Mixed finite volume method for anisotropic problems

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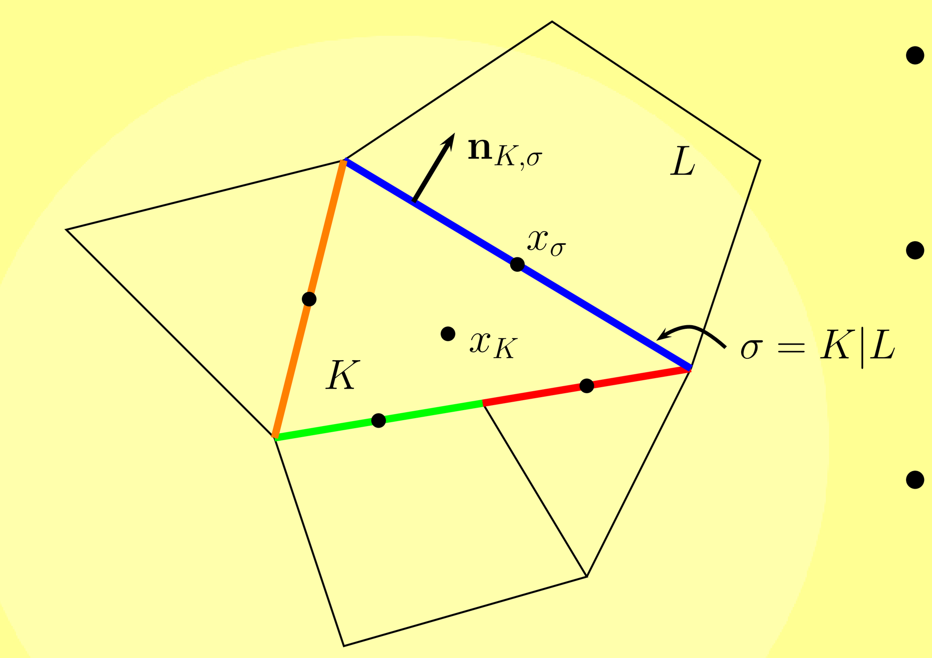
Presentation of the scheme

- **Problem :** $-\mathrm{div}(\mathbb{K}\nabla u) = f$ with Dirichlet-Neumann boundary conditions
- **Discrete unknowns :**



- Approximation of u and of its gradient :

$$u_K \approx u_{|K} \text{ and } \mathbf{v}_K \simeq \nabla u_{|K}$$

- Approximation of the fluxes on the edges :

$$F_{K,\sigma} \simeq \int_\sigma \mathbb{K}\nabla u \cdot \mathbf{n}_{K,\sigma}\,\mathrm{d}s$$

- Approximation of u on the edges :

$$u_\sigma \simeq u(x_\sigma)$$

- **Scheme :** ($h > 0$: size of the mesh ; $r_{K,\sigma} = \sum_{\sigma' \in \mathcal{E}_K} \alpha_{K,\sigma,\sigma'} F_{K,\sigma'}$: penalization term)

$$\mathbf{v}_K \cdot (x_\sigma - x_K) + r_{K,\sigma} = u_\sigma - u_K, \quad \forall K\,,\ \forall \sigma \text{ edge of } K$$
$$F_{K,\sigma} + F_{L,\sigma} = 0, \quad \forall \sigma = K|L \text{ interior edge}$$
$$u_\sigma = 0 \text{ (Dirichlet B.C.) or } F_{K,\sigma} = 0 \text{ (Neumann B.C.)}, \quad \forall \sigma \text{ exterior edge}$$
$$\left(\int_K \mathbb{K}(x)\mathrm{d}x\right)\mathbf{v}_K = \sum_{\sigma\in\mathcal{E}_K} F_{K,\sigma}(x_\sigma - x_K), \quad \forall K$$
$$-\sum_{\sigma\in\mathcal{E}_K} F_{K,\sigma} = \int_K f(x)\mathrm{d}x, \quad \forall K$$

Practical implementation

- **Hybridization** :
  - Elimination of the $(\mathbf{v}_K)$
  - Rewriting the $(F_{K,\sigma})$ and the (u_K) in fonction of the (u_σ) :

$$\begin{pmatrix} B_K & (1)_{\sigma\in\mathcal{E}_K} \\ (1)^T_{\sigma\in\mathcal{E}_K} & 0 \end{pmatrix} \begin{pmatrix} (F_{K,\sigma})_{\sigma\in\mathcal{E}_K} \\ u_K \end{pmatrix} = \begin{pmatrix} (u_\sigma)_{\sigma\in\mathcal{E}_K} \\ -\int_K f(x)\mathrm{d}x \end{pmatrix}$$

where B_K is defined by :

$$(B_K)_{\sigma,\sigma'} = \left(\int_K \mathbb{K}(x)\mathrm{d}x\right)^{-1}(x_{\sigma'} - x_K)\cdot(x_\sigma - x_K) + \alpha_{K,\sigma,\sigma'}$$

  - $\Longrightarrow$ **Linear system of size** $\mathrm{Card}(\mathcal{E}_{\mathrm{int}})$ **on** $(u_\sigma)_{\sigma\in\mathcal{E}_{\mathrm{int}}}$
- **Two separate implementations** for the benchmark : FORTRAN and MATLAB
- **Recursive elimination of the unknowns**
- **Penalization :**
  - no penalization on triangular meshes ($r_{K,\sigma} = 0$)
  - "strong penalization" : $r_{K,\sigma} = 6.\ 10^{-3}\,\dfrac{\mathrm{diam}(K)}{\mathrm{m}(\sigma)} F_{K,\sigma}$ or $r_{K,\sigma} = 10^{-7} F_{K,\sigma}$
  - "weak" penalization, which vanishes at order 1 :

$$r_{K,\sigma} = \sum_{\sigma'\in\mathcal{E}_K} l_{K,\sigma,\sigma'}\left(\frac{F_{K,\sigma'}}{\mathrm{m}(\sigma')} - \left(\int_K \mathbb{K}(x)\mathrm{d}x\right)\mathbf{v_K}\cdot\mathbf{n}_{K,\sigma'}\right)$$

Comments on the numerical results of the benchmark

- **General remarks**
  - Convergence of order 2 for the L^2-norm of the function
  - Convergence of order 1 for the L^2-norm of the gradient
  - Adaptability for the discretization of other kind of equations or coupled systems
  - Usable method for 3D simulations
- **Difference between strong and weak penalizations**
  - Perturbed parallelograms [Test 8, `mesh8`] (exact min=0, exact sumflux=0).

Strong penalization	*Weak penalization*
min=-8.08E-03, sumflux=8.46E-11	min=-2.86E-01, sumflux=2.82E-15

- Vertical fault on non-conforming rectangular mesh [Test 4, `mesh5`].

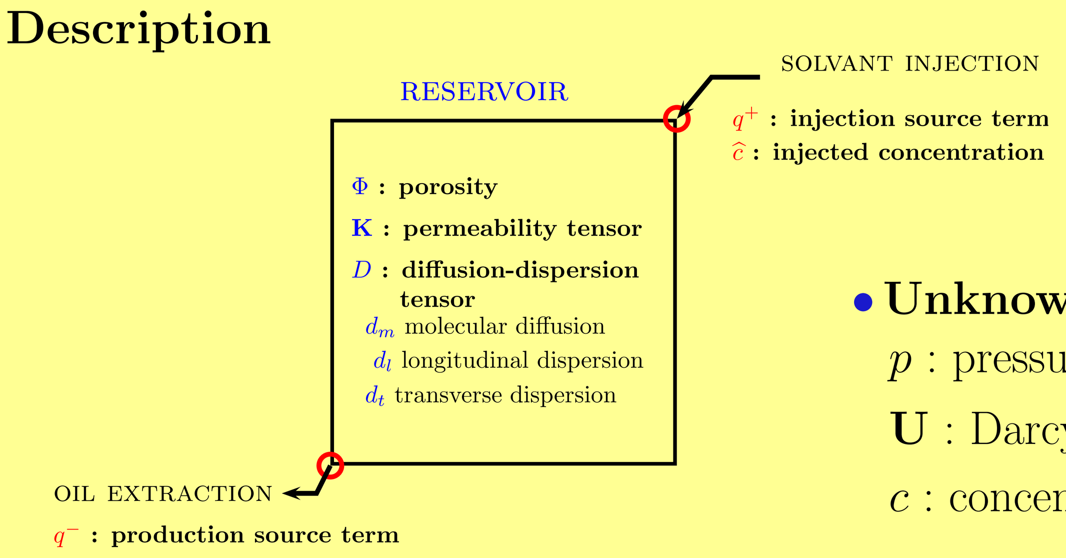
	flux0	fluy1	ener1	ener2	Δener
Strong	-4.40E+01	1.72E+00	4.99E+01	4.99E+01	4.21E-05
Weak	-2.97E+01	1.06E-03	2.18E+01	3.12E+01	3.00E-01
Reference (`mesh5_ref`)	-4.21E+01	8.33E-04	4.32E+01	4.32E+01	1.88E-05

- Oblique drain on oblique non-conforming rectangular mesh, affine exact solution [Test 6, `mesh7`].

	erl2	erflux0	erflum
Strong	6.66E-09	2.43E-05	2.91E-03
Weak	3.38E-15	9.85E-15	3.68E-12

Miscible displacement in porous media

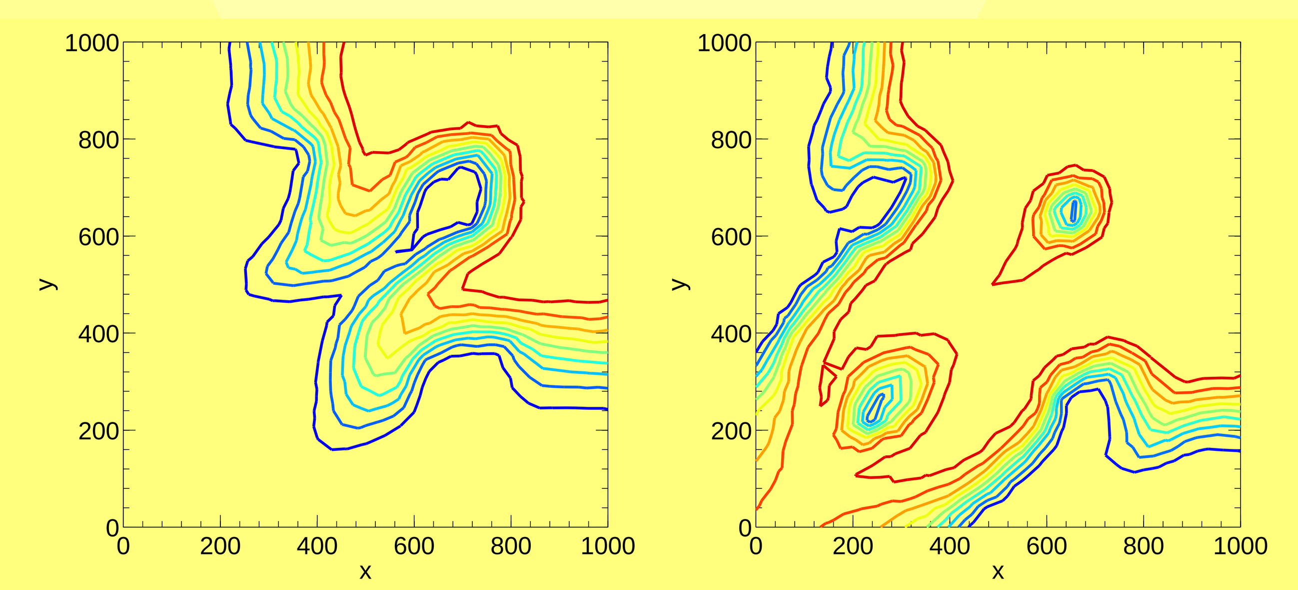
- **Description**



- **Unknowns**
  - p : pressure in the mixture
  - $\mathbf{U}$: Darcy velocity
  - c : concentration of the invading fluid
- **Equations**

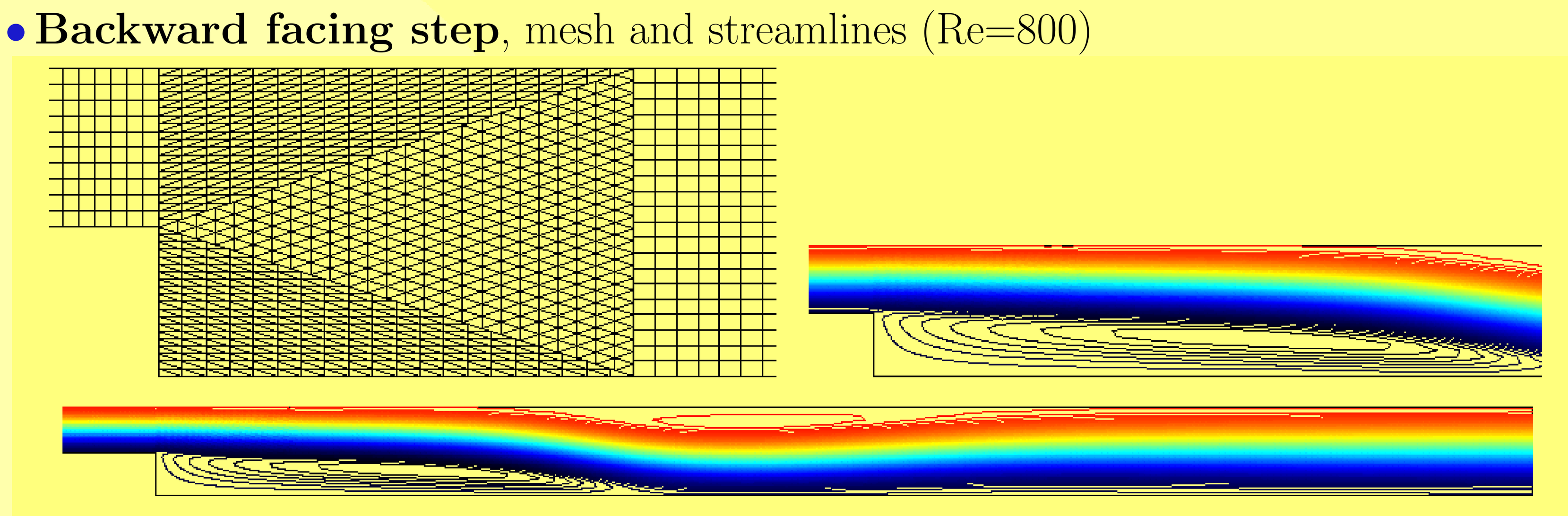
$$\mathrm{div}(\mathbf{U}) = q^+ - q^-, \quad \mathbf{U} = -\frac{\mathbf{K}(x)}{\mu(c)}\nabla p$$
$$\Phi(x)\partial_t c - \mathrm{div}(D(x,\mathbf{U})\nabla c - c\mathbf{U}) + q^- c = q^+\widehat{c}$$
$$D(x,\mathbf{U}) = \Phi(x)\Big(d_m\mathbf{I} + |\mathbf{U}|\Big(d_l E(\mathbf{U}) + d_t(\mathbf{I} - E(\mathbf{U}))\Big)\Big), \quad E(\mathbf{U}) = \left(\frac{\mathbf{U}_i\mathbf{U}_j}{|\mathbf{U}|^2}\right)_{1\le i,j\le d}$$

- **Numerical experiments**, $t = 3$ years and $t = 10$ years

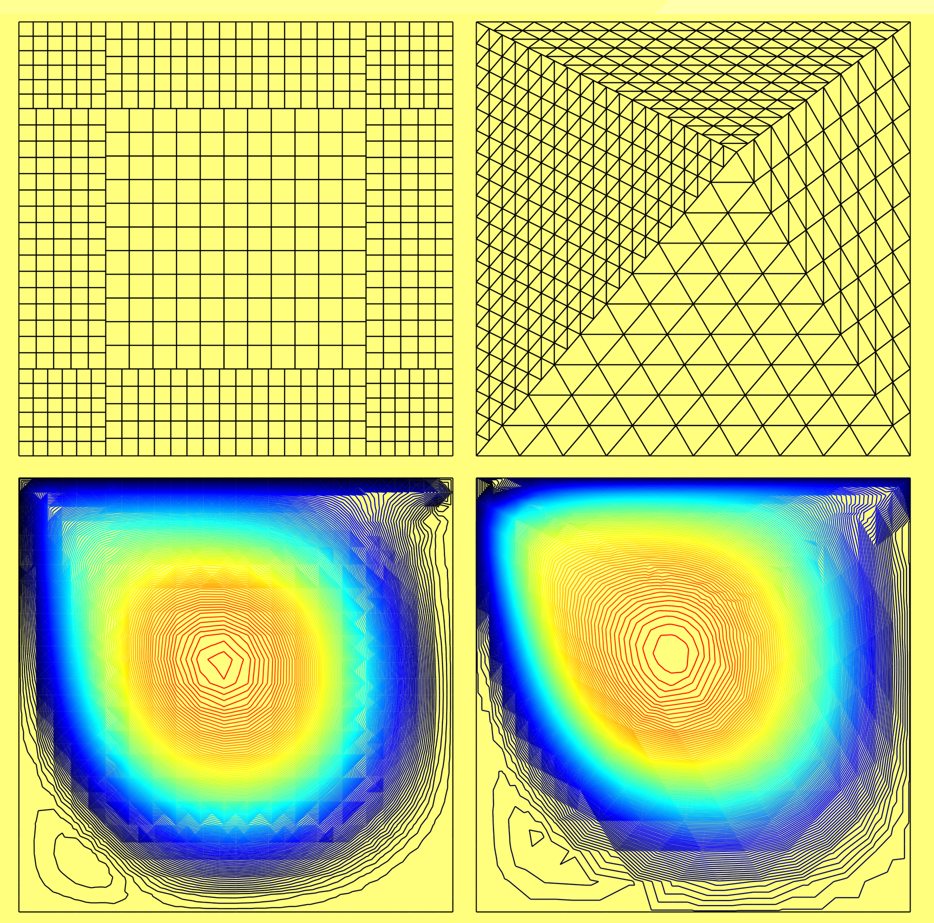


Navier-Stokes equations

- **Description** : incompressible Navier-Stokes equations without temperature
- **Backward facing step**, mesh and streamlines (Re=800)



- **Lid driven cavity**, meshes and streamlines (Re=1000)



Remark : proofs of convergence for all these implementations.