Semi-implicit Diamond-cell Finite Volume Scheme for 3D Nonlinear Tensor Diffusion in Coherence Enhancing Image Filtering

Olga Drblíková, Karol Mikula

Department of Mathematics and Descriptive Geometry
Faculty of Civil Engineering
Slovak University of Technology in Bratislava

9th June 2008

Contents

Nonlinear tensor diffusion model

$$
\begin{aligned}
\frac{\partial u}{\partial t} - \nabla \cdot (D \nabla u) &= 0, \qquad \text{in } Q_T \equiv I \times \Omega, \\
u(x,0) &= u_0(x), \quad \text{in } \Omega, \\
(D \nabla u) \cdot \mathbf{n} &= 0, \qquad \text{on } I \times \partial\Omega.
\end{aligned}
$$

Motivation

- Diffusion tensor D steers the smoothing process such that the diffusion is:
    - strong in preferred directions, e.g. along 2D edge surfaces in 3D images,

Motivation

- Diffusion tensor D steers the smoothing process such that the diffusion is:
  - strong in preferred directions, e.g. along 2D edge surfaces in 3D images,
  - low in the perpendicular direction.

Motivation

- Diffusion tensor D steers the smoothing process such that the diffusion is:
  - strong in preferred directions, e.g. along 2D edge surfaces in 3D images,
  - low in the perpendicular direction.
- Using this type of diffusion one can achieve better edge detection.

Derivation of the diffusion tensor

1. $\nabla u_{\tilde{t}} = (u_{x_1}, u_{x_2}, u_{x_3})^T$,

Derivation of the diffusion tensor

1. $\nabla u_{\tilde{t}} = (u_{x_1}, u_{x_2}, u_{x_3})^T,$
2. $u_{\tilde{t}}(x,t) = (G_{\tilde{t}} * u(\cdot,t))(x), \qquad \tilde{t} > 0,$

Derivation of the diffusion tensor

1. $\nabla u_{\tilde{t}} = (u_{x_1}, u_{x_2}, u_{x_3})^T,$
2. $u_{\tilde{t}}(x,t) = (G_{\tilde{t}} * u(\cdot,t))(x), \qquad \tilde{t} > 0,$
3. $v_1 \parallel \nabla u_{\tilde{t}}, \qquad v_2 \perp \nabla u_{\tilde{t}},$

Derivation of the diffusion tensor

1. $\nabla u_{\tilde{t}} = (u_{x_1}, u_{x_2}, u_{x_3})^T,$
2. $u_{\tilde{t}}(x,t) = (G_{\tilde{t}} * u(\cdot,t))(x), \qquad \tilde{t} > 0,$
3. $v_1 \parallel \nabla u_{\tilde{t}}, \qquad v_2 \perp \nabla u_{\tilde{t}},$
4. $v_3 \perp \nabla u_{\tilde{t}}, \qquad v_2 \perp v_3\,.$

Derivation of the diffusion tensor

1. The direction of vector v_1 represents in every point a direction of the largest change in image intensity.

Derivation of the diffusion tensor

1. The direction of vector v_1 represents in every point a direction of the largest change in image intensity.
2. The other two vectors give a plane which may represent a 2D surface edge in 3D image. We call it a coherence plane $\mathcal{P}$.

Requirements for the diffusion tensor

- For enhancing coherence, the diffusion tensor D must steer a filtering process such that diffusion
  - is strong and increasing with the level of μ along the coherence plane, $\mu = ||\nabla u_{\tilde{t}}||^2$,

Requirements for the diffusion tensor

- For enhancing coherence, the diffusion tensor D must steer a filtering process such that diffusion
  - is strong and increasing with the level of μ along the coherence plane, $\mu = ||\nabla u_{\tilde{t}}||^2$,
  - is small in direction of the vector v_1.

Requirements for the diffusion tensor

- For enhancing coherence, the diffusion tensor D must steer a filtering process such that diffusion
  - is strong and increasing with the level of μ along the coherence plane, $\mu = ||\nabla u_{\tilde{t}}||^2$,
  - is small in direction of the vector v_1.

Derivation of the diffusion tensor

1. To that goal, we choose the eigenvalues of the diffusion tensor D by

$$
\begin{aligned}
\kappa_1 &= \alpha, \quad \alpha \in (0,1),\ \alpha \ll 1, \\
\kappa_2 &= \begin{cases} \alpha, & \text{if } \mu = 0, \\ \alpha + (1-\alpha)\exp\left(\frac{-C}{\mu}\right),\ C > 0 & \text{else} \end{cases}
\end{aligned}
$$

Derivation of the diffusion tensor

1. To that goal, we choose the eigenvalues of the diffusion tensor D by

$$\kappa_1 = \alpha, \quad \alpha \in (0,1),\ \alpha \ll 1,$$
$$\kappa_2 = \begin{cases} \alpha, & \text{if } \mu = 0, \\ \alpha + (1-\alpha)\exp\left(\frac{-C}{\mu}\right),\ C > 0 & \text{else} \end{cases}$$

2. and corresponding eigenspaces are given by v_1 and $\mathcal{P}$.

Diffusion tensor

$$D = G_\rho * D_0, \text{ where } D_0 = \begin{cases} B, & \text{if } \mu = 0, \\ PBP^{-1} & \text{else}, \end{cases} ,$$

$$B = \begin{pmatrix} \kappa_1 & 0 & 0 \\ 0 & \kappa_2 & 0 \\ 0 & 0 & \kappa_2 \end{pmatrix}$$

and P is a transition matrix from the basis (v_1, v_2, v_3) to (e_1, e_2, e_3).

Diffusion tensor

$$\frac{1}{\mu}\begin{pmatrix} u_{x_1}^2\kappa_1 + (u_{x_2}^2 + u_{x_3}^2)\kappa_2 & u_{x_1}u_{x_2}(\kappa_1 - \kappa_2) & u_{x_1}u_{x_3}(\kappa_1 - \kappa_2) \\ u_{x_1}u_{x_2}(\kappa_1 - \kappa_2) & u_{x_2}^2\kappa_1 + (u_{x_1}^2 + u_{x_3}^2)\kappa_2 & u_{x_2}u_{x_3}(\kappa_1 - \kappa_2) \\ u_{x_1}u_{x_3}(\kappa_1 - \kappa_2) & u_{x_2}u_{x_3}(\kappa_1 - \kappa_2) & u_{x_3}^2\kappa_1 + (u_{x_1}^2 + u_{x_2}^2)\kappa_2 \end{pmatrix}$$

Positive definitness of the diffusion tensor

$$0 < \alpha \leq z^T D_0(x) z \Leftrightarrow \sum_{i=1}^{3} \sum_{j=1}^{3} z_i z_j d_{ij}(x) \geq \alpha > 0.$$

$$z^T D(x) z = z^T (G_\rho * D_0(x)) z =$$

$$\sum_{i=1}^{3} \sum_{j=1}^{3} z_i z_j \int_{R^3} G_\rho(x-\xi) d_{ij}(\xi) d\xi =$$

$$\int_{R^3} G_\rho(x-\xi) \sum_{i=1}^{3} \sum_{j=1}^{3} z_i z_j d_{ij}(\xi) d\xi$$

$$\geq \alpha \int_{R^3} G_\rho(x-\xi) d\xi = \alpha > 0.$$

Derivation of the finite volume scheme

1. $\frac{\partial u}{\partial t} - \nabla \cdot (D \nabla u) = 0,$

Derivation of the finite volume scheme

1. $\frac{\partial u}{\partial t} - \nabla \cdot (D \nabla u) = 0,$
2. $\frac{u_K^n - u_K^{n-1}}{k} m(K) - \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_{int}} \int_\sigma (D^{n-1} \nabla u^n) \cdot \mathbf{n}_{K,\sigma} ds = 0,$

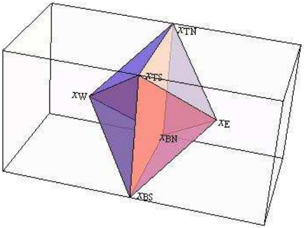
Derivation of the finite volume scheme

1. $\frac{\partial u}{\partial t} - \nabla \cdot (D \nabla u) = 0,$
2. $\frac{u_K^n - u_K^{n-1}}{k} m(K) - \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_{int}} \int_\sigma (D^{n-1} \nabla u^n) \cdot \mathbf{n}_{K,\sigma} ds = 0,$
3. $\frac{u_K^n - u_K^{n-1}}{k} - \frac{1}{m(K)} \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_{int}} \phi_\sigma^n (u_{h,k}^n) m(\sigma) = 0,$

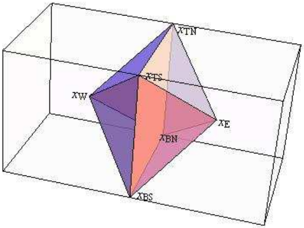
Derivation of the finite volume scheme

1. $\frac{\partial u}{\partial t} - \nabla \cdot (D \nabla u) = 0,$
2. $\frac{u_K^n - u_K^{n-1}}{k} m(K) - \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_{int}} \int_\sigma (D^{n-1} \nabla u^n) \cdot \mathbf{n}_{K,\sigma} ds = 0,$
3. $\frac{u_K^n - u_K^{n-1}}{k} - \frac{1}{m(K)} \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_{int}} \phi_\sigma^n(u_{h,k}^n) m(\sigma) = 0,$
4. $\phi_\sigma^n(u_{h,k}^n) \approx \frac{1}{m(\sigma)} \int_\sigma (D^{n-1} \nabla u^n) \cdot \mathbf{n}_{K,\sigma} ds.$

1. The co-volume χ_σ associated to σ is constructed around each finite volume side by joining four vertices of this side and midpoints of finite volumes which are common to this side.



1. The co-volume χ_σ associated to σ is constructed around each finite volume side by joining four vertices of this side and midpoints of finite volumes which are common to this side.



2. The boundary of co-volume is given by triangles $\bar{\sigma} \subset \partial\chi_\sigma$, we denote their vertices by $N_1(\bar{\sigma})$, $N_2(\bar{\sigma})$ and $N_3(\bar{\sigma})$.

Approximation of the averaged gradient

1. $\frac{1}{m(\chi_\sigma)}\int_{\chi_\sigma}\nabla u^n dx = \frac{1}{m(\chi_\sigma)}\int_{\partial\chi_\sigma} u^n \mathbf{n}_{\chi_\sigma,\bar{\sigma}} ds,$

Approximation of the averaged gradient

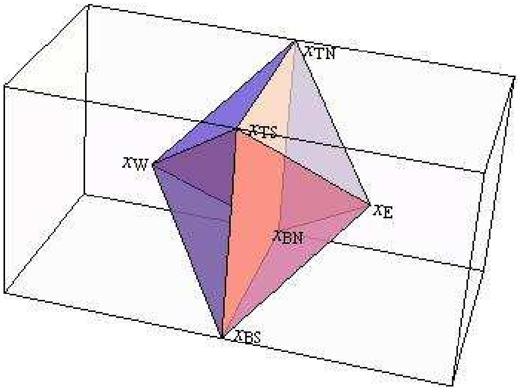
1. $\frac{1}{m(\chi_\sigma)} \int_{\chi_\sigma} \nabla u^n dx = \frac{1}{m(\chi_\sigma)} \int_{\partial\chi_\sigma} u^n \mathbf{n}_{\chi_\sigma,\bar{\sigma}} ds,$
2. $p_\sigma^n(u) =$
$\frac{1}{m(\chi_\sigma)} \sum_{\bar{\sigma}\in\partial\chi_\sigma} \frac{1}{3} \left(u^n_{N_1(\bar{\sigma})} + u^n_{N_2(\bar{\sigma})} + u^n_{N_3(\bar{\sigma})} \right) m(\bar{\sigma}) \mathbf{n}_{\chi_\sigma,\bar{\sigma}},$

Approximation of the averaged gradient

1. $$\frac{1}{m(\chi_\sigma)}\int_{\chi_\sigma}\nabla u^n dx = \frac{1}{m(\chi_\sigma)}\int_{\partial\chi_\sigma} u^n \mathbf{n}_{\chi_\sigma,\bar\sigma} ds,$$
2. $$p_\sigma^n(u) = \frac{1}{m(\chi_\sigma)}\sum_{\bar\sigma\in\partial\chi_\sigma}\frac{1}{3}\left(u^n_{N_1(\bar\sigma)} + u^n_{N_2(\bar\sigma)} + u^n_{N_3(\bar\sigma)}\right) m(\bar\sigma)\mathbf{n}_{\chi_\sigma,\bar\sigma},$$
3. $$p_\sigma^n(u) = \frac{u_E^n - u_W^n}{h}\mathbf{n}_{K,\sigma} + \frac{u^n_{TN}+u^n_{BN}-u^n_{TS}-u^n_{BS}}{2h}\mathbf{t1}_{K,\sigma} + \frac{u^n_{TN}+u^n_{TS}-u^n_{BN}-u^n_{BS}}{2h}\mathbf{t2}_{K,\sigma}.$$

Numerical flux

$\mathbf{t1}_{K,\sigma}$ is a unit vector parallel to $x_{TN} - x_{TS}$ such that $(x_{TN} - x_{TS}) \cdot \mathbf{t1}_{K,\sigma} > 0$ and
$\mathbf{t2}_{K,\sigma}$ is a unit vector parallel to $x_{TN} - x_{BN}$ such that $(x_{TN} - x_{BN}) \cdot \mathbf{t2}_{K,\sigma} > 0$.

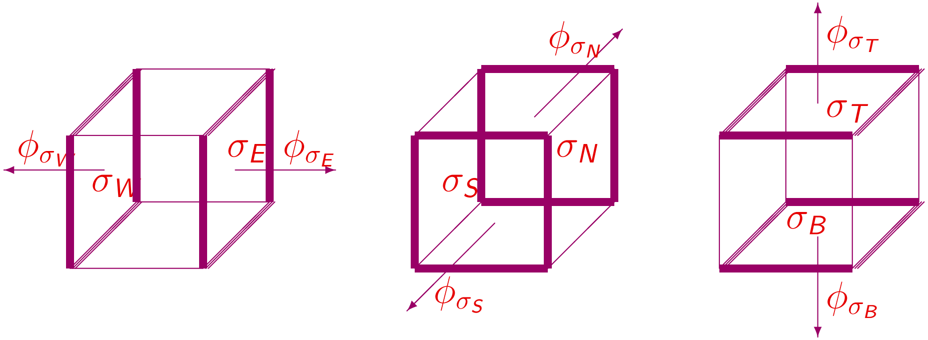

XTN
XTS
XW
XE
XBN
XBS

Numerical flux

$$\phi_\sigma^n(u_{h,k}^n) = (D_\sigma p_\sigma^n(u)) \cdot \mathbf{n}_{K,\sigma},$$

where $D_\sigma = D_\sigma^{n-1} = \begin{pmatrix} \bar{D}_{11}^\sigma & \bar{D}_{12}^\sigma & \bar{D}_{13}^\sigma \\ \bar{D}_{12}^\sigma & \bar{D}_{22}^\sigma & \bar{D}_{23}^\sigma \\ \bar{D}_{13}^\sigma & \bar{D}_{23}^\sigma & \bar{D}_{33}^\sigma \end{pmatrix}.$

D_σ is written in the basis $(\mathbf{n}_{K,\sigma}, \mathbf{t1}_{K,\sigma}, \mathbf{t2}_{K,\sigma})$.


ϕ_{σ_W}
σ_W
σ_E
ϕ_{σ_E}
ϕ_{σ_N}
σ_N
σ_S
ϕ_{σ_S}
ϕ_{σ_T}
σ_T
σ_B
ϕ_{σ_B}

D_σ written in the basis $(\mathbf{n}_{K,\sigma}, \mathbf{t1}_{K,\sigma}, \mathbf{t2}_{K,\sigma})$.

$$\begin{pmatrix} \bar{D}_{11}^\sigma & \bar{D}_{12}^\sigma & \bar{D}_{13}^\sigma \\ \bar{D}_{12}^\sigma & \bar{D}_{22}^\sigma & \bar{D}_{23}^\sigma \\ \bar{D}_{13}^\sigma & \bar{D}_{23}^\sigma & \bar{D}_{33}^\sigma \end{pmatrix} = \begin{pmatrix} D_{11}^\sigma & D_{12}^\sigma & D_{13}^\sigma \\ D_{12}^\sigma & D_{22}^\sigma & D_{23}^\sigma \\ D_{13}^\sigma & D_{23}^\sigma & D_{33}^\sigma \end{pmatrix} \text{ on } \sigma_W \text{ and } \sigma_E.$$

$$\begin{pmatrix} \bar{D}_{11}^\sigma & \bar{D}_{12}^\sigma & \bar{D}_{13}^\sigma \\ \bar{D}_{12}^\sigma & \bar{D}_{22}^\sigma & \bar{D}_{23}^\sigma \\ \bar{D}_{13}^\sigma & \bar{D}_{23}^\sigma & \bar{D}_{33}^\sigma \end{pmatrix} = \begin{pmatrix} D_{22}^\sigma & D_{12}^\sigma & D_{23}^\sigma \\ D_{12}^\sigma & D_{11}^\sigma & D_{13}^\sigma \\ D_{23}^\sigma & D_{13}^\sigma & D_{33}^\sigma \end{pmatrix} \text{ on } \sigma_S \text{ and } \sigma_N.$$

$$\begin{pmatrix} \bar{D}_{11}^\sigma & \bar{D}_{12}^\sigma & \bar{D}_{13}^\sigma \\ \bar{D}_{12}^\sigma & \bar{D}_{22}^\sigma & \bar{D}_{23}^\sigma \\ \bar{D}_{13}^\sigma & \bar{D}_{23}^\sigma & \bar{D}_{33}^\sigma \end{pmatrix} = \begin{pmatrix} D_{33}^\sigma & D_{23}^\sigma & D_{13}^\sigma \\ D_{23}^\sigma & D_{22}^\sigma & D_{12}^\sigma \\ D_{13}^\sigma & D_{12}^\sigma & D_{11}^\sigma \end{pmatrix} \text{ on } \sigma_B \text{ and } \sigma_T.$$

Semi-implicit finite volume scheme

$$\phi_\sigma^n(u_{h,k}^n) =$$

$$\left[\begin{pmatrix} \bar{D}_{11}^\sigma & \bar{D}_{12}^\sigma & \bar{D}_{13}^\sigma \\ \bar{D}_{12}^\sigma & \bar{D}_{22}^\sigma & \bar{D}_{23}^\sigma \\ \bar{D}_{13}^\sigma & \bar{D}_{23}^\sigma & \bar{D}_{33}^\sigma \end{pmatrix} \begin{pmatrix} \frac{u_E^n - u_W^n}{h} \\ \frac{u_{TN}^n + u_{BN}^n - u_{TS}^n - u_{BS}^n}{2h} \\ \frac{u_{TN}^n + u_{TS}^n - u_{BN}^n - u_{BS}^n}{2h} \end{pmatrix} \right] \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \bar{D}_{11}^\sigma \frac{u_E^n - u_W^n}{h} + \bar{D}_{12}^\sigma \frac{u_{TN}^n + u_{BN}^n - u_{TS}^n - u_{BS}^n}{2h}$$

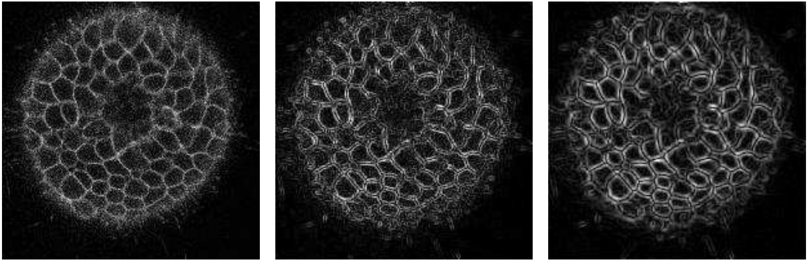
$$+ \bar{D}_{13}^\sigma \frac{u_{TN}^n + u_{TS}^n - u_{BN}^n - u_{BS}^n}{2h}.$$

Semi-implicit finite volume scheme

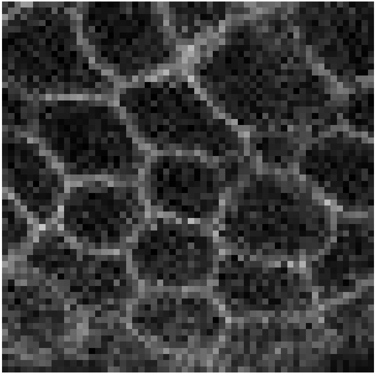
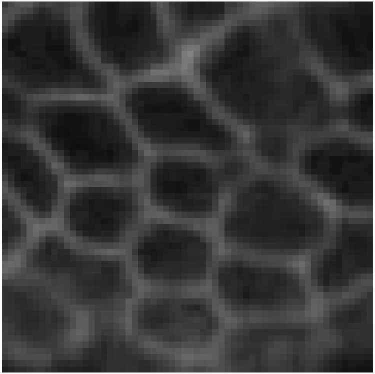
$$\frac{u_K^n - u_K^{n-1}}{k} - \frac{1}{m(K)} \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_{int}} \phi_\sigma^n(u_{h,k}^n) m(\sigma) = 0,$$

where

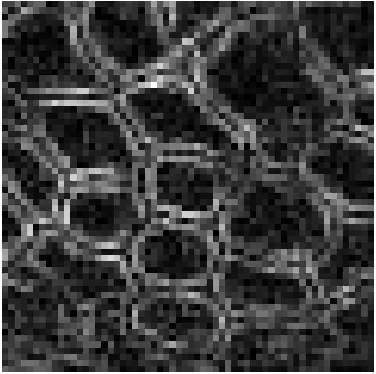
$$\phi_\sigma^n(u_{h,k}^n) = \bar{D}_{11}^\sigma \frac{u_E^n - u_W^n}{h} + \bar{D}_{12}^\sigma \frac{u_{TN}^n + u_{BN}^n - u_{TS}^n - u_{BS}^n}{2h} + \bar{D}_{13}^\sigma \frac{u_{TN}^n + u_{TS}^n - u_{BN}^n - u_{BS}^n}{2h}.$$

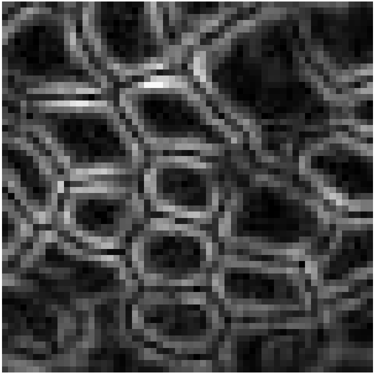


2D slice of 3D membranes image. Original image (left), edge detection of the original image (in the middle), edge detection of the filtered image after 3 steps (right).

Detail (70×70 pixels) of previous image. Original image (left), image filtered by tensor diffusion after 3 steps (right).





Edge detection of the original image (left), edge detection of the filtered image (right).

Y. Coudiere, J. P. Vila and P. Villedieu.
Convergence rate of a finite volume scheme for a two-dimensional convection-diffusion problem.
M2AN Math. Model. Numer. Anal., 33, 493-516, 1999.

Y. Coudiere, J. P. Vila and P. Villedieu.
Convergence rate of a finite volume scheme for a two-dimensional convection-diffusion problem.
M2AN Math. Model. Numer. Anal., 33, 493-516, 1999.

O. Drblíková and K. Mikula.
Convergence Analysis of Finite Volume Scheme for Nonlinear Tensor Anisotropic Diffusion in Image Processing
SIAM, Jour. Num. Anal., 46(1), 37-60, 2007.

- Y. Coudiere, J. P. Vila and P. Villedieu.
  Convergence rate of a finite volume scheme for a two-dimensional convection-diffusion problem.
  M2AN Math. Model. Numer. Anal., 33, 493-516, 1999.
- O. Drblíková and K. Mikula.
  Convergence Analysis of Finite Volume Scheme for Nonlinear Tensor Anisotropic Diffusion in Image Processing
  SIAM, Jour. Num. Anal., 46(1), 37-60, 2007.
- J. Weickert.
  Coherence-enhancing diffusion filtering.
  Int. J. Comput. Vision, Vol. 31, 111–127, 1999.,

Thanks for your attention