Immiscible two-phase flows in heterogeneous porous media involving capillary barriers

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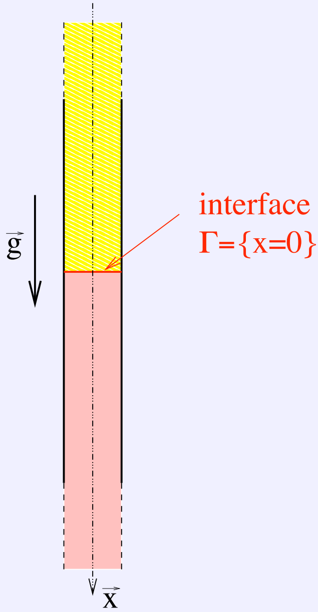
Joint work with T. Gallouët and A. Porretta

Aussois, 2008

Outline



Geometry :

- $\Omega = (-1, 1)$: heterogeneous porous medium,
- $\Omega_1 = (-1, 0)$, $\Omega_2 = (0, 1)$: homogeneous subdomains,
- Γ : interface between Ω_1 and Ω_1.

Flow :

- Immiscible, incompressible two phase flow (oil, water, no gas),
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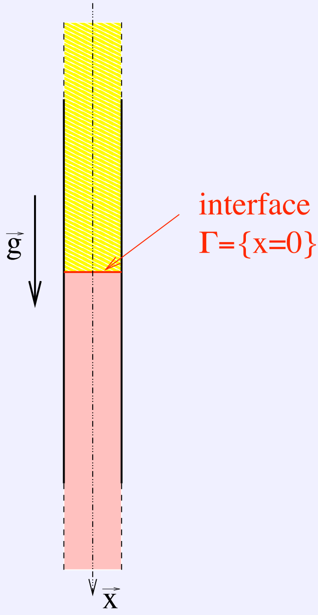

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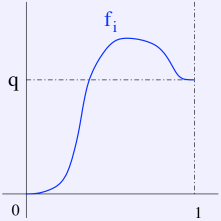
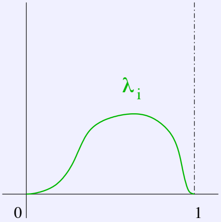
In each Ω_i

$$\partial_t u + \partial_x \Big(f_i(u) - \lambda_i(u)\partial_x \pi_i(u) \Big) = 0,$$

where

- $u \in [0,1]$: saturation, volume rate of water,
- f_i : Lip. functions, $f_i(0) = 0$, $f_i(1) = q$,
- **q** : total flow rate, depends only on time.
- λ_i : Lip. functions, positive on $(0,1)$, $\lambda_i(0) = \lambda_i(1) = 0$,
- π_i : Capillary pressure, increasing Lip. functions,





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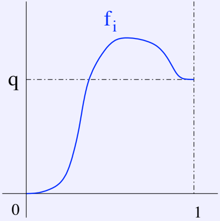

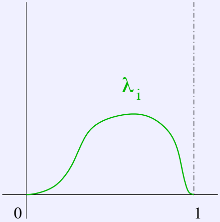

[Enchéry, Eymard & Michel, 06], [C., Gallouët & Porretta] :

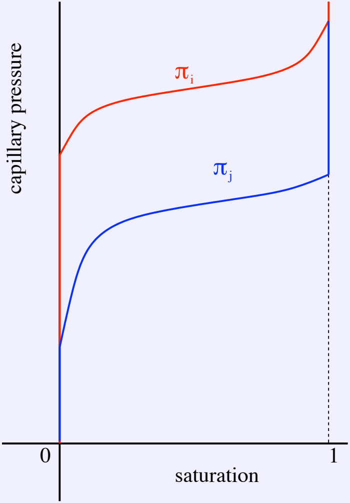

FIG.: Graph of capillary pressures

Let us define the monotonous graph $\tilde{\pi}_i$:
$\tilde{\pi}_i(s) = \pi_i(s)$ if $0 < s < 1$,
$\tilde{\pi}_i(0) =]-\infty, \pi_i(0)]$
$\tilde{\pi}_i(1) = [\pi_i(1), +\infty[$

Thus, at the interface :

$$\tilde{\pi}_1(u_1) \cap \tilde{\pi}_2(u_2) \neq \emptyset.$$

+ Flux connection :

$$F_1(u_1, \partial_x u_1) = F_2(u_2, \partial_x u_2).$$

The problem becomes :

1. $\partial_t u + \partial_x(f_i(u) - \lambda_i(u)\nabla\pi_i(u)) = 0,$ (Ω_i)
2. $\tilde{\pi}_1(u_1) \cap \tilde{\pi}_2(u_2) \neq \emptyset,$ (Γ)
3. $\mathbf{F}_1(u_1, \partial_x u_1) = \mathbf{F}_2(u_2, \partial_x u_2),$ (Γ)
4. "Boundary conditions" $(\partial\Omega)$
5. $u(x, 0) = u_0(x).$ (Ω)

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Let $T > 0$, denoting by $\varphi_i(u) = \int_0^u \lambda_i(s)\pi_i'(s)ds$,

Weak solution

u is said to be a weak solution if :

1. $u \in L^\infty(\Omega \times (0,T))$, $0 \le u \le 1$,
2. $\forall i$, $\varphi_i(u) \in L^2(0,T;H^1(\Omega_i))$,
3. for a.e. $t \in (0,T)$, $\tilde{\pi}_1(u_1(0,t)) \cap \tilde{\pi}_2(u_2(0,t)) \neq \emptyset$,
4. $\forall \psi \in C^\infty(\overline{\Omega} \times [0,T[)$,

$$\sum_{i=1,2} \iint_{\Omega_i \times (0,T)} u \partial_t \psi dxdt + \sum_{i=1,2} \int_{\Omega_i} u_0 \psi(\cdot,0) dx$$
$$+ \sum_{i=1,2} \iint_{\Omega_i \times (0,T)} \partial_x (f_i(u) - \partial_x \varphi_i(u)) \partial_x \psi dxdt$$
$$+\text{"Boundary conditions"} = 0.$$

Outline

Discretization of $\Omega \times (0, T)$:

- Time steps : $0 = t^0 < t^1 < \cdots < t^M = T$,
  - $\delta t^n = t^{n+1} - t^n$.
- Edges : $(x_k)_{k=-N,\ldots N}$, with
  - $x_{-N} = -1$,
  - $x_N = 1$,
  - $x_0 = 0$,
  - $x_{k+1} > x_k$,
- Cell centers : $x_{k+1/2} \in]x_k, x_{k+1}[$, $k = -N, \ldots N-1$,
- Discrete unknowns : $\left(u_{k+1/2}^{n+1}\right)_{k,n}$, $\left(F_k^{n+1}\right)_{k,n}$,
- Finite volume scheme : $-N \le k \le N-1$, $0 \le n \le M-1$,

$$\frac{u_{k+1/2}^{n+1} - u_{k+1/2}^{n+1}}{\delta t^n}(x_{k+1} - x_k) + F_{k+1}^{n+1} - F_k^{n+1} = 0.$$

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- Initial data :

$$u^0_{k+1/2} = \frac{1}{x_{k+1} - x_k} \int_{x_k}^{x_{k+1}} u_0(x)dx,$$

- Inner edges ($k \notin \{-N, 0, N\}$) :

$$F^n_k = G_i\left(u^n_{k-1/2}, u^n_{k+1/2}\right) - \frac{\varphi_i(u^n_{k+1/2}) - \varphi_i(u^n_{k-1/2})}{x_{k+1/2} - x_{k-1/2}}$$

  - G_i Lipschitz continuous w.r.t. to each variable,
  - $G_i(a, b)$ non-decreasing w.r.t a, non-increasing w.r.t. b,
  - $G_i(a, a) = f_i(a)$,

- Boundary conditions :

$$F^n_{-N} = G_1(\overline{u}_1, u^n_{-N+1/2}), \qquad F^n_N = G_2(u^n_{N-1/2}, \overline{u}_2).$$

Flux at the interface :

A Rusanov scheme

$$
\begin{aligned}
F_0^n &= f_1(u^n_{-1/2}) + Lip_{f_1}(u^n_{-1/2} - u^n_{0,1}) + \frac{\varphi_1(u^n_{-1/2}) - \varphi_1(u^n_{0,1})}{x_0 - x_{-1/2}} \quad (1)\\
&= f_2(u^n_{1/2}) + Lip_{f_2}(u^n_{0,2} - u^n_{1/2}) + \frac{\varphi_2(u^n_{0,2}) - \varphi_1(u^n_{1/2})}{x_{1/2} - x_0}, \quad (2)
\end{aligned}
$$

where $(u^n_{0,1}, u^n_{0,2})$ is the unique couple of solutions in $[0,1]$ of

$\rightarrow$ $\tilde{\pi}_1(u^n_{0,1}) \cap \tilde{\pi}_2(u^n_{0,2}) \neq \emptyset$,
$\rightarrow$ flux connection (1)=(2).

- For all $k \in \{-N, N-1\}$, for all $n \in \{0, M\}$,

$$0 \leq u^n_{k+1/2} \leq 1.$$

- There exists a unique $\left(u^n_{k+1/2}\right)_{k,n}$ solution to the scheme.

$\rightarrow$ Discrete solution :

$$\begin{aligned} u_{\mathcal{D}}(x,t) &= u^{n+1}_{k+1/2} \text{ if } x \in (x_k, x_{k+1}) \text{ and } t \in (t^n, t^{n+1}], \\ u_{\mathcal{D}}(x,0) &= u^0_{k+1/2} \text{ if } x \in (x_k, x_{k+1}). \end{aligned}$$

- Discrete L^1-contraction principle : $\forall t \in [0,T]$,

$$\int_\Omega (u_{\mathcal{D}}(x,t) - v_{\mathcal{D}}(x,t))^{\pm} dx \leq \int_\Omega (u_{\mathcal{D}}(x,0) - v_{\mathcal{D}}(x,0))^{\pm} dx.$$

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- Energy estimate :

$$\sum_{n=0}^{M-1} \delta t^n \sum_{k\neq 0} \delta x_k \left(F_k^{n+1}\right)^2 \leq C(f_i, \lambda_i, \pi_i, T).$$

$\Rightarrow$ space and times translates estimates,

$\Rightarrow$ strong convergence of the traces,

Convergence to a weak solution

Up to a subsequence,

$$u_{\mathcal{D}} \to u \text{ a.e. in } \Omega \times (0,T),$$

where u is a weak solution to the problem

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Outline

[Alt & Luckaus (1983), Otto (1996), Carillo (1999)]

Using a doubling variable method :

1. $u \in C([0,T]; L^p(\Omega))$, $p \in [1, +\infty[$.
2. If v is a weak solution with the initial data v_0 :
   $\forall \psi \in \mathcal{D}^+(\overline{\Omega} \times [0,T[)$ with $\psi(0,T) = 0$,

$$\sum_i \int_{\Omega_i \times (0,T)} \phi_i \, (u - v)^+ \, \partial_t \psi dxdt + \sum_i \int_{\Omega_i} \phi_i \, (u_0 - v_0)^+ \, \psi(0) dx$$
$$+ \sum_i \int_{\Omega_i \times (0,T)} \left[\begin{array}{c} sign_+(u - v) \, (f_i(u) - f_i(v)) \\ -\partial_x \left(\varphi_i(u) - \varphi_i(v)\right)^+ \end{array} \right] \partial_x \psi dxdt \geq 0.$$

Let $\varepsilon > 0$, let $\psi_\varepsilon \in \mathcal{D}^+(\Omega)$ with :

- $\psi = 0$ on $\Gamma_{i,j} \times (0, T)$,
- $\psi_\varepsilon \to 1$ in $L^1(\Omega)$ as $\varepsilon \to 0$.

Let $s \in [0, T[$ taking the test function $\chi_{[0,s[}(t)\psi_\varepsilon(x)$ leads to :

$$\sum_i \int_{\Omega_i} (u(x,s) - v(x,s))^+ \, dx \leq \sum_i \int_{\Omega_i} (u_0 - v_0)^+ \, dx$$

$$\underbrace{+ \limsup_{\varepsilon} \sum_i \int_{\Omega_i \times (0,s)} \left[\begin{array}{c} sgn_+(u-v)\,(f_i(u) - f_i(v)) \\ -\partial_x \left(\varphi_i(u) - \varphi_i(v)\right)^+ \end{array} \right] \partial_x \psi_\varepsilon dxdt}_{\leq 0???}$$

$\rightarrow$ Additional regularity is needed :

$$F_i(u, \partial_x u) \in L^\infty(\Omega_i \times (0,T)) \Rightarrow \text{uniqueness.}$$

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Bounded flux solution :

u is said to be a bounded flux solution if

1. u is a weak solution,
2. $F_i(u, \partial_x u) \in L^\infty(\Omega_i \times (0, T))$.

Prepared initial data :

$i)$ $\partial_x \varphi_i(u_0) \in L^\infty(\Omega_i)$,

$ii)$ $\tilde{\pi}_1(u_{0,1}) \cap \tilde{\pi}_2(u_{0,2}) \neq \emptyset$.

$$\Rightarrow \qquad \max_k \left| F_k^0 \right| \leq C(u_0, f_i, \lambda_i, \pi_i).$$

Uniform Bound on the discrete

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$$\left.\begin{array}{l} i)\ \partial_x \varphi_i(u_0) \in L^\infty(\Omega_i), \\ ii)\ \tilde{\pi}_1(u_{0,1}) \cap \tilde{\pi}_2(u_{0,2}) \neq \emptyset. \end{array}\right. \quad \Rightarrow \quad \max_k \left| F_k^0 \right| \leq C(u_0, f_i, \lambda_i, \pi_i).$$

Uniform Bound on the discrete

$$\max_n \max_k \left| F_k^n \right| \leq C(u_0, f_i, \lambda_i, \pi_i)$$

Idea of the proof :

- equation on the flux

$$"\partial_t F_i + f_i'(u_i)\partial_x F_i - \partial_x\left(\varphi_i{}'(u_i)\partial_x F_i\right) = 0"$$

$\Rightarrow$ Maximum principle.

- there exists $(a_{k,k-1}^{n+1}, a_{k,k+1}^{n+1}) \in (\mathbb{R}_+)^2$ such that :

$$\left[F_{k-1}^{n+1}, F_k^{n+1}, F_{k+1}^{n+1}\right] \cdot \left[\begin{array}{c} -a_{k,k-1}^{n+1} \\ 1 + a_{k,k-1}^{n+1} + a_{k,k+1}^{n+1} \\ -a_{k,k+1}^{n+1} \end{array}\right] \leq F_k^n.$$

$$\Rightarrow \max_k F_k^{n+1} \leq \max\left(\max_k F_k^n, F_{-N}^{n+1}, F_N^{n+1}\right).$$

Idea of the proof :

- equation on the flux

$$" \partial_t F_i + f_i'(u_i)\partial_x F_i - \partial_x \left({\varphi_i}'(u_i)\partial_x F_i \right) = 0"$$

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- there exists $(a^{n+1}_{k,k-1}, a^{n+1}_{k,k+1}) \in (\mathbb{R}_+)^2$ such that :

$$\left[F^{n+1}_{k-1}, F^{n+1}_{k}, F^{n+1}_{k+1}\right] \cdot \left[\begin{array}{c} -a^{n+1}_{k,k-1} \\ 1 + a^{n+1}_{k,k-1} + a^{n+1}_{k,k+1} \\ -a^{n+1}_{k,k+1} \end{array}\right] \leq F^n_k.$$

$$\Rightarrow \max_k F^{n+1}_k \leq \max\left(\max_k F^n_k, F^{n+1}_{-N}, F^{n+1}_N\right).$$

Convergence toward a bounded flux solution

Let u_0 be a prepared initial data, then the discrete solution $u_{\mathcal{D}}$ converges to the unique bounded flux solution.

Let $u_0 \in L^\infty$, then $u_{\mathcal{D}}$ converges to the unique SOLA.

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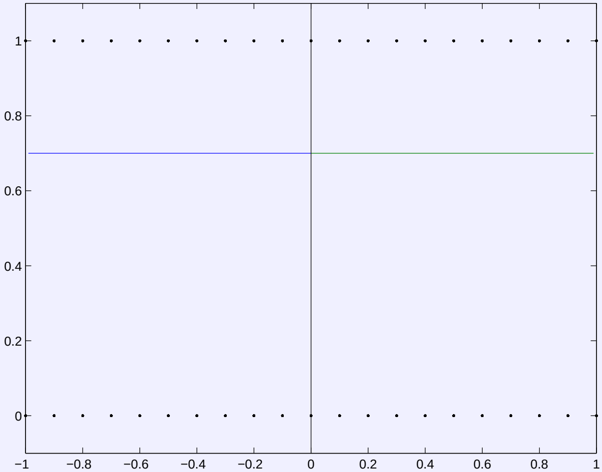
Outline

$u_0 = 0.7$, $q = 0$, $\lambda_i(u) = \kappa_i u(1-u)$, $f_i = qu - \lambda_i(u)$,
$\pi_i(u) = P_i + u/10$, $P_1 = 1$, $P_2 = 2$,

$$\Rightarrow u_1 = 1 \text{ or } u_2 = 0.$$



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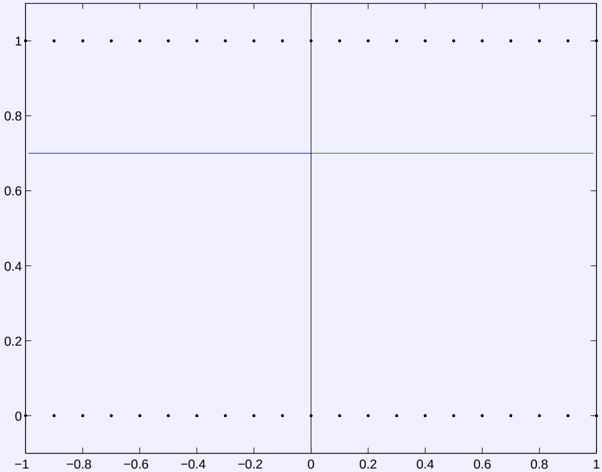

$u_0 = 0.2$, $q = 1$, $\lambda_i(u) = \kappa_i u(1-u)$, $f_i = qu - \lambda_i(u)$,
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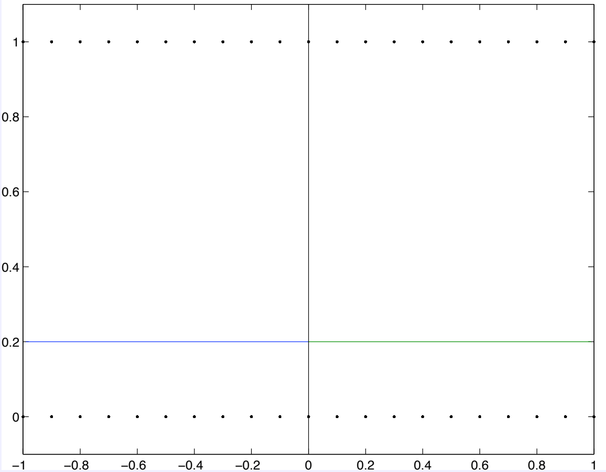

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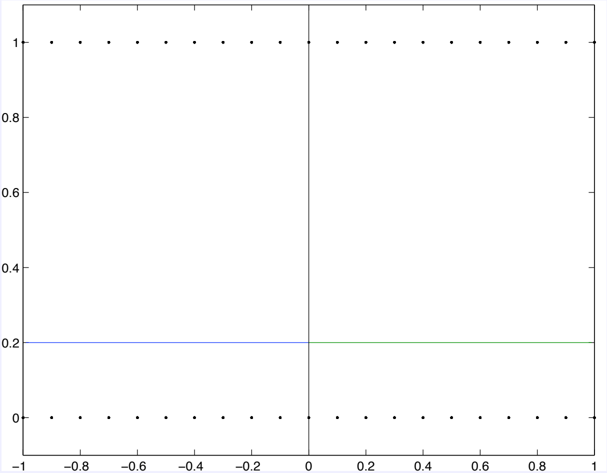

Outline

Conclusion :

- a new simplified model for two phase flow in heterogeneous, allowing the occurrence of capillary barriers,
- a simple convergent scheme to predict the flow.

Prospectives :

- a multidimensional model
  - uniqueness of the solution,
  - interface condition for the pressure equation,
  - numerical simulations.