A Collocated Finite Volume Scheme for the Incompressible Navier-Stokes Equations on General Non-Matching Grids

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1. Objectives of the work

Actual problems

- realistic domains are not shoe boxes,
- no general tools for the meshing of complex 3D domains using Voronoï or Delaunay tessellations,
- generalized hexahedric meshes often used,
- full stress tensors (compressible flows),
- mesh refinement for boundary layers $\rightsquigarrow$ non matching grids.

Drawbacks of the usual schemes for diffusion on general meshes

- non-local stencils,
- cell-centred or face-centred unknowns,
- energy balances not respected,
- no theoretical convergence property,
- no accuracy on some particular grids,
- no "M-matrix".

A new scheme for the Navier-Stokes equations on general meshes

- collocated cell-centred scheme with local stencil,
- s.d.p. matrices for the diffusion operator,
- mathematical convergence properties,
- numerical preservation of the maximum principle in 3D,
- discrete kinetic and energy balances,
- local pressure stabilization.

2. Continuous formulation

Strong formulation

$$-\mathrm{Pr}\Delta\boldsymbol{u}+\boldsymbol{\nabla}p+(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}-\mathrm{Ra}\,\mathrm{Pr}\,T\boldsymbol{e}_3=\boldsymbol{f}(\boldsymbol{x}) \text{ in } \Omega$$

$$-\Delta T+(\boldsymbol{u}\cdot\boldsymbol{\nabla})T=g(\boldsymbol{x}) \text{ in } \Omega$$

$$\mathrm{div}\boldsymbol{u}=0 \text{ in } \Omega$$

$$\begin{cases} \boldsymbol{u}(\boldsymbol{x})=\boldsymbol{0} & \boldsymbol{x}\in\partial\Omega \\ T(\boldsymbol{x})=T_b(\boldsymbol{x}) & \boldsymbol{x}\in\partial\Omega_1 \\ -\boldsymbol{\nabla}T(\boldsymbol{x})\cdot\boldsymbol{n}(\boldsymbol{x})=q_b(\boldsymbol{x}) & \boldsymbol{x}\in\partial\Omega_2 \end{cases}$$

Weak formulation

find $\boldsymbol{u}\in H_0^1(\Omega)^d$, $p\in L^2(\Omega)$ *with* $\int_\Omega p(\boldsymbol{x})\mathrm{d}\boldsymbol{x}=0$, *and* T *with* $T-T_b\in H^1_{\partial\Omega_1,0}(\Omega)$, *such that*

$$\mathrm{Pr}\int_\Omega \boldsymbol{\nabla}\boldsymbol{u}:\boldsymbol{\nabla}\boldsymbol{v}\mathrm{d}\boldsymbol{x}-\int_\Omega p\,\mathrm{div}\boldsymbol{v}\mathrm{d}\boldsymbol{x}+\int_\Omega \mathrm{div}(\boldsymbol{u}\otimes\boldsymbol{u})\cdot\boldsymbol{v}\mathrm{d}\boldsymbol{x} -\mathrm{Ra}\,\mathrm{Pr}\int_\Omega T\boldsymbol{e}_3\cdot\boldsymbol{v}\mathrm{d}\boldsymbol{x}=\int_\Omega \boldsymbol{f}(\boldsymbol{x})\cdot\boldsymbol{v}\mathrm{d}\boldsymbol{x}$$

$$\forall\boldsymbol{v}\in H_0^1(\Omega)^d$$

$$\int_\Omega \boldsymbol{\nabla}T\cdot\boldsymbol{\nabla}\theta\mathrm{d}\boldsymbol{x}+\int_\Omega \mathrm{div}(\boldsymbol{u}T)\theta\mathrm{d}\boldsymbol{x} =\int_\Omega g(\boldsymbol{x})\theta\mathrm{d}\boldsymbol{x}-\int_{\partial\Omega_2} q_b(\boldsymbol{x})\mathrm{trace}_{\partial\Omega_2}\theta(\boldsymbol{x})\mathrm{d}\boldsymbol{x}$$

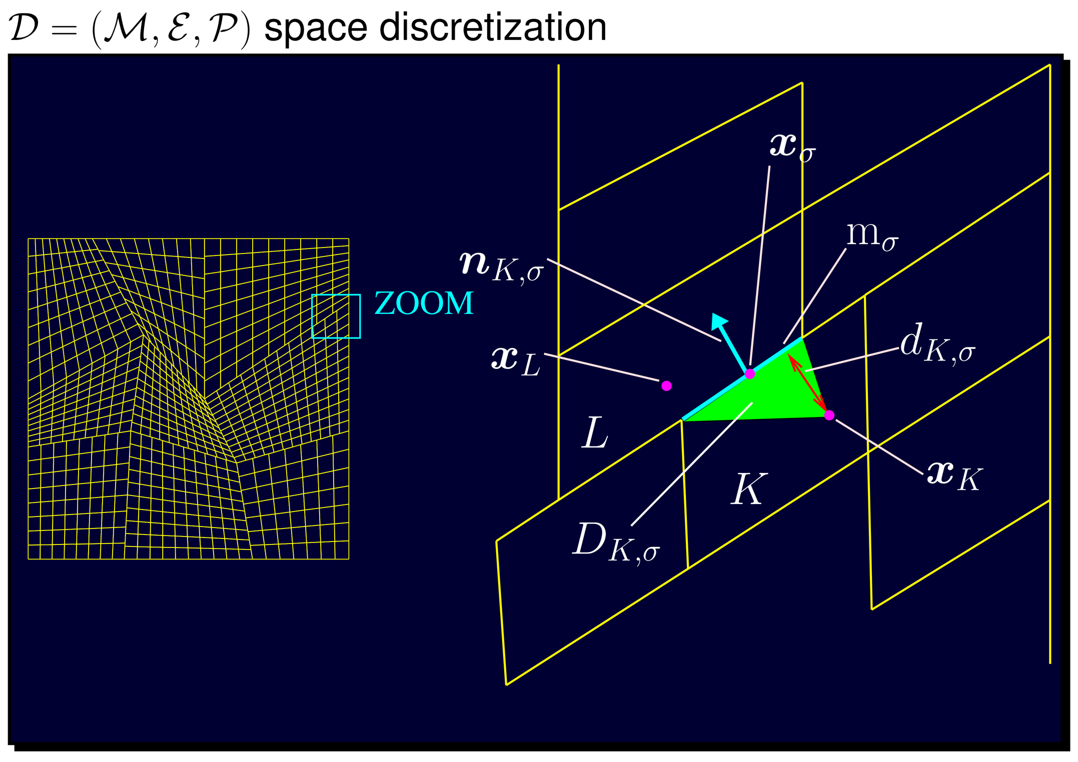
$$\forall\theta\in H^1_{\partial\Omega_1,0}(\Omega)$$

$$\mathrm{div}\boldsymbol{u}(\boldsymbol{x})=0 \text{ for a.e. } \boldsymbol{x}\in\Omega$$

3. Discrete scheme

Basic notations

$\mathcal{D}=(\mathcal{M},\mathcal{E},\mathcal{P})$ space discretization



$K\in\mathcal{M}$, $\sigma\in\mathcal{E}$, $\boldsymbol{x}_K\in\mathcal{P}$

interior face: $\sigma\in\mathcal{E}_{\rm int}$

$\mathcal{M}_\sigma=\{K,L\}$, $\boldsymbol{x}_\sigma=\sum_K a_\sigma^K\boldsymbol{x}_K$, $\Pi_\sigma(u)=\sum_K a_\sigma^K u_K$

boundary face: $\sigma\in\mathcal{E}_{\rm ext}$

$\mathcal{M}_\sigma=\{K\}$

Discretization of viscous terms

$$\boldsymbol{\nabla}_K u=\frac{1}{\mathrm{m}_K}\sum_{\sigma\in\mathcal{E}_K}\mathrm{m}_\sigma(u_\sigma-u_K)\boldsymbol{n}_{K,\sigma}$$

$$R_{K,\sigma}u=\frac{\sqrt{d}}{d_{K,\sigma}}\left(u_\sigma-u_K-\boldsymbol{\nabla}_K u\cdot(\boldsymbol{x}_\sigma-\boldsymbol{x}_K)\right)$$

$$\boldsymbol{\nabla}_{K,\sigma}u=\boldsymbol{\nabla}_K u+R_{K,\sigma}u\,\boldsymbol{n}_{K,\sigma}$$

$$\boldsymbol{\nabla}_{\mathcal{D}}u(\boldsymbol{x})=\boldsymbol{\nabla}_{K,\sigma}u, \text{ for a.e. } \boldsymbol{x}\in D_{K,\sigma},\ \forall K\in\mathcal{M},\ \forall\sigma\in\mathcal{E}_K$$

Pressure-velocity coupling, mass balance and convective contributions

$$\mathrm{div}_K\boldsymbol{u}=\frac{1}{\mathrm{m}_K}\sum_{\sigma\in\mathcal{E}_K}\mathrm{m}_\sigma\boldsymbol{u}_\sigma\cdot\boldsymbol{n}_{K,\sigma},\qquad \forall K\in\mathcal{M}$$

$$\mathrm{div}_{\mathcal{D}}\boldsymbol{u}(\boldsymbol{x})=\mathrm{div}_K\boldsymbol{u}, \text{ for a.e. } \boldsymbol{x}\in K,\ \forall K\in\mathcal{M}$$

$$\Phi^\lambda_{K,\sigma}(\boldsymbol{u},p)=\mathrm{m}_\sigma\left(\boldsymbol{u}_\sigma\cdot\boldsymbol{n}_{K,\sigma}+\lambda_\sigma(p_K-p_L)\right),\ \mathcal{M}_\sigma=\{K,L\}$$

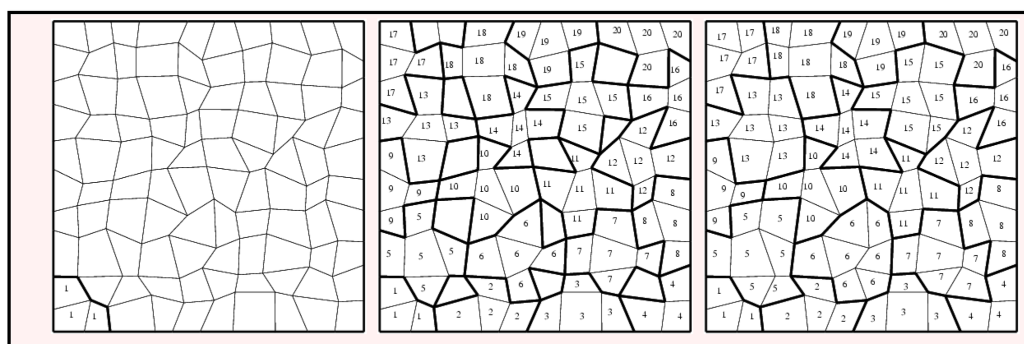
$$\mathrm{div}^\lambda_K(v,\boldsymbol{u},p)=\frac{1}{\mathrm{m}_K}\sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_{\rm int},\mathcal{M}_\sigma=\{K,L\}}\Phi^\lambda_{K,\sigma}(\boldsymbol{u},p)\frac{v_K+v_L}{2}$$

$$\mathrm{div}^\lambda_{\mathcal{D}}(v,\boldsymbol{u},p)(\boldsymbol{x})=\mathrm{div}^\lambda_K(v,\boldsymbol{u},p), \text{ for a.e. } \boldsymbol{x}\in K,\ \forall K\in\mathcal{M}$$

Choice for the parameters $(\lambda_\sigma)_{\sigma\in\mathcal{E}_{\rm int}}$

$\mathcal{G}$ partition of $\mathcal{M}$,

$\lambda_\sigma=\lambda>0$ if $\exists G\in\mathcal{G}$ with $\mathcal{M}_\sigma\subset G$ otherwise $\lambda_\sigma=0$.



Example of the construction of a partition $\mathcal{G}$ of $\mathcal{M}$.

Resulting discrete equations

- $X^{\mathcal{D}}=\left\{u\in\mathbb{R}^{\mathcal{M}}\times\mathbb{R}^{\mathcal{E}},\ \forall\sigma\in\mathcal{E}_{\rm int}, u_\sigma=\Pi_\sigma(u)\right\}$
- $X_0^{\mathcal{D}}=\left\{u\in X^{\mathcal{D}},\ \forall\sigma\in\mathcal{E}_{\rm ext}, u_\sigma=0\right\}$
- $X^{\mathcal{D}}_{\partial\Omega_1,0}=\left\{\theta\in X^{\mathcal{D}},\ \forall\sigma\in\mathcal{E}_{\rm ext}\cap\partial\Omega_1, \theta_\sigma=0\right\}$
- $\mathcal{H}_{\mathcal{M}}(\Omega)\subset L^2(\Omega)$ functions constant in each $K\in\mathcal{M}$
- $\mathcal{F}_{\mathcal{M}}:X^{\mathcal{D}}\to\mathcal{H}_{\mathcal{M}}(\Omega)$:
  $\forall u\in X^{\mathcal{D}}$, $\Big(\mathcal{F}_{\mathcal{M}}(u)\Big)(\boldsymbol{x})=u_K$ for a.e. $\boldsymbol{x}\in K$, $\forall K\in\mathcal{M}$
- $\mathcal{F}_{\mathcal{E}}:X^{\mathcal{D}}\to L^2(\partial\Omega)$:
  $\forall u\in X^{\mathcal{D}}$, $\Big(\mathcal{F}_{\mathcal{E}}(u)\Big)(\boldsymbol{x})=u_\sigma$ for a.e. $\boldsymbol{x}\in\sigma$, $\forall\sigma\in\mathcal{E}_{\rm ext}$

find $\boldsymbol{u}=(u^{(i)})_{i=1,d}\in(X_0^{\mathcal{D}})^d$, $p\in\mathcal{H}_{\mathcal{M}}(\Omega)$ *with* $\int_\Omega p(\boldsymbol{x})\mathrm{d}\boldsymbol{x}=\sum_{K\in\mathcal{M}}\mathrm{m}_K p_K=0$ *and* $T-T_{b,\mathcal{D}}\in X^{\mathcal{D}}_{\partial\Omega_1,0}$ *such that:*

$$\mathrm{Pr}\int_\Omega\boldsymbol{\nabla}_{\mathcal{D}}\boldsymbol{u}:\boldsymbol{\nabla}_{\mathcal{D}}\boldsymbol{v}\mathrm{d}\boldsymbol{x}-\int_\Omega p\,\mathrm{div}_{\mathcal{D}}\boldsymbol{v}\mathrm{d}\boldsymbol{x}+\int_\Omega\mathrm{div}^\lambda_{\mathcal{D}}(\boldsymbol{u},\boldsymbol{u},p)\cdot\mathcal{F}_{\mathcal{M}}(\boldsymbol{v})\mathrm{d}\boldsymbol{x}-\mathrm{Ra}\,\mathrm{Pr}\int_\Omega\mathcal{F}_{\mathcal{M}}(T)\boldsymbol{e}_3\cdot\mathcal{F}_{\mathcal{M}}(\boldsymbol{v})\mathrm{d}\boldsymbol{x}=\int_\Omega\boldsymbol{f}\cdot\mathcal{F}_{\mathcal{M}}(\boldsymbol{v})\mathrm{d}\boldsymbol{x},\ \forall\boldsymbol{v}\in(X_0^{\mathcal{D}})^d$$

$$\int_\Omega\boldsymbol{\nabla}_{\mathcal{D}}T\cdot\boldsymbol{\nabla}_{\mathcal{D}}\theta\mathrm{d}\boldsymbol{x}+\int_\Omega\mathrm{div}^\lambda_{\mathcal{D}}(T,\boldsymbol{u},p)\mathcal{F}_{\mathcal{M}}(\theta)\mathrm{d}\boldsymbol{x}=\int_\Omega g\mathcal{F}_{\mathcal{M}}(\theta)\mathrm{d}\boldsymbol{x}-\int_{\partial\Omega_2}q_b\mathcal{F}_{\mathcal{E}}(\theta)\mathrm{d}s,\ \forall\theta\in X^{\mathcal{D}}_{\partial\Omega_1,0}$$

$$\mathrm{div}^\lambda_{\mathcal{D}}(1,\boldsymbol{u},p)=0 \text{ a.e. in } \Omega$$

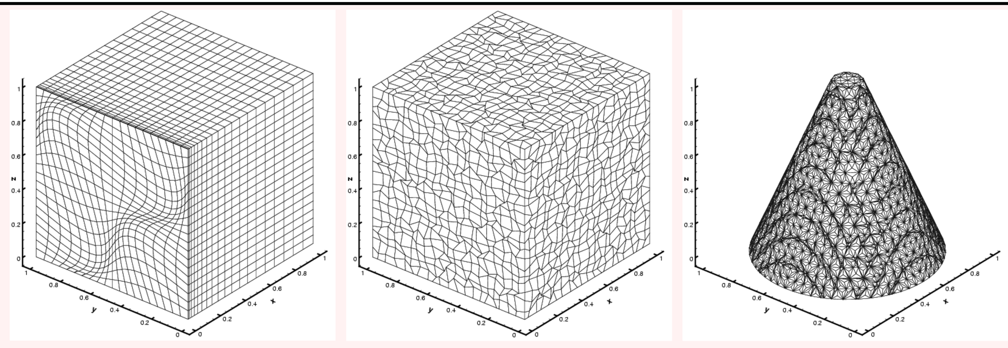
Mathematical properties

- existence of discrete solution,
- estimates on the kinetic energy,
- convergence of the scheme.

4. Numerical validation

Use of scalable linear solvers package *HYPRE* [1] with a preconditioning based on *Euclid*.

Geometry and meshes

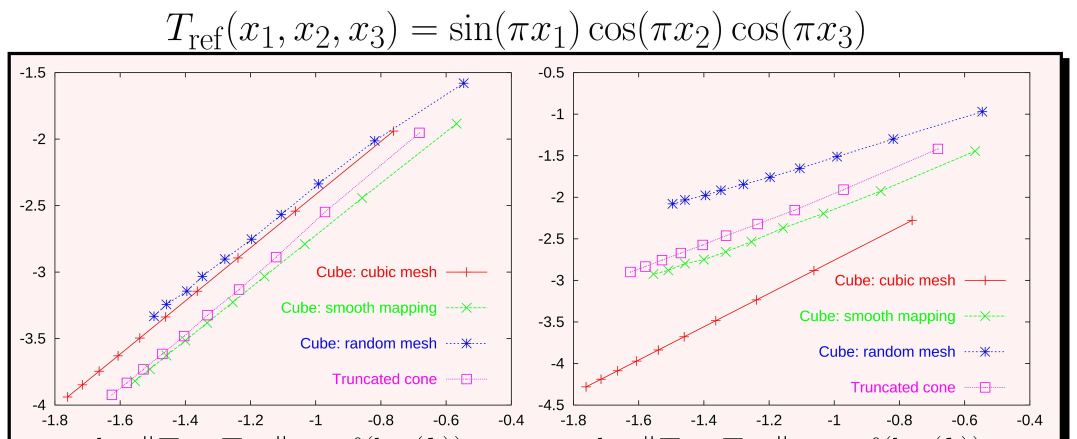


Smooth meshes Random meshes Truncated cone

n_i^3 (eventually distorted or cut) cubes in an unit 3D-box.

Case of an analytical solution in pure diffusion

$T_{\rm ref}(x_1,x_2,x_3)=\sin(\pi x_1)\cos(\pi x_2)\cos(\pi x_3)$



Solution accuracy for different norms, $h=\max_{K\in\mathcal{M}}h_K$ where h_K measuare the diameter of K.

Enclosure	Cubic			Truncated conic
Meshes	cubic	smooth	random	cube-based
L^2-norm	2 (2)	1.96 (1.97)	1.87 (1.93)	2.09 (2.03)
L^∞-norm	1.99 (2)	1.81 (1.68)	1.74 (1.88)	1.92 (1.58)
H^1-norm	2 (2)	1.50 (1.28)	1.16 (1.07)	1.55 (1.48)

Slopes of the linear approximations of convergence curves for $10\le n_i\le 100$ ($50\le n_i\le 100$).

Case of an isothermal Navier-Stokes analytical solution in cubic enclosure

$$\boldsymbol{u}_{\rm ref}(\boldsymbol{x})=\boldsymbol{\nabla}\wedge\sum_{i=1}^d(4x_1(x_1-1))^3(4x_2(x_2-1))^4(4x_3(x_3-1))^5\boldsymbol{e}_i,$$

$p_{\rm ref}(\boldsymbol{x})=\cos(\pi x_1)\cos(\pi x_2)\cos(\pi x_3)$ and $\mathrm{Pr}=1,\ \mathrm{Ra}=0.$

		Cubic	Smooth	Random
		meshes		
L^2	$u^{(1)}$	2 (2)	1.94 (1.99)	1.82 (1.95)
	$u^{(2)}$	1.99 (2)	1.96 (1.99)	1.84 (1.94)
	$u^{(3)}$	2 (2)	1.95 (1.98)	1.87 (1.94)
	p	2 (2)	1.09 (0.76)	0.85 (0.93)
L^∞	$u^{(1)}$	2 (2)	1.67 (2.02)	1.61 (1.84)
	$u^{(2)}$	1.75 (1.78)	1.39 (1.64)	1.66 (1.72)
	$u^{(3)}$	1.88 (1.74)	1.55 (1.59)	1.73 (1.86)
	p	1.72 (1.93)	−−	−−
H^1	$u^{(1)}$	1.98 (2)	1.80 (1.85)	1.43 (1.31)
	$u^{(2)}$	2 (2)	1.81 (1.84)	1.41 (1.27)
	$u^{(3)}$	1.95 (1.99)	1.76 (1.83)	1.40 (1.25)
	p	1.91 (1.97)	−−	−−

Slopes of the linear approximations of convergence curves for $10\le n_i\le 60$ ($40\le n_i\le 60$).

Natural convection problem in a unit and differentially heated cubic enclosure

Non-uniform cubic meshes

n_i	$e(u^{(1)})$	$e(u^{(2)})$	$e(u^{(3)})$	$e(\mathrm{Nu})$
20	-13%	-15%	-0.15%	0.23%
30	-3.1%	-5.1%	-0.91%	0.14%
40	-1.6%	-2.7%	-0.91%	0.11%
50	-0.95%	-1.4%	-0.11%	0.086%
60	-0.88%	-0.93%	-0.031%	0.065%

Random meshes

n_i	$e(u^{(1)})$	$e(u^{(2)})$	$e(u^{(3)})$	$e(\mathrm{Nu})$
20	30%	340%	13%	-2.0%
30	9.3%	230%	1.2%	-0.88%
40	4.3%	81%	1.5%	-0.51%

Relative differences to reference values[a] for $\|u^{(i)}\|_\infty$ ($i\in[1,d]$) and for the average Nusselt number $\mathrm{Nu}=\int_0^1\int_0^1(\boldsymbol{\nabla}T\cdot\mathbf{n})_{x_1=0}\,\mathrm{d}x_2\mathrm{d}x_3$ for $Ra=10^7$, $Pr=0.71$.

[a] $\|u^{(1)}\|_\infty\simeq 383.8357$, $\|u^{(2)}\|_\infty\simeq 83.3885$, $\|u^{(3)}\|_\infty\simeq 768.1393$, $\mathrm{Nu}\simeq 16.3427$ [2]

References

[1] HYPRE 2.0.0, Copyright (c) 2006 The Regents of the University of California. Produced at the Lawrence Livermore National Laboratory. Written by the HYPRE team. UCRL-CODE-222953. All rights reserved.

[2] E. Tric, G. Labrosse, M. Betrouni, A first incursion into the 3D structure of natural convection of air in a differentially heated cubic cavity, from accurate numerical solutions, Int. J. Heat Mass Transfer 43 (2000) 4043–4056.