Preconditioning Navier-Stokes problem discretized by a finite volume method.

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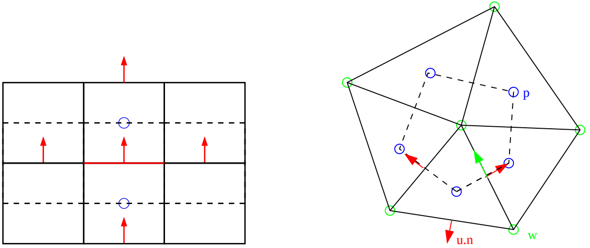
Resolution of elliptic problems on "arbitrary" meshes



In order to avoid orthogonality constraints, we add unknowns : the velocity $\mathbf{u}$ is defined on the edges with its two components, and the pressure p and the vorticity ω are defined both on the primal and dual cells.

PRIMAL AND DUAL MESHES

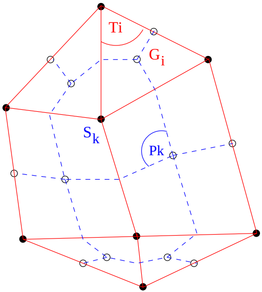


FIG.: A primal mesh and its dual mesh

Hypothesis : The primal boundary cells have only one edge on the boundary.

Diamond mesh

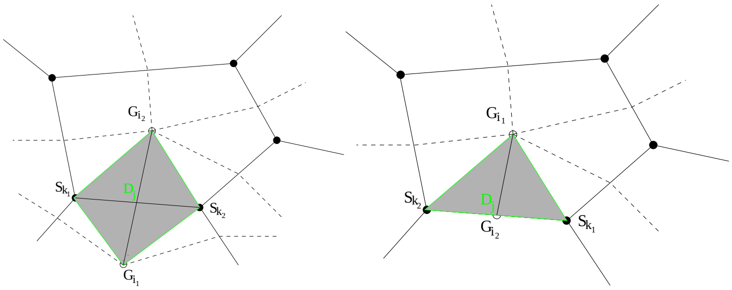
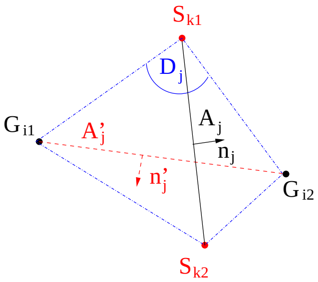


Fig.: Interior and boundary diamond cells

Construction discrete gradient operator

$$\langle \nabla p \rangle_{|D_j} = \frac{1}{|D_j|} \int_{D_j} \nabla p = \frac{1}{|D_j|} \int_{\partial D_j} p\, n$$



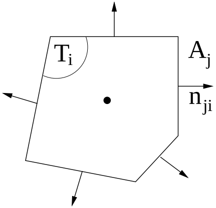
Using $\int_{[SG]} p \approx \ell_{SG} \frac{[p(S) + p(G)]}{2}$ and the relations in the triangle, we obtain *the definition* of the discrete gradient ∇_h^D on D_j :

$$(\nabla_h^D p)_j := \frac{1}{2\,|D_j|} \left\{ \left[p_{k_2}^P - p_{k_1}^P\right] |A'_j|\, \mathbf{n}'_j + \left[p_{i_2}^T - p_{i_1}^T\right] |A_j|\, \mathbf{n}_j \right\}$$

Construction of the discrete divergence operator

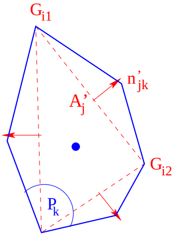
Definition of the discrete divergence on the primal cells :

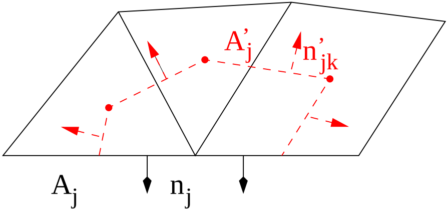
$$(\nabla_h^T \cdot \mathbf{u})_i := \frac{1}{|T_i|} \sum_{j \in \mathcal{V}(i)} |A_j| \, \mathbf{u}_j \cdot \mathbf{n}_{ji}$$



Definition of the discrete divergence on the interior dual cells :

$$(\nabla_h^P \cdot \mathbf{u})_k := \frac{1}{|P_k|} \sum_{j \in \mathcal{E}(k)} |A'_j| \, \mathbf{u}_j \cdot \mathbf{n}'_{jk}$$





Definition of the discrete divergence on the boundary dual cells :

$$(\nabla_h^P \cdot \mathbf{u})_k := \frac{1}{|P_k|}\left(\sum_{j\in\mathcal{E}(k)} |A'_j|\ \mathbf{u}_j \cdot \mathbf{n}'_{jk} + \sum_{j\in\mathcal{E}(k)\cap\partial\Omega} \frac{1}{2}|A_j|\ \mathbf{u}_j \cdot \mathbf{n}_j\right)$$

We have the following discrete Green formula :

$$-(\mathbf{u}, \boldsymbol{\nabla}_h^D p)_\Omega + (\mathbf{u}\cdot\mathbf{n}, p)_{\partial\Omega} = \frac{1}{2}\left[(\nabla_h^T \cdot\ \mathbf{u}, p^T)_\Omega + (\nabla_h^P \cdot\ \mathbf{u}, p^P)_\Omega\right]$$

Discrete curl operators

In the same way, we define a discrete vector curl operator (acting on a scalar) on the diamond cells :

$$(\boldsymbol{\nabla}_h^D \times \phi)_j := -\frac{1}{2\,|D_j|}\left\{\left[\phi_{k_2}^P - \phi_{k_1}^P\right]|A_j'|\,\boldsymbol{\tau}_j' + \left[\phi_{i_2}^T - \phi_{i_1}^T\right]|A_j|\,\boldsymbol{\tau}_j\right\}$$

and, a discrete scalar curl operator (acting on a vector) on the primal and dual cells :

$$(\nabla_h^T \times \mathbf{u})_i := \frac{1}{|T_i|}\sum_{j\in\mathcal{V}(i)} |A_j|\;\mathbf{u}_j \cdot \boldsymbol{\tau}_{ji}$$

$$(\nabla_h^P \times \mathbf{u})_k := \frac{1}{|P_k|}\sum_{j\in\mathcal{E}(k)} |A_j'|\;\mathbf{u}_j \cdot \boldsymbol{\tau}_{jk}'$$

$$(\nabla_h^P \times \mathbf{u})_k := \frac{1}{|P_k|}\left(\sum_{j\in\mathcal{E}(k)} |A_j'|\;\mathbf{u}_j \cdot \boldsymbol{\tau}_{jk}' + \sum_{j\in\mathcal{E}(k)\cap\partial\Omega} \frac{1}{2}|A_j|\;\mathbf{u}_j \cdot \boldsymbol{\tau}_j\right)$$

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Discretization

We are interested in the stationary Navier-Stokes problem :

$$-\nu\, \boldsymbol{\Delta} \mathbf{u} + \mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u} + \boldsymbol{\nabla} p = \mathbf{f},\ \ \nabla \cdot \mathbf{u} = 0 \text{ in } \Omega,$$

$$\mathbf{u} = 0 \text{ on } \Gamma, \quad \int_\Omega p = 0.$$

As $\mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u} = \boldsymbol{\nabla}(\frac{\mathbf{u}^2}{2}) + (\nabla \times \mathbf{u})\ \mathbf{u} \times \mathbf{e}_z$, using the "Bernoulli pressure" $\pi = p + \frac{\mathbf{u}^2}{2}$, we solve :

$$-\nu\, \boldsymbol{\Delta} \mathbf{u} + (\nabla \times \mathbf{u})\ \mathbf{u} \times \mathbf{e}_z + \boldsymbol{\nabla} \pi = \mathbf{f}, \quad \nabla \cdot \mathbf{u} = 0 \text{ in } \Omega,$$

$$\mathbf{u} = 0 \text{ on } \Gamma, \quad \int_\Omega \pi = 0.$$

Approximation of $(\nabla \times \mathbf{w})_{|D_j}$:

$$(\nabla\times\mathbf{w})_{|D_j} \approx \frac{(\nabla_h^T \times \mathbf{w})_{i_1} + (\nabla_h^T \times \mathbf{w})_{i_2} + (\nabla_h^P \times \mathbf{w})_{k_1} + (\nabla_h^P \times \mathbf{w})_{k_2}}{4}$$

For continuous operators, $-\boldsymbol{\Delta}\mathbf{u} = \boldsymbol{\nabla} \times \nabla \times \mathbf{u} - \boldsymbol{\nabla}\nabla \cdot \mathbf{u}$.

Unknowns : $(\mathbf{u}, \pi) = (\mathbf{u}_j, \pi_i^T, \pi_k^P)$

$$\begin{aligned}
\nu\left[(\boldsymbol{\nabla}_h^D \times \nabla_h^{T,P} \times \mathbf{u})_j - (\boldsymbol{\nabla}_h^D \nabla_h^{T,P} \cdot \mathbf{u})_j\right] & \\
+(\nabla \times \mathbf{w})_{|D_j} \mathbf{u}_j \times \mathbf{e}_z + (\boldsymbol{\nabla}_h^D \pi)_j &= \mathbf{f}_j^D, \quad \forall D_j \notin \Gamma \\
(\nabla_h^T \cdot \mathbf{u})_i &= 0, \quad \forall T_i \\
(\nabla_h^P \cdot \mathbf{u})_k &= 0, \quad \forall P_k \\
\mathbf{u}_j &= 0, \quad \forall D_j \in \Gamma \\
\sum_{i\in[1,I]} |T_i|\ \pi_i^T = \sum_{k\in[1,K]} |P_k|\ \pi_k^P &= 0
\end{aligned}$$

Existence and uniqueness of the solution $(\mathbf{u}_j, \pi_i^T, \pi_k^P)$:
we use the essential property

$$\mathbf{u}_j \times \mathbf{e}_z \cdot \mathbf{u}_j = 0.$$

We can deduce (p_i^T, p_k^P) computing : $p = \pi - \dfrac{\widetilde{\mathbf{u}}^2}{2}$, where $\widetilde{\mathbf{u}}$ is a quadrature formula defined on the primal and dual cells, according to the $\mathbf{u}_j$ defined on the diamond cells. At last, we project the (p_i^T, p_k^P) in order to vanish the mean-value.

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Preconditioning the linear system

(Work in collaboration with Delphine Jennequin)

We are led to solve the following saddle–point problem :

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ p \end{pmatrix} = \begin{pmatrix} \mathbf{F} \\ g \end{pmatrix}.$$

which is equivalent to the following system (Uzawa method) :

$$\begin{aligned} A\mathbf{u} + B^T p &= \mathbf{F} \\ -BA^{-1}B^T p &= g - BA^{-1}\mathbf{F} \end{aligned}$$

Preconditioning the Schur complement :

$$S = -BA^{-1}B^T.$$

Elman (1996) proposed (for the finite elements) to take :

$$S^{-1} \approx -(BB^T)^{-1}(BAB^T)(BB^T)^{-1}.$$

There exists also another formulation with weights :

$$S^{-1} \approx -(BM_2^{-1}B^T)^{-1}(BM_2^{-1}AM_1^{-1}B^T)(BM_1^{-1}B^T)^{-1},$$

where the possible choices of M_1 and M_2 can be diag$(A), X$...

Numerical illustration :

$$\Omega = [0,1]^2 \quad \text{and} \quad \mathbf{w} = \begin{pmatrix} 2(2y-1)(1-(2x-1)^2) \\ -2(2x-1)(1-(2y-1)^2) \end{pmatrix}.$$

Numerical illustration with $\nu = 1$ and $M_1 = M_2 = \text{diag}(A)$

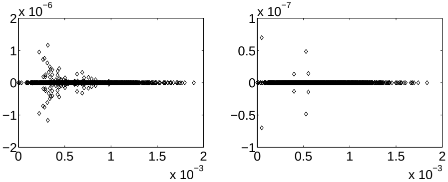

Fig.: Eigenvalues for the Schur complement and for the Elman preconditioner

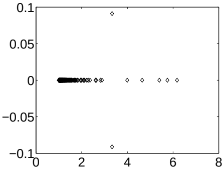

Fig.: Eigenvalues for the Schur complement preconditioned by the Elman preconditioner

Numerical illustration with $\nu = 0.01$ and $M_1 = M_2 = \text{diag}(A)$

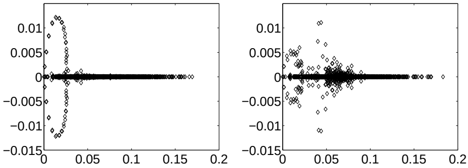

Fig.: Eigenvalues for the Schur complement and the Elman preconditioner

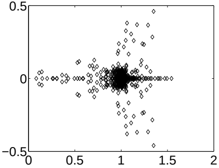

Fig.: Eigenvalues of the Schur complement preconditioned by the Elman preconditioner

The linear system is solved by a Uzawa method with a preconditioned Bicgstab such as the relative residual is lower down 10^{-8}.

h	Precond.	$\nu = 1$	$\nu = 10^{-1}$	$\nu = 10^{-2}$	$\nu = 10^{-3}$
0.0398	X	15	15	42	NC
	diag(A)	17	18	35	223
0.0212	X	19	22	73	NC
	diag(A)	32	34	40	175
0.01129	X	28	32	101	NC
	diag(A)	66	61	69	160

Tab.: Number of iterations according to the mesh step h and the viscosity ν with the Elman preconditioner.

Conclusion

- Applications to fluid dynamics problems with "arbitrary" meshes
- Preconditioners (Fortran 90, Sparskit2, PETSC)
- Strategy for higher Reynolds numbers
- Extension to the 3D-problems