12}$$

Therefore a Time derivative approximation for $u$ of order $h+k$ (instead of $\frac{h+k}{\sqrt{k}}$ in (9)), in a discrete $L^\infty(0,T;L^2(\Omega))$ (instead of $L^2(0,T;L^2(\Omega))$ in (9)) , could be derived from (12)

## Idea of the proof of (11)–(12)

The main idea of the proof of estimates (11)–(12) is to compare the finite volume solution defined by (4) and (10) with the finite volume approximation $\{\bar{u}_K^n : K \in \mathcal{T}, n \in [\![0, N+1]\!]\}$ defined by:

$$-\sum_{\sigma\in\mathcal{E}_K} \tau_\sigma (\bar{u}_L^n - \bar{u}_K^n) = -\int_K \Delta u(x,t_n)dx,\ \forall n \in [\![0, N+1]\!]. \tag{13}$$

## In progress

The following considerations are subject of the extended paper [2]:

- Using the so called Discrete Gradient for Elliptic Equation introduced in [4], and thanks to (11), we could obtain approximation for the gradient of the exact solution $u$ of (1)–(3)
- Extention to implicit finite volume schemes for more general parabolic equations for different cases of initial value approximations
- Previous error estimates under assumptions of Bochner integrability of $u$.
- Using present results to obtain approximations of order $(h+k)^2$.




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