The influence of the boundary discretization on the MPFA L-method and its application for two-phase flow

Yufei Cao , Rainer Helmig , Barbara Wohlmuth

Universität Stuttgart

cao@ians.uni-stuttgart.de

http://www.ians.uni-stuttgart.de/nmh/

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Outline

1. **Motivation**

2. **MPFA L-method**

3. **Influence of Dirichlet boundary discretization on the MPFA L-method**
   - Rectangular mesh
   - General quadrilateral mesh

4. **Application for two-phase flow**
   - Mathematical formulation
   - Solution procedure
   - Numerical simulation

5. **Conclusion**

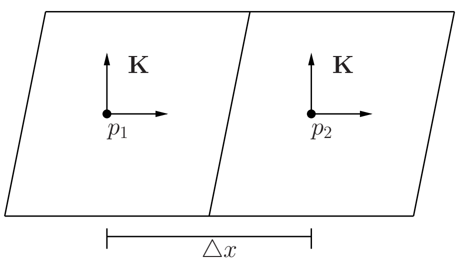
Motivation

Finite volume method: popular → multiphase flow equations in reservoir simulation

- local conservative
- simple discretization stencils

Classical cell-centered finite volume (CCFV) method: TPFA

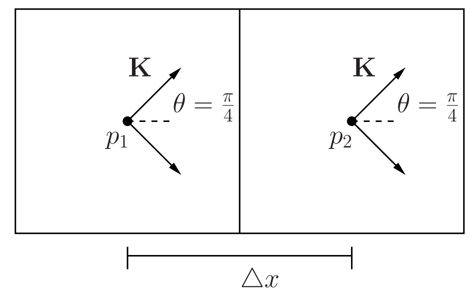
grid effect



$$\mathbf{K} = \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$-\mathbf{K}\nabla p \cdot \mathbf{n} \neq -\frac{p_2 - p_1}{\triangle x}$$

permeability orientation



$$\mathbf{K} = \begin{pmatrix} 1.5 & 0.5 \\ 0.5 & 1.5 \end{pmatrix}$$

TPFA does not work properly for :

⋆ non-orthogonal grids: at faults; near-well regions

⋆ general orientation of $\mathbf{K}$: upscaling model→full permeability tensor

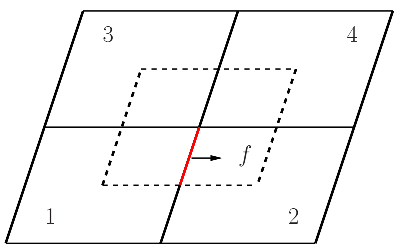
$\Longrightarrow$ **Different MPFA methods were developed!**

- **Introduction to the different MPFA methods**
  - ⋆ I.Aavatsmark, T.Barkve, Ø.Bøe, T.Mannseth [96]
  - ⋆ I.Aavatsmark, T.Barkve, T.Mannseth [98]
  - ⋆ M.G.Edwards, C.F.Rogers [98]
  - ⋆ I.Aavatsmark [02]
  - ⋆ M.G.Edwards [02]
  - ⋆ J.M.Nordbotten, G.T.Eigestad [05]
  - ⋆ I.Aavatsmark, G.T.Eigestad, B.T.Mallison, J.M.Nordbotten [07]
- **Convergence study**
  - ⋆ G.T.Eigestad, R.A.Klausen [05]
  - ⋆ I.Aavatsmark, G.T.Eigestad, R.A.Klausen [06]
  - ⋆ I.Aavatsmark, G.T.Eigestad [06]
  - ⋆ R.A.Klausen, R.Winther [06]
  - ⋆ M.F.Wheeler, I.Yotov [06]
  - ⋆ I.Aavatsmark, G.T.Eigestad, R.A.Klausen, M.F.Wheeler, I.Yotov [07]

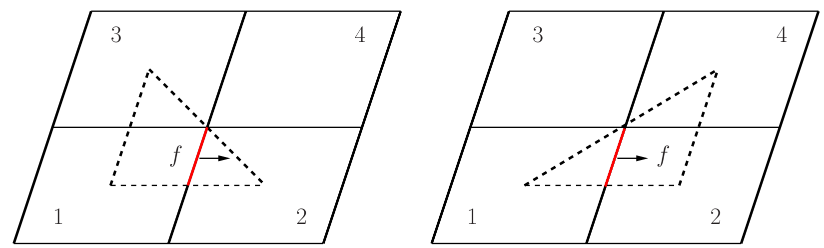
There are many variants of the MPFA method: O, U, Z, L, etc.

- **Most popular**: MPFA O-method

MPFA O-method



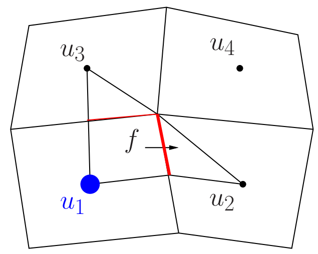
MPFA L-method



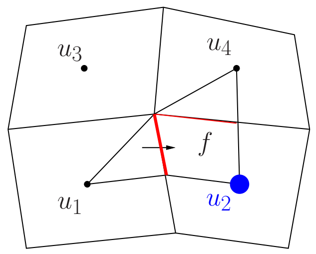
- **Our interest**: MPFA L-method
  (recently developed by I.Aavatsmark et al. [07])
  - ⋆ flux stencils: smaller
  - ⋆ domain of convergence: larger
  - ⋆ domain of monotonicity: larger

MPFA L-method

- **Step1.** Choose the proper L triangle to compute the transmissibility t_i^j.
  If $|t_1^1| < |t_2^2|$, choose T$_1$; otherwise, choose T$_2$.



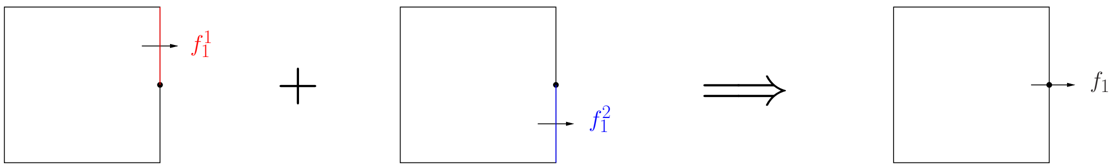
$$f = -\mathbf{K}\nabla u \cdot \mathbf{n}$$

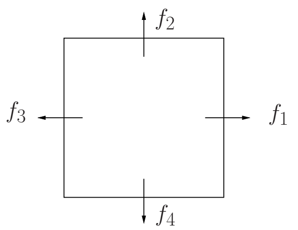


T$_1$: $f = t_1^1 u_1 + t_2^1 u_2 + t_3^1 u_3$

T$_2$: $f = t_1^2 u_1 + t_2^2 u_2 + t_4^2 u_4$

- **Step2.** Calculate the fluxes through each half edge f_1^1 and f_1^2 as in Step 1





- **Step3.** Insert flux expressions derived in Step 2 into the local control -volume discretization $f_1 + f_2 + f_3 + f_4 = \int_K q dx$

Numerical example

- **In this section**
  - ⋆ study and compare the convergence of the L-method with O-method
  - ⋆ illustrate the influence of the Dirichlet boundary discretization on the L-method

- **We consider**

$$\begin{cases} -\nabla \cdot (\mathbf{K}\nabla u) = q, & \text{in} \quad \Omega, \\ u = 0, & \text{on} \quad \Gamma_D = \partial\Omega, \end{cases} \tag{1}$$

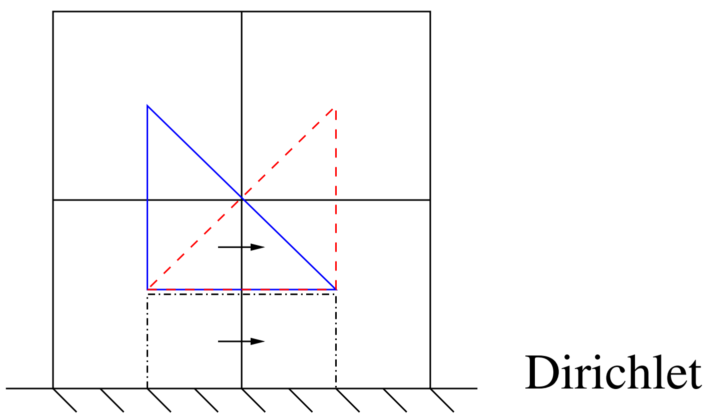
  - ⋆ $\mathbf{K}$ is a homogeneous anisotropic tensor:

$$\mathbf{K} = \begin{pmatrix} 1.5 & 0.5 \\ 0.5 & 1.5 \end{pmatrix}.$$

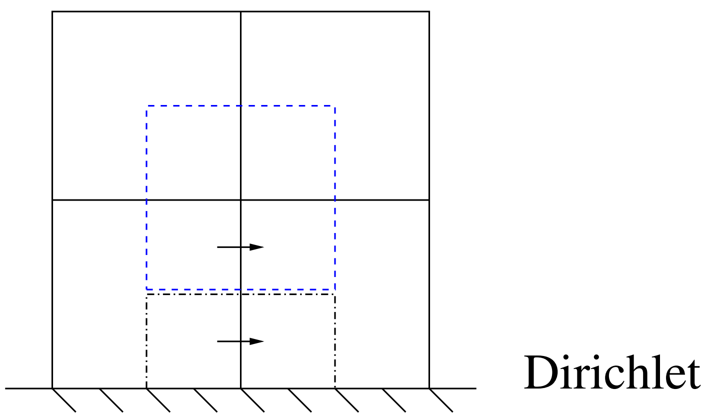
  - ⋆ exact solution $u = 16x(1-x)y(1-y)$
  - ⋆ source term $q = 48x(1-x) - 16(1-2x)(1-2y) + 48y(1-y)$

Discretization strategies

- **MPFA O-method**
- **MPFA L-method: two possibilities for Dirichlet boundary discretization**
  - ⋆ Case I – MPFA L: L+O
  - ⋆ Case II – MPFA L : full O



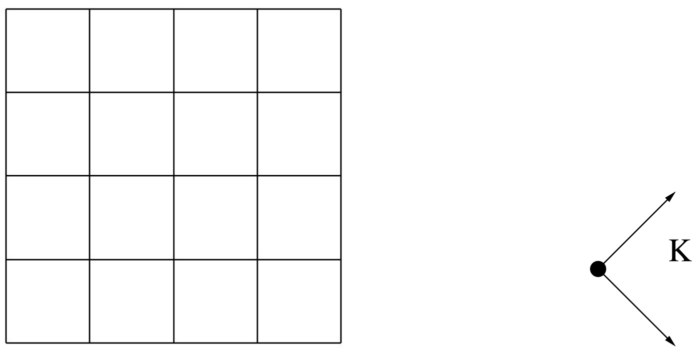
MPFA L: L+O



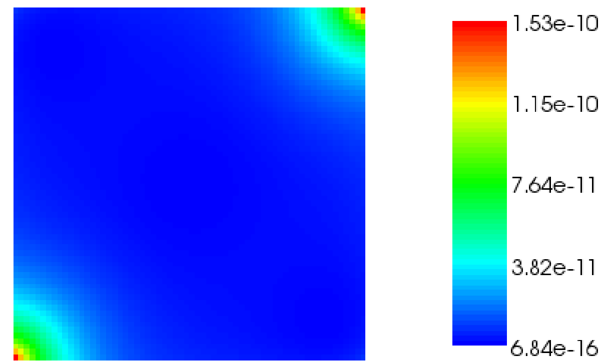
MPFA L: full O

Local relative error graphs for normal velocity

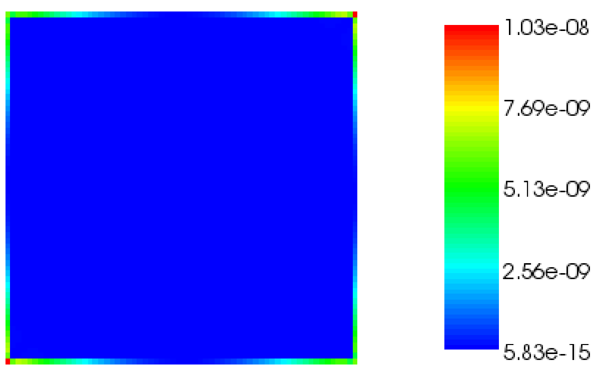
Rectangular mesh



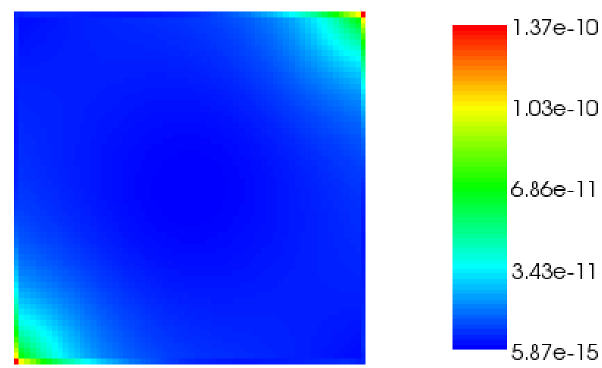
MPFA O (error max. 10^{-10})



MPFA L: L+O (error max. 10^{-8})

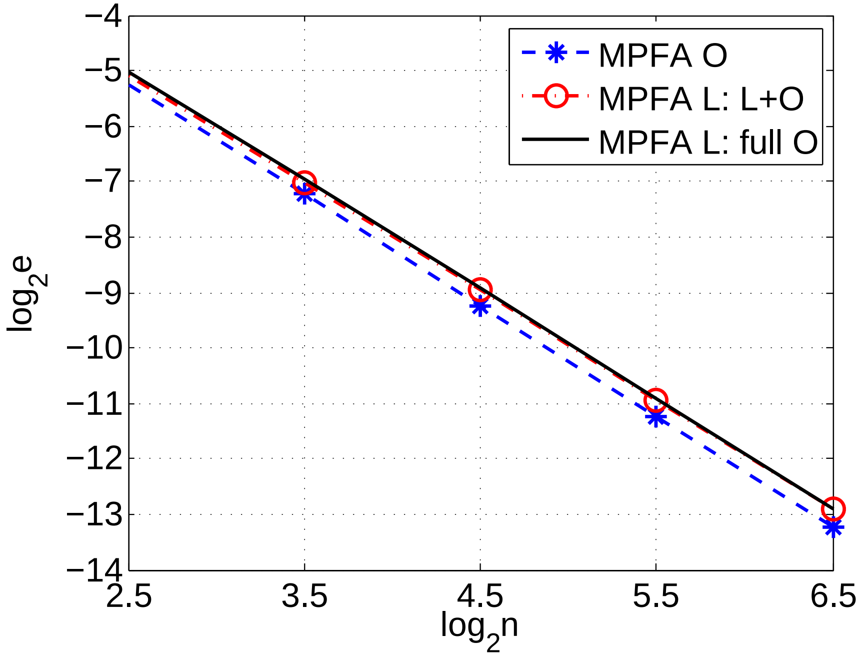


MPFA L: full O (error max. 10^{-10})

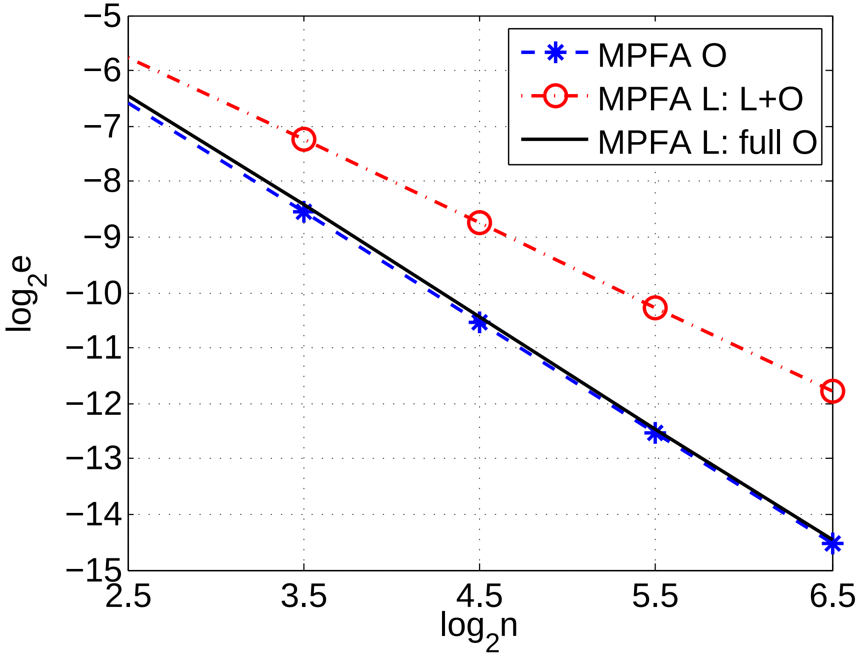


Convergence order graph

potential

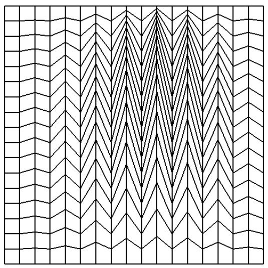


normal velocity

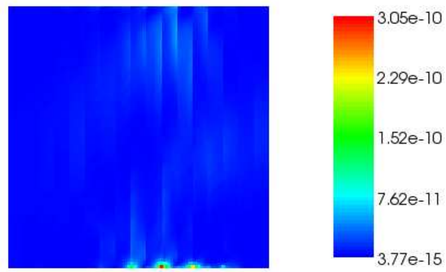


Local relative error & convergence order graphs

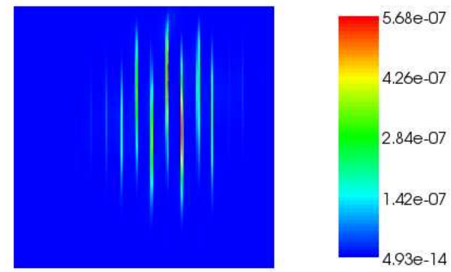
General quadrilateral mesh



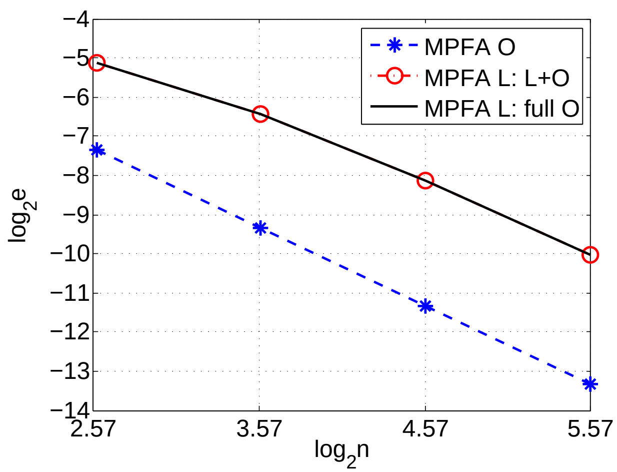
MPFA O (error max. 10^{-10})



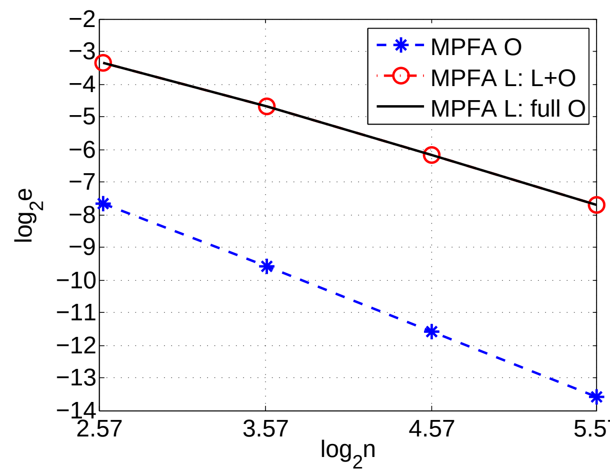
MPFA L (error max. 10^{-7})



potential



normal velocity



Mathematical model

Mass balance equations: $$\frac{\partial(\Phi\rho_\alpha S_\alpha)}{\partial t} + \nabla\cdot(\rho_\alpha \mathbf{v}_\alpha) - \rho_\alpha q_\alpha = 0, \quad \alpha \in \{w, n\}$$

Darcy's law: $$\mathbf{v}_\alpha = -\frac{k_{r\alpha}}{\mu_\alpha}\mathbf{K}(\nabla p_\alpha - \rho_\alpha \mathbf{g}), \quad \alpha \in \{w, n\}$$

Assumptions: isothermal system; $p_c = 0$, $q = 0$; no gravity;
incompressible fluids & porous media: $\varrho_\alpha =$ const, $\Phi =$ const.

Fully coupled formulation: $$\Phi\frac{\partial S_\alpha}{\partial t} - \nabla\cdot(\lambda_\alpha \mathbf{K}\nabla p_\alpha) = 0, \quad \alpha \in \{w, n\}$$

Closure relations: $$S_w + S_n = 1, \quad p_n = p_w$$

Φ - porosity
$\mathbf{K}$ - permeability
$k_{r\alpha}$ - rel. permeability
μ_α - viscosity

$\lambda_\alpha = \frac{k_{r\alpha}}{\mu_\alpha}$ - mobility
S_α - saturation
p_α - pressure

Fractional Flow Formulation

Pressure equation - elliptic

$$\nabla \cdot \mathbf{v}_t = 0, \quad \mathbf{v}_t = -\lambda_t(S_w)\mathbf{K}\nabla\bar{p} = -\lambda_t(S_w)\mathbf{K}\nabla p_w$$

- total velocity: $\mathbf{v}_t = \mathbf{v}_w + \mathbf{v}_n$; total mobility: $\lambda_t = \lambda_w + \lambda_n$
- global pressure: $\bar{p} = p_w = p_n$ (if no p_c)
- $\star$**Solution Procedure**: **L-method** $\rightarrow \bar{p} \rightarrow$ **reuse L-method** $\rightarrow \mathbf{v}_t$

Saturation equation - hyperbolic

$$\Phi\frac{\partial S_w}{\partial t} + \nabla \cdot (f_w(S_w)\mathbf{v}_t) = 0$$

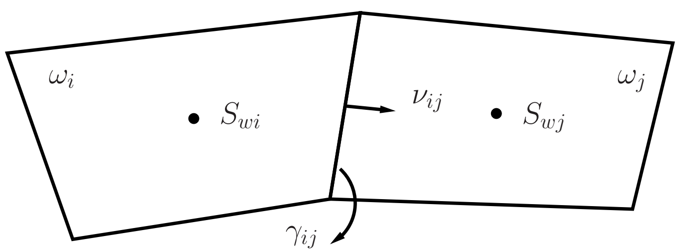
- fractional flow function: $f_\alpha = \frac{\lambda_\alpha}{\lambda_t}$
- In 1D, the saturation equation $\Longrightarrow$ **Buckley-Leverett equation**.

Weakly coupled $\Longrightarrow$ IMPES (IMplicit-Pressure-Explicit-Saturation) method

Discretization: saturation equation

- **Notations**

  $\mathcal{J} = \{t^0, t^1, \cdots, t^M\}$: partition of the time interval $[0,T], \triangle t^n = t^{n+1} - t^n$



ω_i:	cell i with boundary $\partial\omega_i$
γ_{ij}:	either $\partial\omega_i \cap \partial\omega_j$ or $\partial\omega_i \cap \partial\Omega$
$\lvert\gamma_{ij}\rvert, \lvert\omega_i\rvert$:	measure of γ_{ij}, ω_i
$\mathbf{v}_{tij}^n$:	total velocity on γ_{ij} at t^n
ν_{ij}:	unit outer normal of γ_{ij}

- **Upwind cell-centered finite volume scheme**

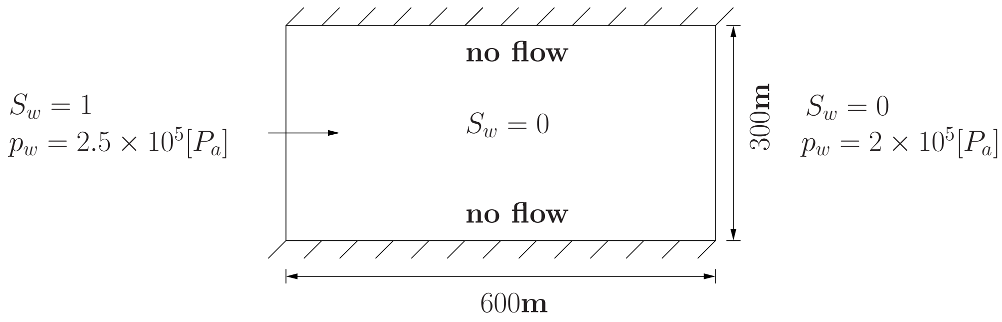
$$S_{wi}^{n+1} = S_{wi}^n - \frac{\triangle t^n}{\Phi} \sum_{\gamma_{ij}} \frac{|\gamma_{ij}|}{|\omega_i|} (f_w(S_{wi}^n) \mathrm{max}(0, \mathbf{v}_{tij}^n \cdot \nu_{ij}) + f_w(S_{wj}^n) \mathrm{max}(0, -\mathbf{v}_{tij}^n \cdot \nu_{ij}))$$

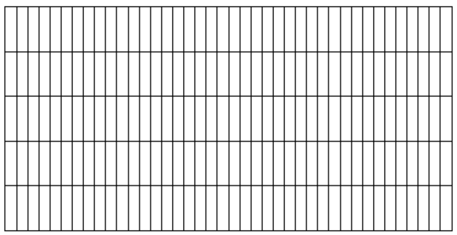
- **CFL condition**

  IMPES: conditionally stable $\Longrightarrow C_r = \frac{|\mathbf{v}_{tij}^n| \triangle t^n}{\triangle x} < 1$

Buckley-Leverett Problem

- Buckley-Leverett problem: displacement of a non-wetting phase by a wetting phase
- Boundary and initial conditions (left) and K-orthogonal grid (right)

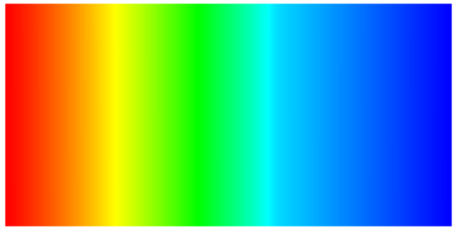




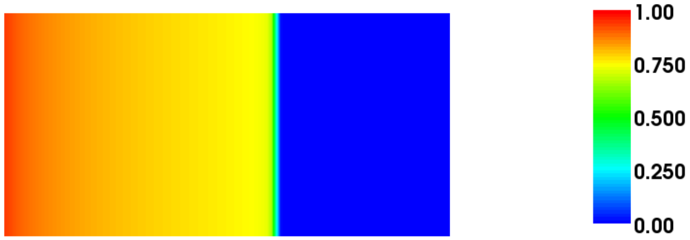
- Simulation parameters:

  $\mathbf{K} = 10^{-10}\mathbf{I}$, $\mu_w = \mu_n = 0.001$, $\Phi = 0.2$, nonlinear Brooks-Corey k_r-S_w relation
- Reference solution at time $t = 1.0 \times 10^7$s

pressure

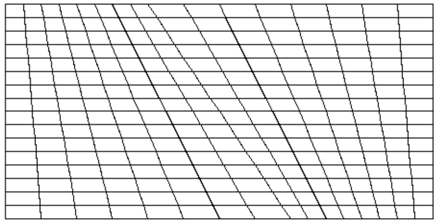


saturation



Simulation on General Quadrilateral Grid 1

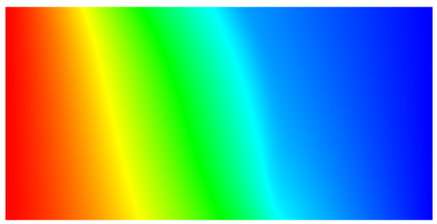
qudrilateral grid 1

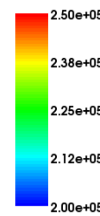


TPFA method

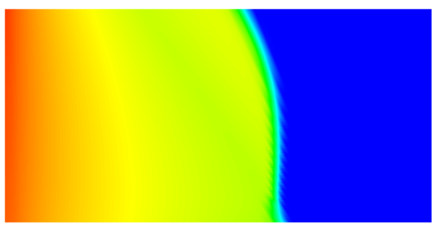
MPFA L-method

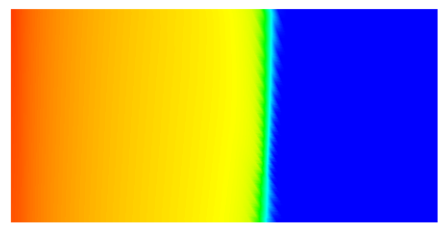
pressure

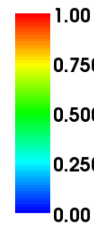




saturation



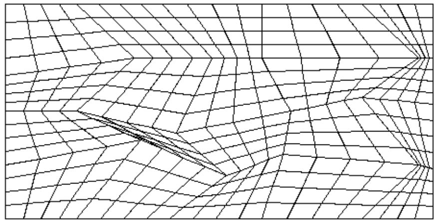




- **TPFA method**: saturation—more smearing; saturation front—wrong
- **MPFA L-method**: converge well

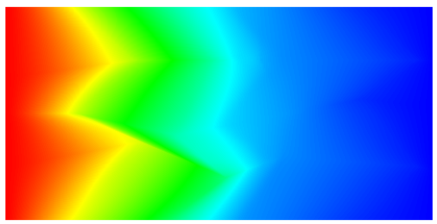
Simulation on General Quadrilateral Grid 2

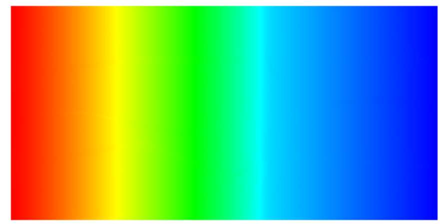
quadrilateral grid 2

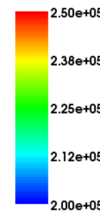


TPFA method | MPFA L-method

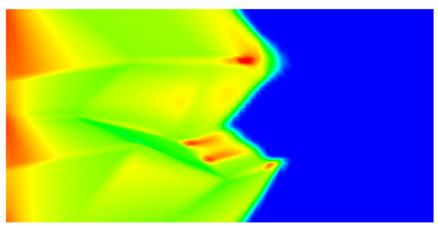
pressure

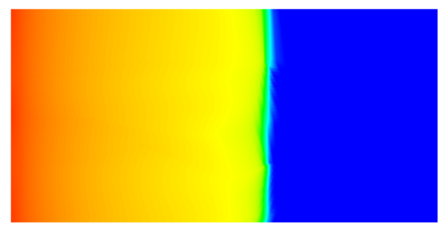


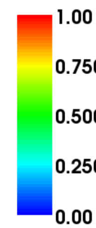




saturation







- **TPFA method**: convergence—completely lost; solution—worse
- **MPFA L-method**: work well; convergence rate—reduced

Conclusion

- TPFA method: inconsistent scheme in the case of non-$\mathbf{K}$-orthogonal grid
- Compared to MPFA O-method, MPFA L-method:
  - ⋆ its superconvergence behavior more depends on the Dirichlet boundary discretization and the shape of the grid
  - ⋆ fewer flux stencils
  - ⋆ a larger domain of convergence
  - ⋆ a larger domain of monotonicity
- On a $\mathbf{K}$-orthogonal grid, TPFA $=$ MPFA O $=$ MPFA L
- For some cases, MPFA O-method is good enough, while for some special cases, MPFA L-method is superior.
- For multi-phase flow, it is important to choose a suitable discretization method for pressure equation$\Longrightarrow \mathbf{v}_t \Longrightarrow$ transport, saturation, concentration equations

Many thanks to

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Thank you for your attention!