0mm.

Stability in $L^{1,\infty}$ of **odd** *order transport schemes* *Consequences in CFD*

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Starting point

The transport equation is

$$\boxed{\partial_t u + a\partial_x u = 0, \qquad a > 0.}$$

- It is a fundamental equation in CFD (Computational Fluid Mechanics),
- but also for waves

$$\begin{cases} \partial_t v + \partial_x w = 0, \\ \partial_t w + \partial_x v = 0, \end{cases} \Longleftrightarrow \begin{cases} \partial_t (v+w) + \partial_x (v+w) = 0, \\ \partial_t (v-w) - \partial_x (v-w) = 0, \end{cases}$$

- or for particles $\partial_t f + v.\nabla_x f = Q(f)$, and kinetic equations.

Discrete case

For the transport equation it is evident that

$$||u(t)||_{L^p} = ||u(0)||_{L^p}, \qquad \forall p \in [1, \infty].$$

Is true at the discrete level for linear FV or DF schemes ?

- in L^2 : stability of numerical methods (arbitrary order) is usually proved in L^2 under CFL

$$||u_h^{n+1}||_{L^2} \leq ||u_h^n||_{L^2}$$

- in L^∞ : a classical obstruction result of Godunov states that the only linear schemes which satisfy the maximum principle are **order one**

$$||u_h^{n+1}||_{L^\infty} \leq ||u_h^n||_{L^\infty} \implies \text{First order.}$$

Standard numerical methods

The standard Finite Volume scheme for transport writes

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + a \frac{u^n_{i+\frac{1}{2}} - u^n_{i-\frac{1}{2}}}{\Delta x} = 0, \quad \nu = a\frac{\Delta t}{\Delta x}.$$

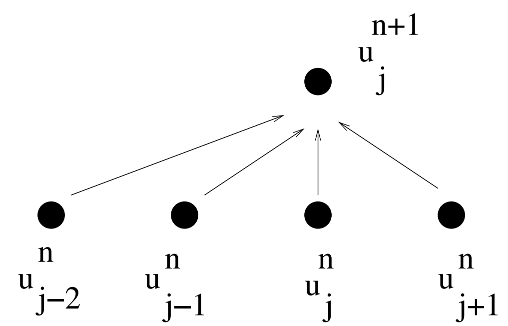
Flux	Formula	CFL (in L^2)
Upwind	$u_{i+\frac{1}{2}} = u_i$	$\nu \leq 1$
Lax-Wendroff	$u_{i+\frac{1}{2}} = u_i + \frac{1}{2}(1-\nu)(u_{i+1} - u_i)$	$\nu \leq 1$
Beam-Warming	$u_{i+\frac{1}{2}} = u_i + \frac{1}{2}(1-\nu)(u_i - u_{i-1})$	$\nu \leq 2$
O3	$u_{i+\frac{1}{2}} = \frac{2-\nu}{3} u_i^{LW} + \frac{1+\nu}{3} u_i^{BW}$	$\nu \leq 1$

The $p-k$ family

- We limit ourselves to explicit and compact schemes with a stencil of $p+1$ contiguous cells

$$u_j^{n+1} = \sum_{r=k-p}^{k} \alpha_r(\nu) u_{j+r}^n.$$

The order in space and time is p. The coefficients $\alpha_r(\nu)$ are unique. The O3 scheme $(p,k)=(3,1)$ is



- Upwind= (1,0). L-W=(2,1). B-W=(2,0).
- Explicit formulas exist for all (p,k) : see ADER (Toro and al, Munz and al), Delpino-Jourdren, ...

The Strang-Iserles theorem

Theorem 1. (Strang-Iserles) *The only pairs (p, k) for which the L^2 stability is true for all $\nu \leq 1$ are : $p = 2k + 1$, $p = 2k$ and $p = 2k + 2$.*

See the seminal works of Strang[1962] and Strang-Iserles[1983], see also Iserles-Norsett[1991] (Order stars theory) and Jeltsch-and-al[1989,1996].

Our new proof of the Strang-Iserles theorem is based on the following formula. Set $f_k(z) = (1+z)^{k+\nu}$. The amplification factor of the scheme with order p and stencil k is

$$\alpha(\theta) = \left[f_k(z) - \int_{t=0}^{1} \frac{(1-t)^p}{p!} f_k^{(p+1)}(tz)dt \right] e^{-k\mathbf{i}\theta}, \quad z = e^{\mathbf{i}\theta} - 1, \quad \theta \in [0, \pi$$

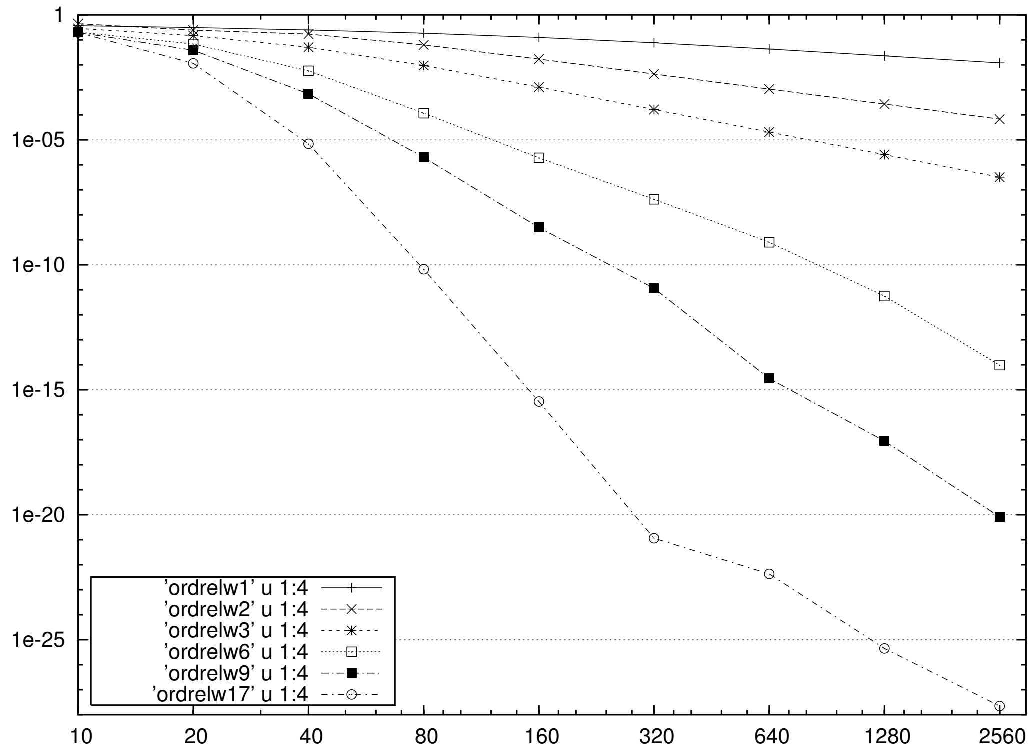
Our result is slightly more precise.

Theorem 2. *For $p = 2k + 2$, L^2 stability is true for all $\nu \leq 2$.*

• What is remarkable is the CFL which is independant of the order p of the scheme.

• At the same time all schemes are explicit ! ! !

Convergence for smooth functions



translation of $\sin(x)^8$

One has $2^{17} = 131072 \approx 10^5$. The theoretical order is reached.
Results with Havé-Delpino-Jourdren.

Stability in L^1 and L^∞

- Define $||u||_{L^d} = \left(\Delta x \sum |u_i|^d\right)^{\frac{1}{d}}$ for $1 \le d < \infty$ and $||u||_{L^\infty} = \sup |u_i|$. In practice $d = 1$, 2 or ∞ are the most interesting cases

Definition 1. Asymptotic-stability *We say a scheme is A-stable (asymptotic stability) in L^d if there exists a bound $K > 0$ which does not depend on Δx, and does not depend on ν (in the L^2 stability domain), such that*

$$||u^n||_{L^d} = ||A^n u^0||_{L^d} \le K||u^0||_{L^d}, \quad \forall n, \quad \forall u^0.$$

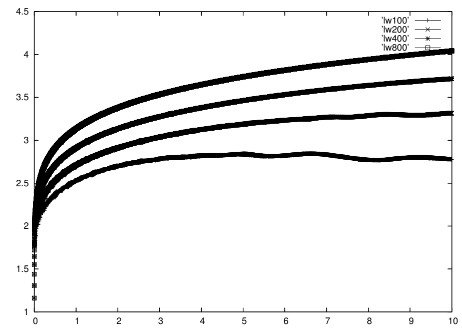
Set for convenience $\beta = \alpha e^{-\mathbf{i}\nu\theta}$

where $\alpha(\theta)$ is the amplification factor in Fourier. Then $|\beta| \leq 1$ for $\nu \leq 1$ (stability), and
$|\beta - 1| \leq C_1 \nu \theta^{p+1} + O(\theta^{p+2})$ (consistency). Set $q \in \mathbb{N}$ the order of dissipativity :
$|\beta| \leq 1 - \nu C_2 \theta^{q+1}$ for $0 \leq \theta \leq \pi$.

Flux	β	p	q
Up	$1 - \frac{\nu(1-\nu)}{2}\theta^2 + O(\theta^3)$	1	1
LW	$1 - \mathbf{i}\frac{\nu(1-\nu^2)}{6}\theta^3 - \frac{\nu^2(1-\nu^2)}{8}\theta^4 + O(\theta^5)$	2	3
BW	$1 - \mathbf{i}\frac{\nu(\nu-1)(\nu-2)}{6}\theta^3 - \frac{\nu(1-\nu)^2(2-\nu)}{8}\theta^4 + O(\theta^5)$	2	3
O3	$1 - \frac{\nu(1-\nu)(1+\nu)(2-\nu)}{2}4\theta^4 + O(\theta^5)$	3	3

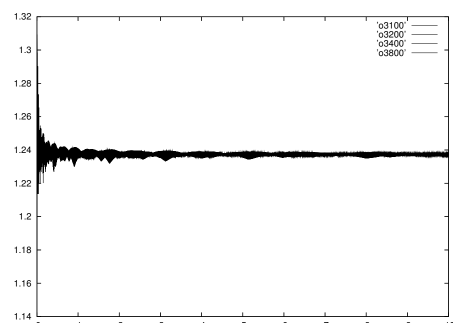
Theorem 3. *Assume $p = q$ and some technical conditions on the amplification factor are true ($C_2 > 0$), then the scheme is A-stable in L^1 and L^∞. (Numer. Math. 2008)*

Illustration : LW versus O3



$$||A^n_{\mathrm{LW}}||_{L^1} \approx n^{\frac{1}{10}}.$$

Non A-stability in L^1 of the Lax-Wendroff scheme. The norm increases with respect to T and $\frac{1}{\Delta x}$. Computations done with 100, 200, 400 and 800 cells on a periodic domain.



$$||A^n_{\mathrm{O3}}||_{L^1} \leq C \approx 1.3$$

A-stability in L^1 of the order-3 scheme. The norm is uniformly bounded with respect to T and $\frac{1}{\Delta x}$.

Convergence for BV solutions

Theorem 4. *Assume $p = 2k+1$ is odd and the scheme is A-stable in L^1. Assume $u_0 \in L^\infty \cap BV$*

$$||u^n - u(n\Delta t)||_{L^1} \le D_p |u_0|_{BV} \left(\Delta x^a T^b + \Delta x\right), \qquad a = \frac{p}{p+1},\ b = \frac{1}{p+1}.$$

- Very high order schemes are optimal for BV functions since

$$\frac{p}{p+1} \to 1.$$

- In L^2 one only gets for all p

$$||u^n - u(n\Delta t)||_{L^2} \le \left(C ||u||_{L^\infty}^{\frac12} |u|_{BV}^{\frac12}\right) \times \left(\Delta x^a T^b + \Delta x^{\frac12}\right)$$

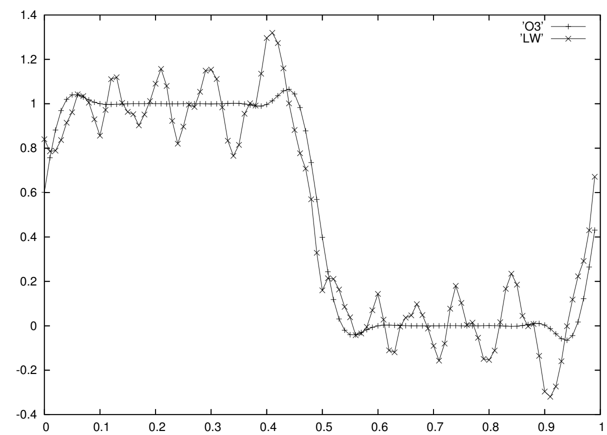
with $a = \frac{p}{2(p+1)} < \frac12$ and $b = \frac{1}{2(p+1)}$.

Numerical results

Oscillations and default of stability in L^1 are correlated

cells	LW (L^1)	LW (L^2)	O3 (L^1)	O3 (L^2)
100	0.136120	0.183949	0.040989	0.102097
800	0.056499	0.094048	0.008625	0.047275
order	$\frac{1}{3} < 0.42 < \frac{2}{3}$	$\frac{1}{3}$	$\frac{3}{4}$	$\frac{3}{8}$

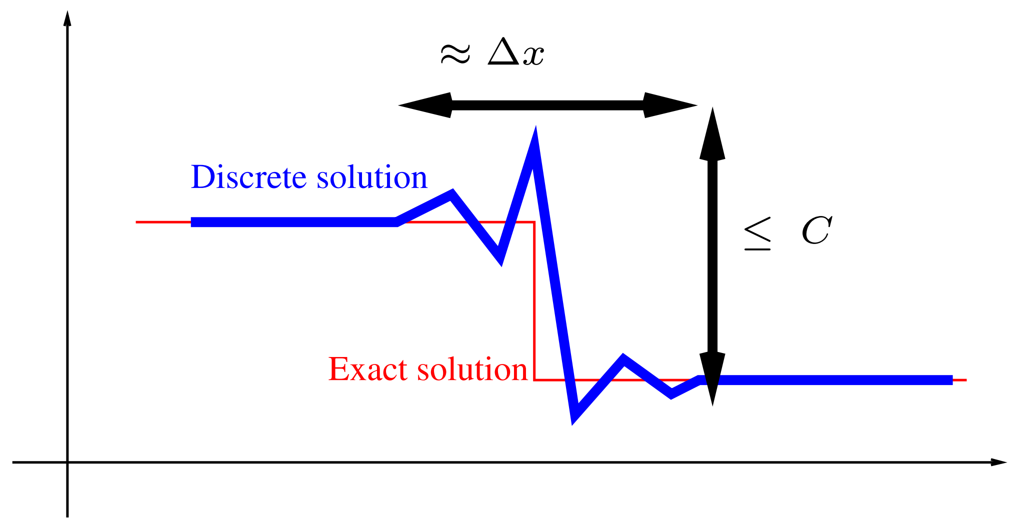
Errors and order of convergence in L^1 and L^2. CFL=0.001.



Huge oscillations for the Lax-Wendroff scheme for $\nu = 0.01$.
O3 is OK.

Control of the dispersion

- The error is $\approx \Delta x$ in L^1 for $p = 2k+1$ large
- The vertical oscillations are bounded



- The error in Fourier is controlled in L^∞ since

$$||\widehat{e}||_{L^\infty} \leq \frac{1}{2\pi} ||e||_{L^1}.$$

This is much stronger than the Parseval equality $||\widehat{e}||_{L^2} = ||e||_{L^2}$.
Therefore the structure of oscillations is controled.

Control of the dissipation

The error estimate is

$$||u^n - u(n\Delta t)||_{L^1} \le D_p |u_0|_{BV} \left(\Delta x^a T^b + \Delta x\right), \qquad a = \frac{p}{p+1}, \ b = \frac{1}{p+1}.$$

- Assume $p = 17$. Then the time dependance is $T^{\frac{1}{p+1}} = T^{\frac{1}{18}}$. That is the extra amount of dissipation is negligeable.
- The age of universe if

$$T_{\text{universe}} \approx 60 \times 60 \times 24 \times 365 \times 14 \times 10^9 \approx 4.41 \times 10^{17} s.$$

The dissipation time-scale of the scheme compares with the age of universe ! !

The Godunov obstruction theorem

- Assume the order is odd $p = 2k + 1$: then the scheme is A-stable in L^∞ (up to the verification of the technical conditions)

$$||u^n||_{L^\infty} \leq K_p ||u^0||_{L^\infty}, \quad \forall n.$$

  Assume

$$K_p \leq 1.$$

  Since the scheme is conservative, then it also preserves the maximum principle. Therefore

$$p = 1.$$

- And also

$$p = 2k + 1 > 1 \Longrightarrow K_p > 1.$$

- The oscillations are bounded for $p = 2k + 1 > 1$. In some sense, the new result bypasses the standard obstruction result of Godunov.

Linear waves

We solve the linear wave system

$$\begin{cases} \partial_t v + \partial_x w = 0, \\ \partial_t w + \partial_x v = 0, \end{cases}$$

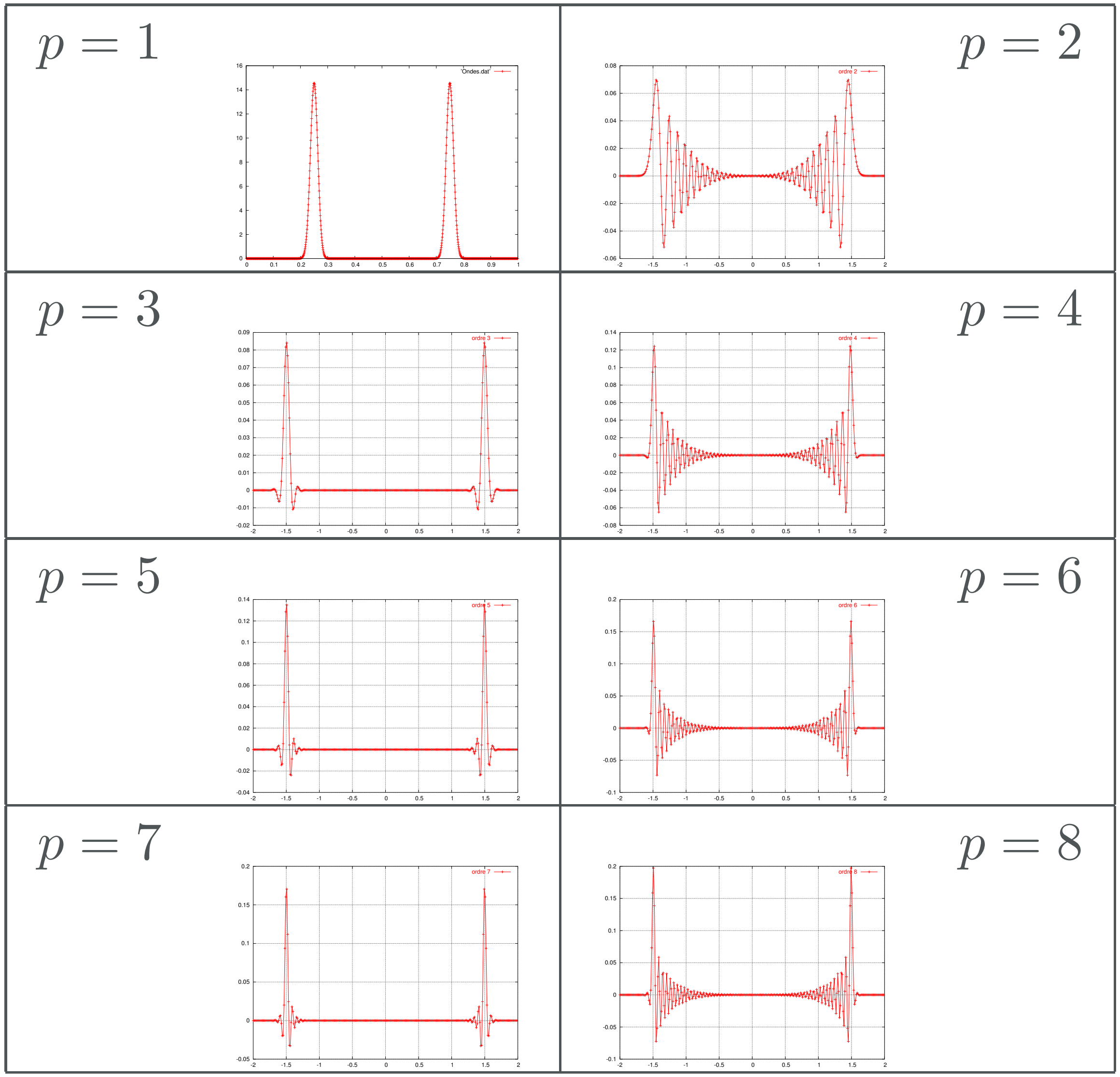
as 2 decoupled transport equations

$$\begin{cases} \partial_t u^+ + \partial_x u^+ = 0, & u^+ = v + w, \\ \partial_t u^- - \partial_x u^- = 0, & u^- = v - w. \end{cases}$$

This is archetypal of linearized Riemann solvers for hyperbolic systems of conservation laws.
For the initial data $v(0) = \delta(x)$, $w(0) = 0$, the exact solution is

$$v(t) = \frac{1}{2}\delta(x-t) + \frac{1}{2}\delta(x+t), \quad v(t) = \frac{1}{2}\delta(x-t) - \frac{1}{2}\delta(x+t).$$

Results


$p = 1$
'Ondes.dat'
$p = 2$
ordre 2
$p = 3$
ordre 3
$p = 4$
ordre 4
$p = 5$
ordre 5
$p = 6$
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$p = 7$
ordre 7
$p = 8$
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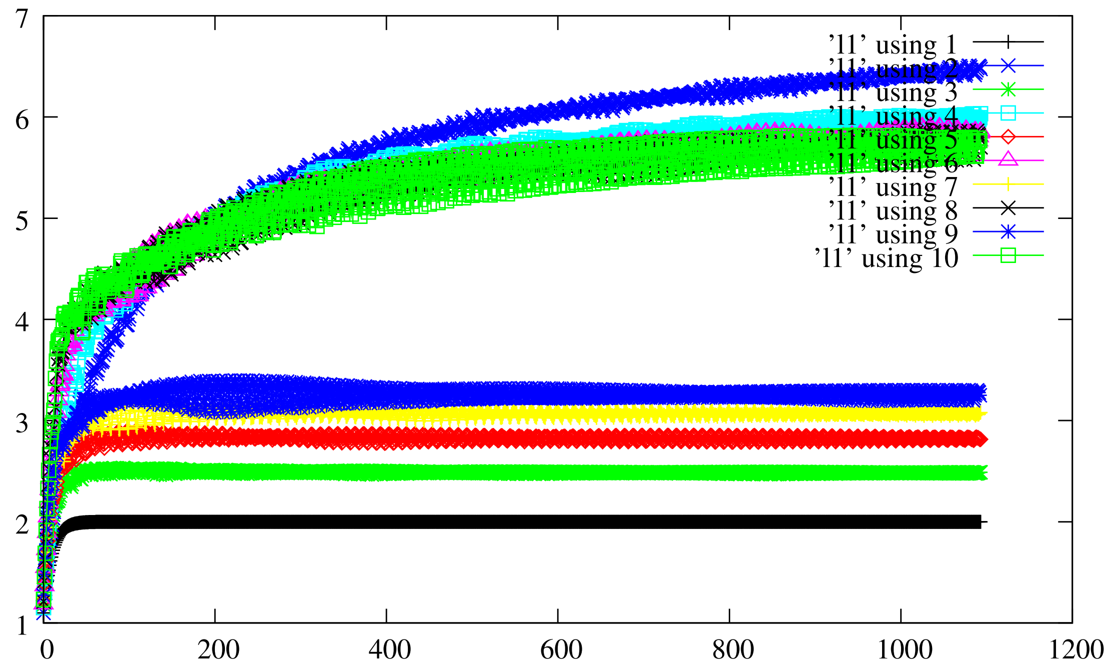
Results

The even order (2, 4, 6, . . .) fluxes are not A-stable in L^1.

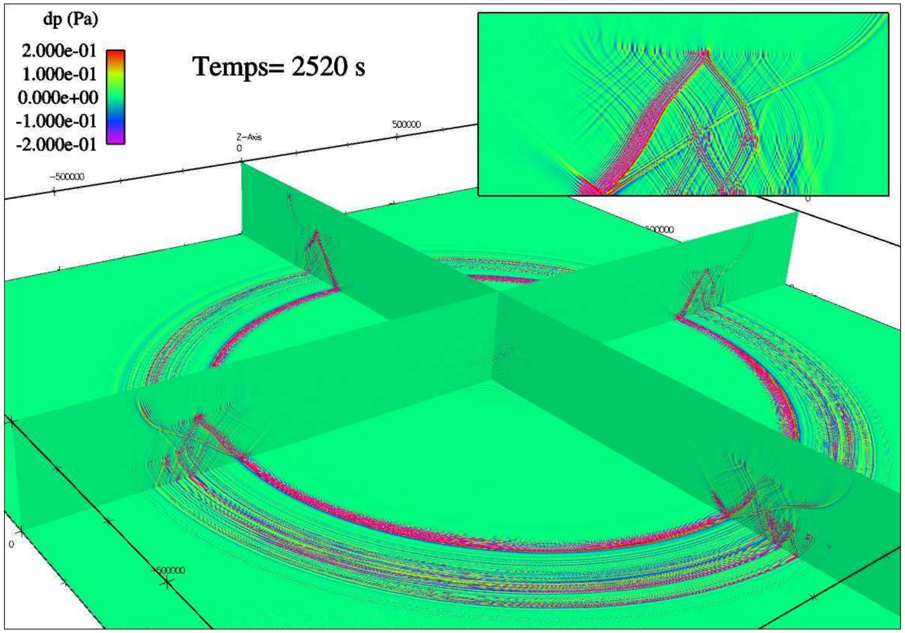


The odd order (1, 3, 5, . . .) fluxes are experimentally A-stable in L^1 : The technical conditions are true (a rigorous proof that also gives $p \mapsto K_p$ is in preparation).

3D : Dragster code

Solve linearized Euler equations

$$\begin{cases} \partial_t p & +a_0\partial_x p & +\rho_0 c_0^2(\partial_x u + \partial_y v) & = 0 \\ \rho_0\partial_t u & +a_0\partial_x u & +\partial_x p & = 0 \\ \rho_0\partial_t v & +a_0\partial_x v & +\partial_y p & = 0. \end{cases}$$



- High order FV + variable coefficients + x-y-z Splitting.
- Performance of 11.7 Tflops on the 4096 Itanium2 processors computer (the TERA10 computer at the CEA) for a 10^{10} cells structured mesh
- With P. Havé, S. Delpino an H. Jourdren.

Conclusion

Numerical analysis of such methods is at the beginning

- High order odd finite volume schemes have “just enough” dissipation. The main theoretical idea is to prove the stability in L^1 (and L^∞). It is possible for $p = q = 2k + 1$.
- It extends the standard obstruction result of Godunov.
- These schemes are accurate for the transport of pikes and Dirac functions.

Open problems

- Extend the theory in dimension > 1 for the wave equations, the Maxwell equations, . . ., Maxwell+particules (non linear RHS).
- Extend to Finite Volume in multiD (completely open).
- Extend 1D DG for transport (order 3, 5, . . .) : reasonnable.
- Adapt to convection-diffusion equations (?).
- Adapt these methods in non linear CFD codes (Kluth presentation on Monday).