A recipe to couple two Finite Volume schemes for elliptic problems

J. Droniou

Université Montpellier II
France

FVCA5
Aussois
June, 08-13 2008.

Contents

Contents

Model problem

Elliptic linear equation:

$$\begin{cases} -\mathrm{div}(A\nabla u) = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega \end{cases}$$

- Ω bounded polygonal open set of $\mathbb{R}^d$,
- $A : \Omega \to M_N(\mathbb{R})$ bounded uniformly elliptic tensor,
- $f \in L^2(\Omega)$, $g \in H^{1/2}(\partial\Omega)$.

Remark: also possible: convection terms, less regular right-hand side, etc.

2-points Finite Volume schemes

Advantages

- good properties (conservation of fluxes, maximum principle)
- no complex geometric function to compute, accept generic grid elements
- cost-effective and easy to implement

Drawbacks

- the mesh must satisfy orthogonality conditions depending on the diffusion tensor A.
- strong non-linearity not obvious to handle

"Advanced" FV schemes

(VF-Hybrid, MultiPoint Flux Approximation, Discrete Duality Finite Volume, Mixed Finite Volume, etc... also: Discontinuous Galerkin, Mimetic Finite Difference...)

Advantages

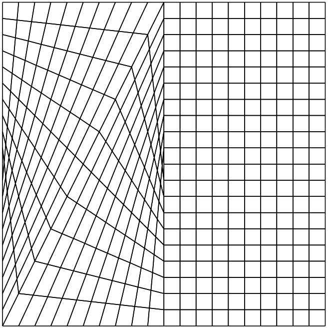
- nearly any kind of grid
- anisotropic, heterogeneous diffusion tensor
- sometimes even strongly non-linear problems

Drawbacks

- loss of good properties (mainly the maximum principle)
- very expensive (number of unknows)

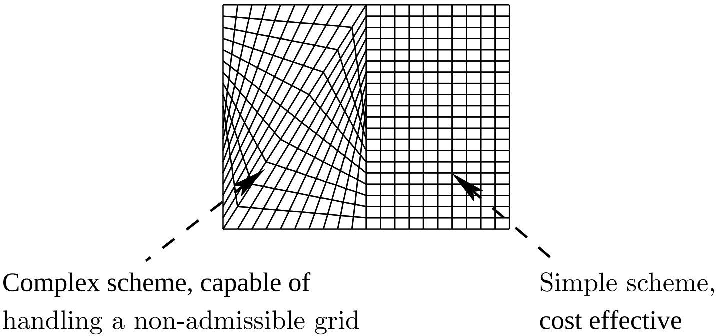
"Advanced" FV schemes built to accept anisotropy or non-admissible grids.

These issues are however not necessarily spread throughout the domain Ω:



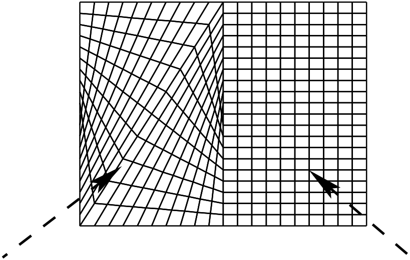
Ideas

Idea 1 (not original): use a simple scheme where possible, and a complex scheme only where necessary.



Ideas

Idea 1 (not original): use a simple scheme where possible, and a complex scheme only where necessary.



Complex scheme, capable of handling a non-admissible grid

Simple scheme, cost effective

Idea 2 (a little bit more original ?): do this in a systematic way.

About mixing two schemes for a single equation...

- Quarteroni-Valli 1999 (Domain Decomposition)
- Lazarov-Pasciak-Vassilevski 1999 (MFE+FV) *[FVCA2]*
- Achdou-Japhet-Maday-Nataf 2002 (mortar for FV)

▶ Mainly performed on specific schemes, using their precise expressions.

▶ No reflexion, especially for FV methods, on a mixing based on generic properties and principles of the schemes.

Objectives

- Extract the "core" elements of FV methods:
  - basic *unknowns* and *relations* used to write the schemes
  - basic *properties* proved on the schemes to study their convergence.

Objectives

- Extract the "core" elements of FV methods:
  - basic *unknowns* and *relations* used to write the schemes
  - basic *properties* proved on the schemes to study their convergence.

- Using only the core unknowns and relations: find a way to couple two FV schemes (each one being applied in a subregion of Ω).

Objectives

- Extract the "core" elements of FV methods:
  - basic *unknowns* and *relations* used to write the schemes
  - basic *properties* proved on the schemes to study their convergence.
- Using only the core unknowns and relations: find a way to couple two FV schemes (each one being applied in a subregion of Ω).
- Using only the core properties of the schemes: study the convergence of the coupling of the schemes.

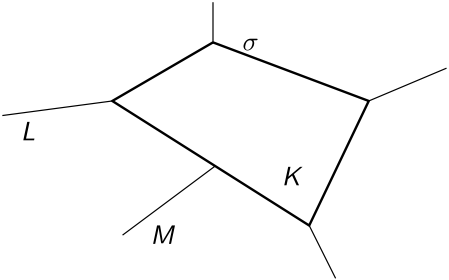
$\rightsquigarrow$ *generic coupling method working with many FV schemes.*

Contents

Core unknows of FV schemes for $-\mathrm{div}(A\nabla u) = f$

Mesh: partition $\mathcal{M}$ of Ω in polygonal control volumes K.

- $\mathcal{E}_K$ set of edges σ of K.
- $\mathcal{E}_{\rm int}$ and $\mathcal{E}_{\rm ext}$ interior and boundary edges.



Core unknows of FV schemes for $-\mathrm{div}(A\nabla u) = f$

Mesh: partition $\mathcal{M}$ of Ω in polygonal control volumes K.

- $\mathcal{E}_K$ set of edges σ of K.
- $\mathcal{E}_{\mathrm{int}}$ and $\mathcal{E}_{\mathrm{ext}}$ interior and boundary edges.

Unknows: approximate values...

- $(u_K)_K$ of the solution in the control volumes,
- $(u_\sigma)_\sigma$ of the solution on the boundary edges $\mathcal{E}_{\mathrm{ext}}$,
- $(F_{K,\sigma})_{K,\sigma}$ of the flux $\int_\sigma A\nabla u \cdot \mathbf{n}_{K,\sigma}$.

In some schemes, some of these unknowns can be directly eliminated and/or other unknowns can be needed.

Core relations of FV schemes for $-\mathrm{div}(A\nabla u) = f$

Conservativity of the fluxes:

$$F_{K,\sigma} + F_{L,\sigma} = 0 \quad \text{for all } \sigma \text{ between } K \text{ and } L$$

Balance of fluxes:

$$-\sum_{\sigma\in\mathcal{E}_K} F_{K,\sigma} = \int_K f \quad \text{for all } K$$

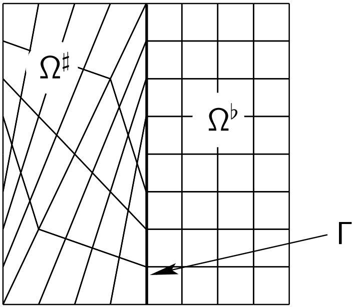
Boundary conditions:

$$u_\sigma = \int_\sigma g \quad \text{for all } \sigma \in \mathcal{E}_{\mathrm{ext}}$$

Definition of the coupling of two schemes $S^\flat$ and $S^\sharp$

Setting:

- Ω cut in two: $\Omega = \Omega^\flat \sqcup \Omega^\sharp$
- $\mathcal{M}^\flat$ mesh on $\Omega^\flat$, adapted to a FV scheme $S^\flat$.
- $\mathcal{M}^\sharp$ mesh on $\Omega^\sharp$, adapted to a FV scheme $S^\sharp$.
- the edges of $\mathcal{M}^\flat$ and $\mathcal{M}^\sharp$ are the same on $\Gamma = \partial\Omega^\flat \cap \partial\Omega^\sharp$.



Definition of the coupling of two schemes $S^\flat$ and $S^\sharp$

Equations inside each subdomain:

- The equations of $S^\flat$ on $\mathcal{M}^\flat$, of $S^\sharp$ on $\mathcal{M}^\sharp$.

$\rightsquigarrow 2 \times \mathrm{Card}(\{\sigma \subset \Gamma\})$ equations cannot be written: boundary equations on Γ for each schemes.

Definition of the coupling of two schemes $S^\flat$ and $S^\sharp$

Equations inside each subdomain:

- The equations of $S^\flat$ on $\mathcal{M}^\flat$, of $S^\sharp$ on $\mathcal{M}^\sharp$.

$\rightsquigarrow$ $2 \times \mathrm{Card}(\{\sigma \subset \Gamma\})$ equations cannot be written: boundary equations on Γ for each schemes.

Equations coupling the schemes:

- No jump of the solution at Γ:

$$u_\sigma^\flat = u_\sigma^\sharp \quad \text{for all } \sigma \subset \Gamma$$

- conservativity of the fluxes:

$$F_{K,\sigma}^\flat + F_{L,\sigma}^\sharp = 0 \quad \text{for all } \sigma \subset \Gamma \text{ between } K \in \mathcal{M}^\flat \text{ and } L \in \mathcal{M}^\sharp.$$

Works with other kinds of schemes, if they have fluxes...

"FV technique of proof"

Find an adequate discrete H^1 (semi-)norm: multiply the left-hand side of the flux balance by u_K.

- Existence and uniqueness of an approximate solution.

"FV technique of proof"

Find an adequate discrete H^1 (semi-)norm: multiply the left-hand side of the flux balance by u_K.

▶ Existence and uniqueness of an approximate solution.

A priori estimates: using the discrete norm.

▶ Compactness of the approximate solution, convergence in L^2 up to a subsequence to a function $\bar{u} \in H^1$.

"FV technique of proof"

Find an adequate discrete H^1 (semi-)norm: multiply the left-hand side of the flux balance by u_K.

▶ Existence and uniqueness of an approximate solution.

A priori estimates: using the discrete norm.

▶ Compactness of the approximate solution, convergence in L^2 up to a subsequence to a function $\bar{u} \in H^1$.

Convergence: take φ regular, multiply the flux balance by $\varphi_K \approx \varphi$ on K, and perform discrete integrate-by-parts.

▶ $\bar{u}$ is the solution to the PDE.

Technique also efficient for non-linear equations, coupled systems with low regularity on the solutions, etc.

Discrete H^1 semi-norm

We denote $U = (u_K, F_{K,\sigma}, u_\sigma)$ the vector of core unknowns.

$$|U|_S^2 = \sum_{\sigma \in \mathcal{E}_{\text{int}}} F_{K,\sigma}(u_L - u_K) + \sum_{\sigma \in \mathcal{E}_{\text{ext}}} F_{K,\sigma}(u_\sigma - u_K) \qquad (1)$$

Discrete H^1 semi-norm

We denote $U = (u_K, F_{K,\sigma}, u_\sigma)$ the vector of core unknowns.

$$|U|_S^2 = \sum_{\sigma \in \mathcal{E}_{\mathrm{int}}} F_{K,\sigma}(u_L - u_K) + \sum_{\sigma \in \mathcal{E}_{\mathrm{ext}}} F_{K,\sigma}(u_\sigma - u_K) \qquad (1)$$

Definition (Property N)

A scheme S satisfies Property N if, for any solution U, the right-hand side of (1) is indeed non-negative, and if $U = 0$ as soon as $|U|_S$ and one u_σ are zero.

Proposition

If $S^\flat$ and $S^\sharp$ satisfy Property N, then the coupling $S^\flat$–$S^\sharp$ has one and only one solution $U^{\flat\sharp}$.

A priori estimates ($g = 0$)

Definition (Properties P_Λ and T)

Let $\Lambda \subset \partial\omega$. A scheme S on ω satisfies Property P_Λ or T if there exists C such that, for any solution $U = (u_K, F_{K,\sigma}, u_\sigma)$,

$$\begin{aligned} ||u||_{L^2(\Omega)} &\leq C|U|_S + C||t(U)||_{L^2(\Lambda)} && \text{(Property } P_\Lambda\text{)} \\ ||t(U)||_{L^2(\partial\Omega)} &\leq C|U|_S + C||u||_{L^2(\Omega)} && \text{(Property } T\text{).} \end{aligned}$$

with $t(U) = (u_\sigma)_{\sigma \in \mathcal{E}_{\rm ext}}$.

A priori estimates ($g = 0$)

Definition (Properties P_Λ and T)

Let $\Lambda \subset \partial\omega$. A scheme S on ω satisfies Property P_Λ or T if there exists C such that, for any solution $U = (u_K, F_{K,\sigma}, u_\sigma)$,

$$\begin{aligned} ||u||_{L^2(\Omega)} &\leq C|U|_S + C||t(U)||_{L^2(\Lambda)} && \text{(Property } P_\Lambda\text{)} \\ ||t(U)||_{L^2(\partial\Omega)} &\leq C|U|_S + C||u||_{L^2(\Omega)} && \text{(Property } T\text{).} \end{aligned}$$

with $t(U) = (u_\sigma)_{\sigma \in \mathcal{E}_{\mathrm{ext}}}$.

Proposition

If $S^\flat$ satisfies Property $P_{\partial\Omega^\flat\setminus\Gamma}$ and if

- *$S^\sharp$ satisfies Property $P_{\partial\Omega^\sharp\setminus\Gamma}$, or*
- *$S^\sharp$ satisfies Properties $P_{\partial\Omega^\sharp}$ and $S^\flat$ satisfies Property T*

then the L^2 and discrete H^1 norms of $U^{\flat\sharp}$ are bounded.

Convergence

Definition (Properties C and L)

A couple $(S, (\mathcal{M}_n)_{n\geq 1})$ *of scheme/discretizations on* ω *satisfies Property C if* $\mathrm{size}(\mathcal{M}_n) \to 0$ *and, with* U_n *solution to* S *on* $\mathcal{M}_n$*, if* $(|U_n|_S + ||u_n||_{L^2})_n$ *is bounded then, up to a subsequence,* $u_n \to \bar{u}$ *and* $t(U_n) \to \gamma(\bar{u})$ *weakly in* L^2*, with* $\bar{u} \in H^1$*.*

It satisfies Property L if, moreover, for all regular φ*,*

$$\sum_{\sigma\in\mathcal{E}^n_{\mathrm{int}}} F_{K,\sigma}(\varphi_L - \varphi_K) + \sum_{\sigma\in\mathcal{E}^n_{\mathrm{ext}}} F_{K,\sigma}(\varphi_\sigma - \varphi_K) \to \int A\nabla\bar{u}\cdot\nabla\varphi.$$

Proposition

If $(S^\flat, (\mathcal{M}^\flat_n)_{n\geq 1})$ *and* $(S^\sharp, (\mathcal{M}^\sharp_n)_{n\geq 1})$ *satisfy Properties C and L, then* $U^{\flat\sharp}$ *converges to the solution of the PDE.*

Contents

Framework

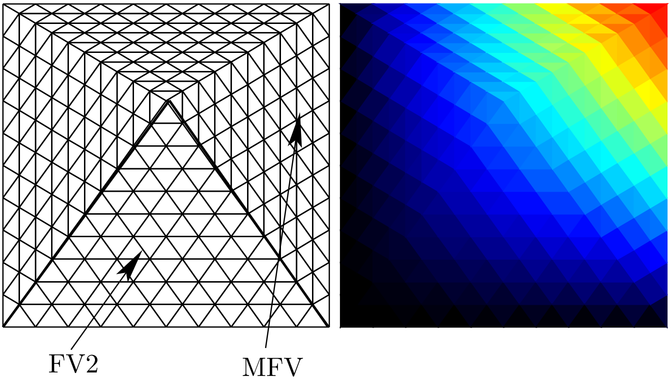
Following numerical tests realized by coupling:

- Standard 2-point Finite Volume scheme (FV2).
- Mixed Finite Volume scheme (MFV) [“advanced” scheme].

Qualitative behavior of the coupling

Equation: $-\Delta u = f$ on $\Omega = (0,1)^2$

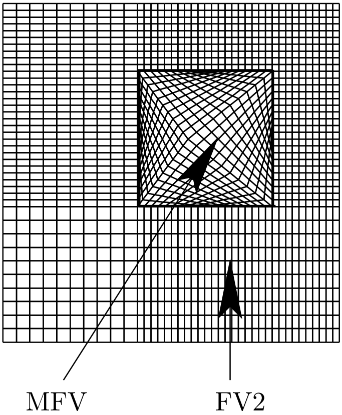
Exact solution: $u(x,y) = \sin(xy)$.

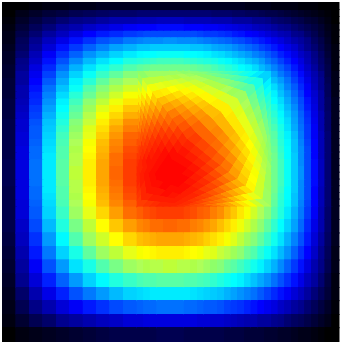


Visualy completely similar to the pure MFV scheme

Equation: $-\Delta u = f$ on $\Omega = (0,1)^2$

Exact solution: $u(x,y) = x(1-x)y(1-y)$.

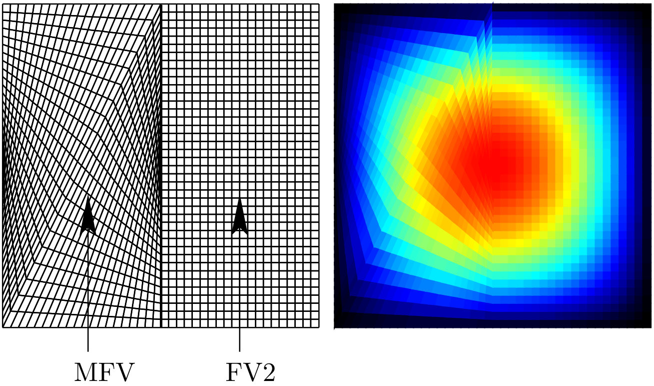




Equation: $-\mathrm{div}(A\nabla u) + \mathrm{div}(Vu) = f$ on $\Omega = (0,1)^2$ with:

- $A = Id$ in the FV2 zone.
- $A(x,y) = R_{2\pi x} \begin{pmatrix} 0.1 & 0 \\ 0 & 1 \end{pmatrix} R^T_{2\pi x}$ in the MFV zone.
- $V(x,y) = 10(-y,x)$.

Exact solution: $u(x,y) = x(1-x)y(1-y)$.



Comparison of the number of unknowns and errors

Triangular grids (≈ −8% unknowns):

MFV			Coupling MFV-FV2		
#UNK	L^2	L^∞	#UNK	L^2	L^∞
14900	5.76E-4	1.15E-3	13725	6.04E-4	1.15E-3
33600	2.56E-4	5.12E-4	30900	2.68E-4	5.12E-4
59800	1.44E-4	2.88E-4	54950	1.51E-4	2.88E-4

Quadrangular grids, "well" (≈ −35% unknowns):

MFV			Coupling MFV-FV2		
#UNK	L^2	L^∞	#UNK	L^2	L^∞
19800	2.75E-4	3.38E-4	12600	2.92E-4	3.37E-4
43512	1.25E-4	1.54E-4	29032	1.26E-4	1.49E-4
79600	6.98E-5	8.63E-5	50200	7.31E-5	8.99E-5

A remark on singularities

Equation: $-\mathrm{div}(A\nabla u) = f$ with $u(x, y) = x^2$ and

$$A = k_1 \mathbf{1}_{x<0.5} + k_2 \mathbf{1}_{x\geq 0.5}.$$

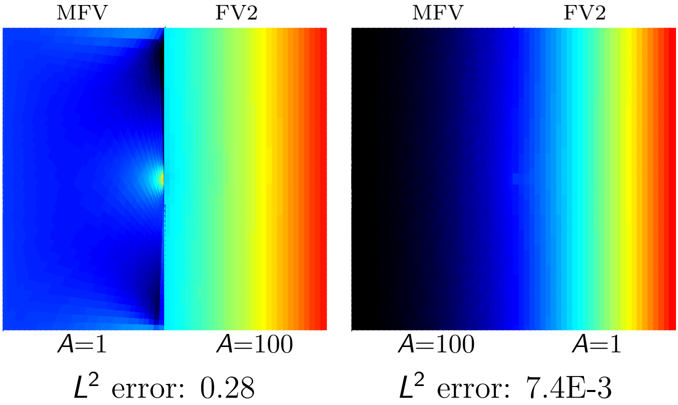
- f is singular: measure on $x = 0.5$, put in the left domain.

A remark on singularities

Equation: $-\mathrm{div}(A\nabla u) = f$ with $u(x,y) = x^2$ and

$$A = k_1 \mathbf{1}_{x<0.5} + k_2 \mathbf{1}_{x \geq 0.5}.$$

- f is singular: measure on $x = 0.5$, put in the left domain.



Conclusion

- Generic coupling method, using only the core elements of FV schemes, not their particular expression.

- Convergence of the coupling ensured by properties satisfied by each scheme separately *(if you know your scheme, you do not need to make a specific study of the coupling)*.

- Gain of computational cost, no degradation of the qualitative and quantitative convergence properties.

- The solution of the coupling can be computed by an iterative process, using pre-existing “black boxes” implementations of each scheme.