Mimetic Discretizations

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Mimetic Finite Difference Method

THE mimetic finite difference (**MFD**) method mimics the underlying properties of the original continuum differential equations such as conservation laws, symmetries, monotonicity of solutions, and the fundamental identities and theorems of vector and tensor calculus.

Define the space $\mathcal{N}$ of node-based unknowns (nodal values of scalar functions), the space $\mathcal{E}$ of edge-based unknowns (tangential components of vector functions), the space $\mathcal{F}$ of face-based unknowns (normal components of vector functions), and the space $\mathcal{P}$ of element-based unknowns (mean values of scalar functions). Then,

$$\begin{array}{ccccccc} C^\infty(\Omega) & \xrightarrow{\nabla} & \left(C^\infty(\Omega)\right)^3 & \xrightarrow{\text{curl}} & \left(C^\infty(\Omega)\right)^3 & \xrightarrow{\text{div}} & C^\infty(\Omega) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathcal{N} & \underset{\mathcal{GRAD}^h}{\longrightarrow} & \mathcal{E} & \underset{\mathcal{CURL}^h}{\longrightarrow} & \mathcal{F} & \underset{\mathcal{DIV}^h}{\longrightarrow} & \mathcal{P} \\ & & & & & & \\ \mathcal{N} & \underset{\widetilde{\mathcal{DIV}}^h}{\longleftarrow} & \mathcal{E} & \underset{\widetilde{\mathcal{CURL}}^h}{\longleftarrow} & \mathcal{F} & \underset{\widetilde{\mathcal{GRAD}}^h}{\longleftarrow} & \mathcal{P} \end{array}$$

The **MFD** method uses first principle (e.g. the divergence and Stokes theorems) to approximate the primary operators ($\mathcal{GRAD}^h$, $\mathcal{CURL}^h$, and $\mathcal{DIV}^h$) and then builds the corresponding derived operators ($\widetilde{\mathcal{DIV}}^h$, $\widetilde{\mathcal{CURL}}^h$, and $\widetilde{\mathcal{GRAD}}^h$) using discrete analogs of Green theorems. For instance,

$$\left(\mathcal{DIV}^h \mathbf{u}^h\right)_E = \frac{1}{|E|} \sum_{f \in \partial E} |f|\, u_f^h,$$

$$\widetilde{\mathcal{GRAD}}^h = -\mathbb{M}_{\mathcal{F}}^{-1} \left(\mathcal{DIV}^h\right)^T \mathbb{M}_{\mathcal{P}},$$

where $|E|$ is the volume of element E, $|f|$ is the area of face f, u_f^h is the average flux through f, and matrix $\mathbb{M}_X$ defines an inner product on space X.

- $\mathbb{M}_{\mathcal{N}}$ and $\mathbb{M}_{\mathcal{P}}$ are relatively easy to build.
- $\mathbb{M}_{\mathcal{F}}$ and $\mathbb{M}_{\mathcal{E}}$ are hard to build for general meshes.
- $\mathbb{M}_X$ is assembled from elemental matrices $\mathbb{M}_{X,E}$.
- Material properties are blended into construction of $\mathbb{M}_{X,E}$.

Steady Diffusion Equation

CONSIDER the steady diffusion equation with a full tensor $\mathbb{K}$,

$$\operatorname{div} \vec{u} = Q, \qquad \vec{u} = -\mathbb{K}\nabla p,$$

subject to appropriate boundary conditions. The **MFD** method is to find pressure $\mathbf{p}^h \in \mathcal{P}$ and velocity $\mathbf{u}^h \in \mathcal{F}$ such that

$$\mathcal{DIV}^h \mathbf{u}^h = \mathbf{Q}^h, \qquad \mathbf{u}^h = -\widetilde{\mathcal{GRAD}}^h \mathbf{p}^h.$$

The **MFD** discretization...

- is locally conservative;
- is 2nd-order accurate for $\mathbf{p}^h$ (element-based unknowns), at least 1st-order accurate for $\mathbf{u}^h$ (face-based unknowns) [3];

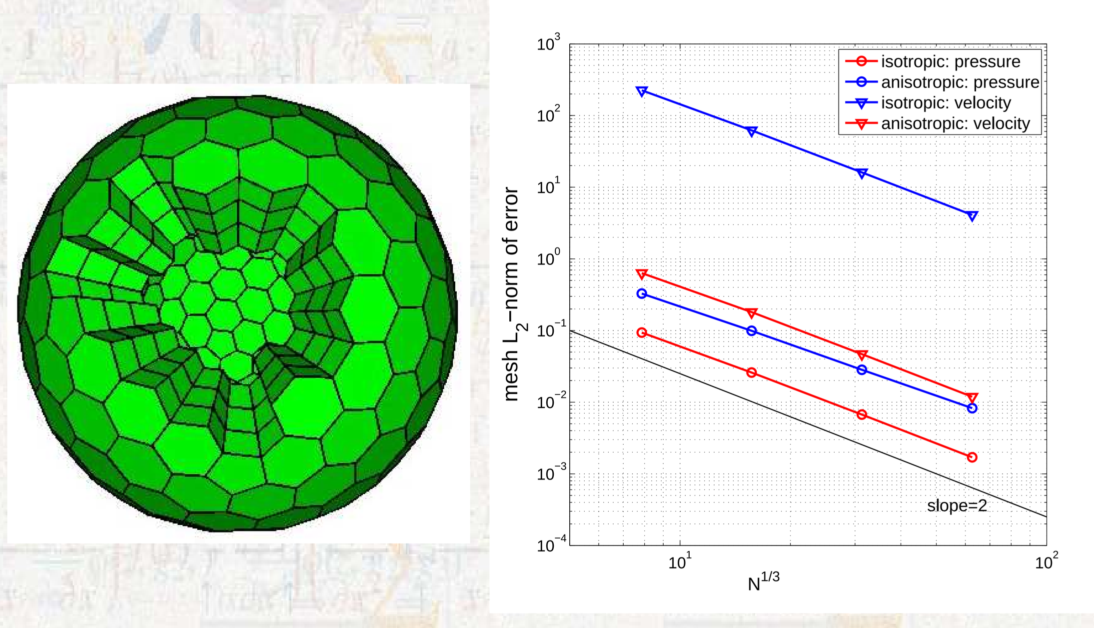
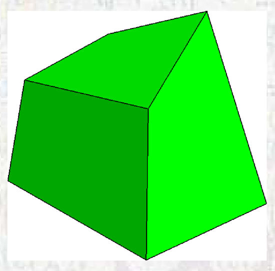


Figure 1: *The polyhedral mesh has three spherical layers of prisms. The second-order convergence for $\mathbf{p}^h$ is achieved for both isotropic and anisotropic (10^3) diffusion tensors.*

- naturally accommodates a full diffusion tensor;
- allows arbitrary meshes (unstructured, polyhedral, AMR, generalized polyhedral, with non-convex elements);
- does NOT require subdivision of elements into simplexes, contrary to the Kuznetsov-Repin FE method;
- *can be implemented on polyhedral meshes in exactly the same manner as on tetrahedral meshes* [1, 2];
- results in an algebraic problem with the symmetric positive definite matrix.

Generalized Polyhedral Meshes

A polyhedron is a three-dimensional solid bounded by a collection of polygons, usually joined at their edges. A generalized polyhedron is a topological polyhedron. Its faces are non-planar.



The generalized polyhedral meshes appear in moving mesh (Lagrangian, ALE) methods, in methods fitting a mesh to curved material interfaces, etc.

- The **MFD** method uses three degrees of freedom on a *strongly* curved face to approximate velocity $\mathbf{u}^h$.
- Continuity of a full velocity vector on strongly curved faces is required for method convergence [3].
- The developed theory predicts the second-order convergence rate for $\mathbf{p}^h$ and the first-order for $\mathbf{u}^h$.

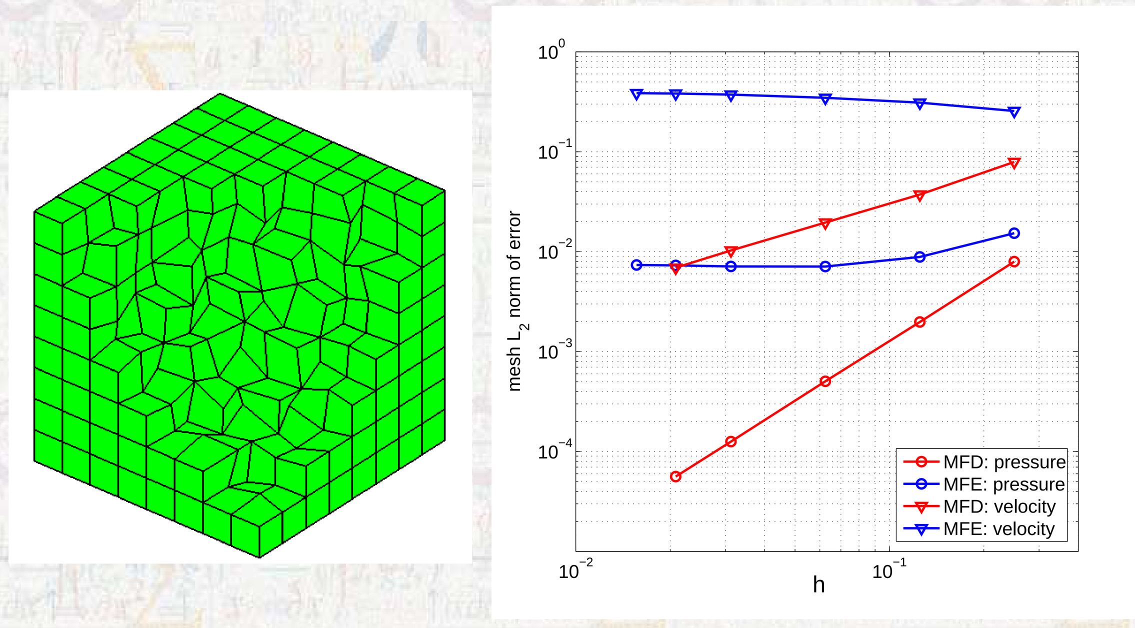
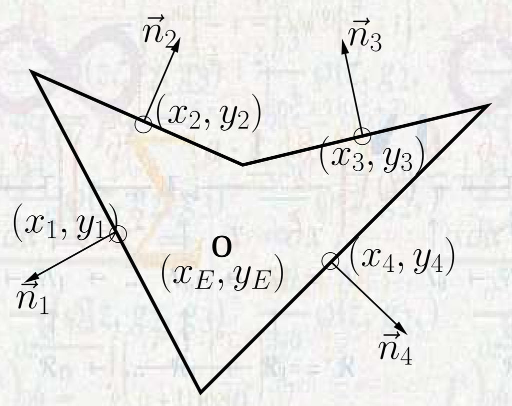


Figure 2: *A part of a logically cubic mesh with randomly perturbed interior nodes. The graphs show optimal convergence rates for the MFD method with three flux unknowns per strongly curved mesh face (red), and lack of convergence for the mixed finite element with one flux unknown (blue) method.*

Inner Product Matrix $\mathbb{M}_{\mathcal{F}}$

A family of inner product matrices $\mathbb{M}_{\mathcal{F}}$ does exist (e.g., a 6-parameter family for hexahedral meshes) [1]. This family can be analyzed to tackle other computational problems such as monotonicity and optimal accuracy for a fixed mesh.



The unknowns are defined by

$$p_E^h = \frac{1}{|E|} \int_E p \,\mathrm{d}V$$

and

$$u_{f_i}^h = \frac{1}{|f_i|} \int_{f_i} \vec{u} \cdot \vec{n}_i \,\mathrm{d}S.$$

Let us introduce auxiliary unknowns, $p_i^h = \frac{1}{|f_i|} \int_{f_i} p \,\mathrm{d}S$. For a polyhedron with k faces, we assume that the following linear relations,

$$\begin{pmatrix} u_{f_1}^h \\ u_{f_2}^h \\ \vdots \\ u_{f_k}^h \end{pmatrix} = -\mathbb{W}_{\mathcal{F},E} \begin{pmatrix} |f_1|\,(p_1^h - p_E^h) \\ |f_2|\,(p_2^h - p_E^h) \\ \vdots \\ |f_k|\,(p_k^h - p_E^h) \end{pmatrix}, \qquad \mathbb{M}_{\mathcal{F},E} \equiv \mathbb{W}_{\mathcal{F},E}^{-1},$$

are exact for any linear function p and the corresponding constant vector $\vec{u}$. There are four linearly independent functions p:

1. $p(x,y,z) = 0$ and $\vec{u} = -\mathbb{K}_E(0;0;0)^T$ (the formula is trivial);
2. $p(x,y,z) = x$ and $\vec{u} = -\mathbb{K}_E(1;0;0)^T$;
3. $p(x,y,z) = y$ and $\vec{u} = -\mathbb{K}_E(0;1;0)^T$;
4. $p(x,y,z) = z$ and $\vec{u} = -\mathbb{K}_E(0;0;1)^T$.

The non-trivial cases give three matrix equations

$$\mathbf{n}_\alpha = \mathbb{W}_{\mathcal{F},E}\,\mathbf{r}_\alpha, \qquad \alpha = x, y, z.$$

Theorem. *Let* $\mathbb{N} = -[\mathbf{n}_x; \mathbf{n}_y; \mathbf{n}_z]$ *and* $\mathbb{R} = [\mathbf{r}_x; \mathbf{r}_y; \mathbf{r}_z]$,

$$\mathbb{N} = \begin{bmatrix} \vec{n}_1^T \\ \vec{n}_2^T \\ \vdots \\ \vec{n}_k^T \end{bmatrix} \mathbb{K}_E \quad \text{and} \quad \mathbb{R} = \begin{bmatrix} |f_1|\,(\vec{x}_1 - \vec{x}_E)^T \\ |f_2|\,(\vec{x}_2 - \vec{x}_E)^T \\ \vdots \\ |f_k|\,(\vec{x}_k - \vec{x}_E)^T \end{bmatrix}.$$

Then, $\mathbb{R}^T\mathbb{N} = \mathbb{N}^T\mathbb{R} = |E|\,\mathbb{K}_E$. *Furthermore, let the columns of matrix* $\mathbb{C}$ *span* $\ker(\mathbb{R}^T)$, *i.e.* $\mathbb{R}^T\mathbb{C} = 0$. *Then*

$$\mathbb{W}_{\mathcal{F},E} = \frac{1}{|E|} \mathbb{N}\mathbb{K}_E^{-1}\mathbb{N}^T + \mathbb{C}\,\mathbb{U}\,\mathbb{C}^T$$

defines an inner product on E *for any* $\mathbb{U} = \mathbb{U}^T > 0$.

- Matrix $\mathbb{U}$ can be replaced by a scalar matrix $u_E\mathbb{I}$ and $\mathbb{C}\,\mathbb{C}^T$ can be computed without explicit construction of $\mathbb{C}$ [1]:

$$\mathbb{W}_{\mathcal{F},E} = \frac{1}{|E|} \mathbb{N}\mathbb{K}_E^{-1}\mathbb{N}^T + u_E\left(\mathbb{I} - \mathbb{R}\left(\mathbb{R}^T\mathbb{R}\right)^{-1}\mathbb{R}^T\right),$$

where $u_E = \operatorname{trace}(\mathbb{K}_E)/|E|$.

- A similar formula holds for $\mathbb{M}_{\mathcal{F},E}$. However, the diffusion solver needs only $\mathbb{W}_{\mathcal{F},E}$.
- Only areas, normals, and mass centers of faces of E are required to compute $\mathbb{W}_{\mathcal{F},E}$ and $\mathbb{M}_{\mathcal{F},E}$. *Shape-regularity affects constants but not the order of convergence of the method.*
- The complexity of building $\mathbb{W}_{\mathcal{F},E}$ is

$$(2d+1)k^2 + 4d^2k \quad \text{flops}, \qquad d = 2, 3.$$

- Extension to generalized polyhedra has been made in [2].

Multi-point Flux MFD Method

A MULTI-POINT flux version of the **MFD** method results in a symmetric scheme with a local expression for flux u_f^h. This is achieved by enforcing a sparsity structure of the matrix $\mathbb{M}_{\mathcal{F},E}$. This implies that the discrete elliptic operator

$$\mathcal{L}^h = \mathcal{DIV}^h\, \widetilde{\mathcal{GRAD}}^h$$

has a local stencil.

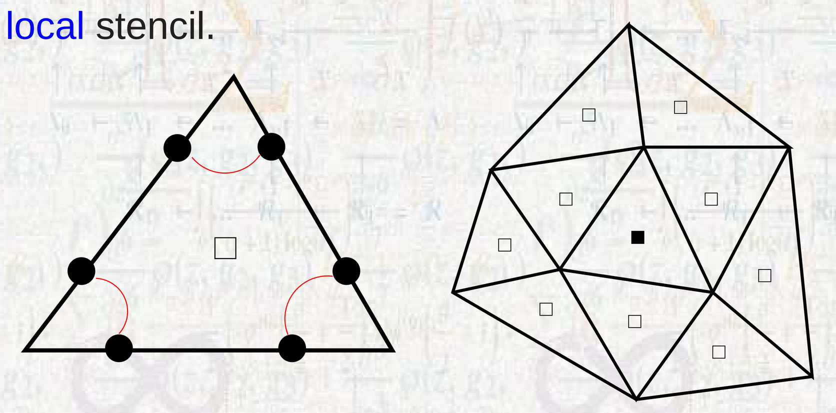


Figure 3: *Velocity degrees of freedom are marked by solid circles. Red lines show non-zero off-diagonal entries of matrix $\mathbb{M}_{\mathcal{F},E}$. Left picture shows the stencil of $\mathcal{L}^h$ for the pressure unknown marked by the black square.*

- Optimal error estimates has been proved for simplicial meshes and full diffusion tensors [4].
- Extension to polyhedral meshes results in lose of symmetry. However, optimal convergence estimates can be proved for a class of meshes [4].

2D Tensor Viscosity Method

ONE of the applications of the developed MFD methods is the tensor viscosity method for a gasdynamics. A dissipative term added to the momentum equation is

$$\rho\frac{\partial \vec{u}}{\partial t} = -\nabla p + \operatorname{div}\left(\mu(\vec{u})\nabla\vec{u}\right).$$

The *primary* operator, gradient $\mathcal{GRAD}^h$, is defined on mesh edges. The *derived* operator is $\widetilde{\mathcal{DIV}}^h$:

$$\left(\mathcal{GRAD}^h \mathbf{u}^h\right)_e = \frac{\mathbf{u}_i^h - \mathbf{u}_j^h}{|e|} \approx \nabla\mathbf{u}\cdot\boldsymbol{\tau}_e,$$

$$\widetilde{\mathcal{DIV}}^h = -\mathbb{M}_{\mathcal{N}}^{-1}\left(\mathcal{GRAD}^h\right)^T \operatorname{diag}\{\mathbb{M}_{\mathcal{E}}, \mathbb{M}_{\mathcal{E}}\}.$$

- In two-dimensions, the inner product matrix $\mathbb{M}_{\mathcal{E}}$ is known:

$$\mathbb{M}_{\mathcal{E}} = \mathbb{M}_{\mathcal{F}}.$$

- New method provides more accurate nodal viscous forces for non-convex mesh elements.

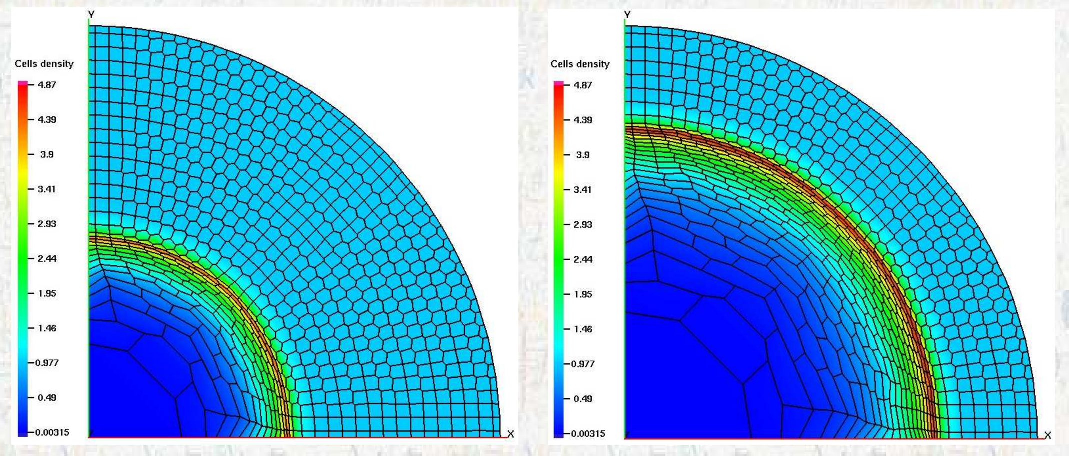


Figure 4: *The Sedov explosion problem: a high energy gas is released in a cold gas at point $(0,0)$. The density profiles at 0.3s and 0.7s illustrate propagation of a shock wave. Lagrangian simulation without artificial viscosity fails at 0.1s.*

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