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joint work with
S. Funken (Ulm), D. Praetorius (Vienna)

Adaptive Finite Volume Method

A posteriori error estimate and adaptive mesh refinement for the cell-centered FVM and coupling with BEM

Elliptic Model Problem

$\Omega \subset \mathbb{R}^2$: bounded, connected domain with Lipschitz boundary $\Gamma := \partial\Omega$

$$-\Delta u = f \in L^2(\Omega) \quad \text{in } \Omega$$

with mixed boundary conditions

$$\begin{aligned} u = u_D &\in H^1(\Gamma_D) \quad \text{on } \Gamma_D \\ \partial u / \partial \mathbf{n} = g &\in L^2(\Gamma_N) \quad \text{on } \Gamma_N \end{aligned}$$

where Γ_D as well as Γ_N are connected.

Discretization by cell-centered Finite Volume Method using the Diamond Path [Coudiére, Villedieu, M2AN 2000].

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Discretization by cell-centered Finite Volume Method using the Diamond Path [Coudiére, Villedieu, M2AN 2000].

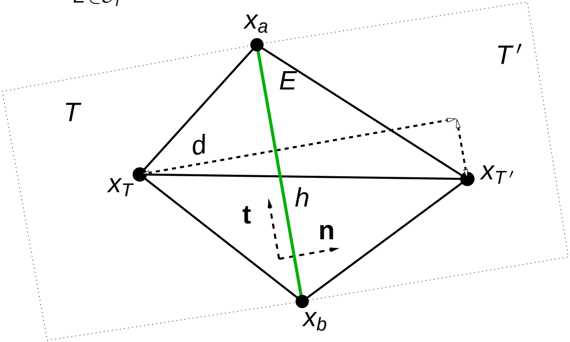
Develop a posteriori estimator based on a so-called Morley Interpolant,

Extended from [Nicaise, SIAM 2005] to the case of hanging nodes and mixed boundary conditions.

Discretization

Find $u_h \in \mathcal{P}_0(\mathcal{T})$ such that

$$-\sum_{E \in \mathcal{E}_T} F_E(u_h) = \int_T f\,dx, \qquad \text{for all } T \in \mathcal{T}.$$



$$F_{\mathbf{E}}(u_h) := h\left(\frac{u_{T'} - u_T}{d} - \alpha \frac{u_a - u_b}{h}\right)$$

$$\text{with} \qquad \alpha = \frac{(x_{T'} - x_T)\cdot \mathbf{t}}{(x_{T'} - x_T)\cdot \mathbf{n}}, \qquad d = (x_{T'} - x_T)\cdot \mathbf{n}.$$

Node Evaluation

For each node $a \in \mathcal{N}$, we define

$$u_a = \begin{cases} \displaystyle\sum_{T \in \widetilde{\omega}_a} \psi_T(a)\, u_T\,, & \text{for all } a \in \mathcal{N}_I \\ u_D(a), & \text{for all } a \in \mathcal{N}_D \\ \overline{u}_a + \overline{g}_a, & \text{for all } a \in \mathcal{N}_N \end{cases}$$

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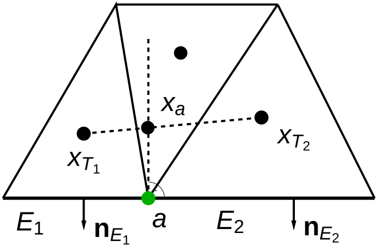

For example (special case):

$$\overline{u}_a = \frac{u_{T_2} - u_{T_1}}{|x_{T_2} - x_{T_1}|} |x_a - x_{T_1}| + u_{T_1}$$

If $\mathbf{n}_{E_1} = \mathbf{n}_{E_2}$:

$$\overline{g}_a = |x_a - a| \Big(\frac{1}{|E_1|} \int_{E_1} g\, ds + \frac{1}{|E_2|} \int_{E_2} g\, ds \Big) / 2$$

Morley Interpolant

- $\mathcal{I}u_h \in \mathcal{P}^2(\mathcal{T})$ is uniquely defined (defined on Morley Element).

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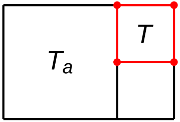
- $\mathcal{I}u_h$ is continuous in all nodes $a \in \mathcal{N}$ but not globally continuous in Ω.



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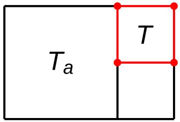

- The flux over an edge of $\mathcal{I}u_h$ is equal to the numerical flux.

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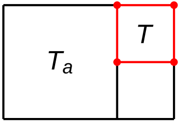

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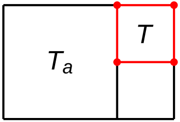


- The flux over an edge of $\mathcal{I}u_h$ is equal to the numerical flux.
- The residual $R := f + \Delta(\mathcal{I}u_h)$ is L^2-orthogonal to $\mathcal{P}_0(\mathcal{T})$.
- Some more properties such as jump-relations which are needed for the proof of an error estimator.

Reliability

Helmholtz Decomposition

Given $\Phi \in L^2(\Omega)^2$, there are $v, w \in H^1(\Omega)$ with $\Phi = \nabla v + \operatorname{curl} w$ such that $v|_{\Gamma_D} = 0$ as well as $w|_{\Gamma_N} = 0$. In particular, there holds

$$\|\Phi\|^2_{L^2(\Omega)} = \|\nabla v\|^2_{L^2(\Omega)} + \|\operatorname{curl} w\|^2_{L^2(\Omega)}. \qquad \square$$

Refinement indicator

$$\eta_T^2 := h_T^2 \|f - f_{\mathcal{T}}\|^2_{L^2(T)} + \sum_{E \in \mathcal{E}_T \cap \mathcal{E}_E} h_E \|[\![\nabla_{\mathcal{T}}(\mathcal{I}u_h)]\!]\|^2_{L^2(E)}$$
$$+ \sum_{E \in \mathcal{E}_T \cap \mathcal{E}_N} h_E \left\| \frac{\partial(u - \mathcal{I}u_h)}{\partial \mathbf{n}_E} \right\|^2_{L^2(E)} + \sum_{E \in \mathcal{E}_T \cap \mathcal{E}_D} h_E \left\| \frac{\partial(u - \mathcal{I}u_h)}{\partial \mathbf{t}_E} \right\|^2_{L^2(E)}.$$

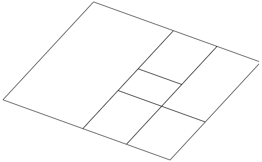
Theorem

There holds ($C_{rel} > 0$ only depends on the shape of the elements in $\mathcal{T}$)

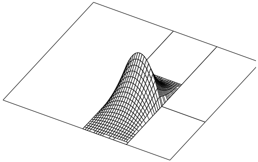
$$C_{rel}^{-1} \|\nabla_{\mathcal{T}}(u - \mathcal{I}u_h)\|_{L^2(\Omega)} \leq \eta := \Big(\sum_{T \in \mathcal{T}} \eta_T^2 \Big)^{1/2}. \qquad \square$$

Efficiency

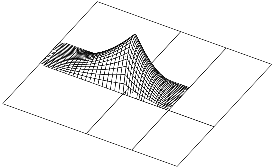
Bubble Functions b_E



Triangulation.



b_E on $E \in \mathcal{E}_0$.



b_E on $E \in \mathcal{E}_E \backslash \mathcal{E}_0$.

Edge Lifting Operator: $F_{ext} : \mathcal{P}_p(E) \to H^1(\omega_E^*)$

Theorem

There is a constant $C_{eff} > 0$ which depends only on c_E and the shape of the elements in $\mathcal{T}$ but neither on the size nor the number of elements such that

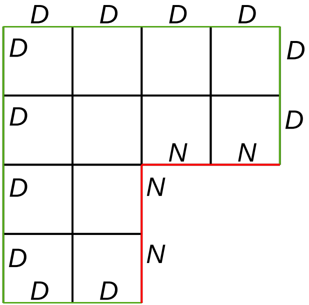
$$\eta_T^2 \leq C_{eff}\left(\|\nabla_{\mathcal{T}}(u - \mathcal{I}u_h)\|^2_{L^2(\omega_T)} + h_T^2\|f - f_{\mathcal{T}}\|^2_{L^2(\omega_T)}\right), \quad \textit{for all } T \in \mathcal{T}. \quad \square$$

Laplace Problem with Generic Singularity

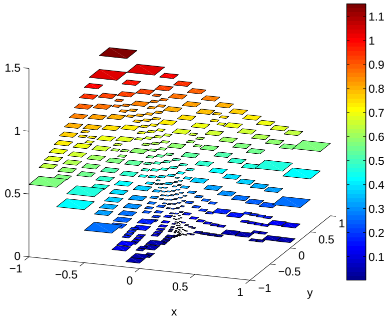
L-shaped domain

$$\Omega = (-1,1)^2 \backslash \big([0,1]\times[-1,0]\big)$$

$$u(x,y) = r^{2/3}\,\sin(2\varphi/3) \quad \text{with} \quad (x,y) = r\,(\cos\varphi, \sin\varphi).$$

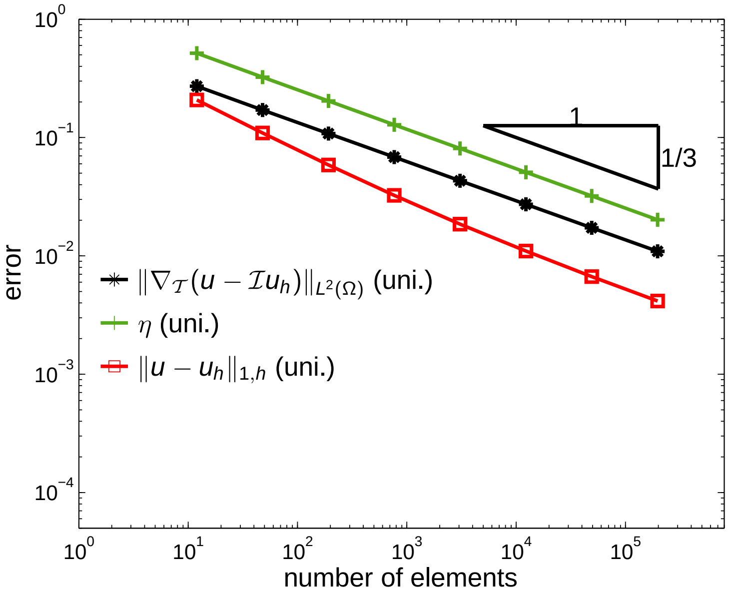


Initial Mesh



Solution: 369 elements

Error Comparison (Rectangle)



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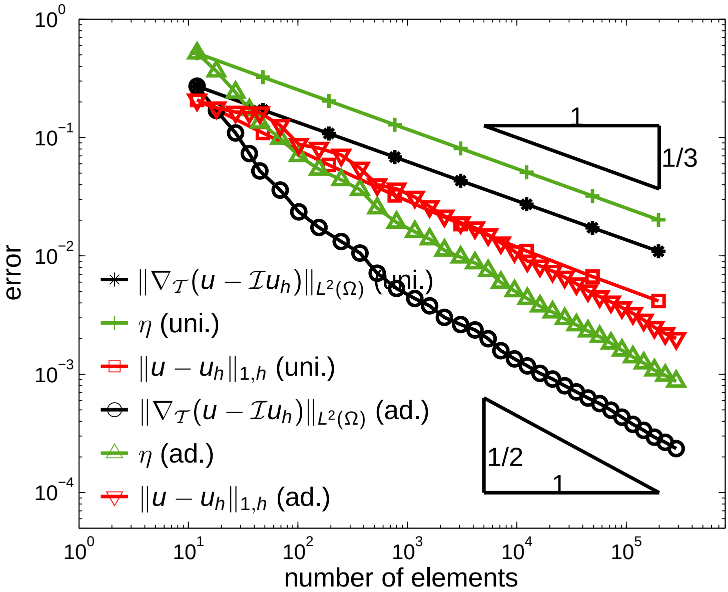
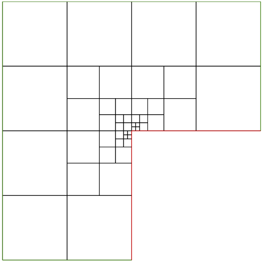


45 elements



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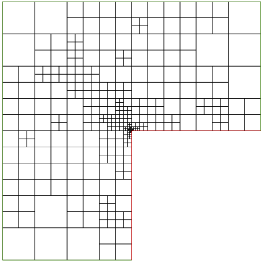
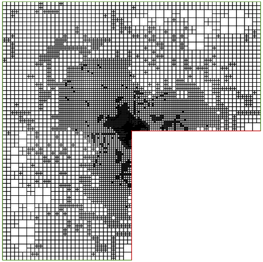
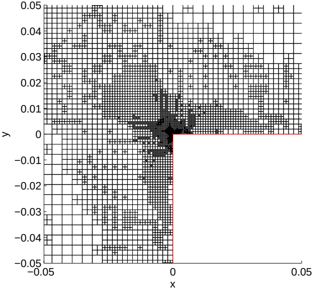


9510 elements

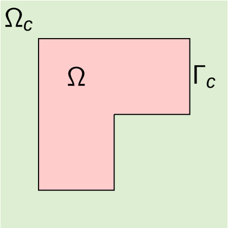


Zoom: 35004 elements



Coupling Problem

$$
\begin{aligned}
-\Delta u &= f && \text{in } \Omega \\
\Delta v &= 0 && \text{in } \Omega_c \\
\lim_{|x|\to\infty} \{v(x) - b\log(|x|)\} &= a && \\
u &= v + u_0 && \text{on } \Gamma_c \\
\frac{\partial u}{\partial \mathbf{n}_{ext}} &= \frac{\partial v}{\partial \mathbf{n}_{ext}} + t_0 && \text{on } \Gamma_c
\end{aligned}
$$



$a, b \in \mathbb{R}$ for the radiation condition

u_0 and t_0 denote jumps

The Coupling

FVM part:

Apply divergence theorem and we get for all $T \in \mathcal{T}$

$$-\int_{\partial T} \nabla u \cdot \mathbf{n}_T \, ds = \int_T f \, dx .$$

$$-\int_{\partial T \setminus \Gamma_C} \nabla u \cdot \mathbf{n}_T \, ds - \int_{\partial T \cap \Gamma_C} \nabla u \cdot \mathbf{n}_T \, ds = \int_T f \, dx ,$$

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$$-\int_{\partial T \setminus \Gamma_C} \nabla u \cdot \mathbf{n}_T \, ds - \int_{\partial T \cap \Gamma_C} \frac{\partial v}{\partial \mathbf{n}_{ext}} \, ds = \int_T f \, dx + \int_{\partial T \cap \Gamma_C} t_0 \, ds .$$

BEM part: Calderon System (exterior problem)

Cauchy data $(\xi, \phi) := (v, \partial v / \partial \mathbf{n}_{ext})$

$$\begin{pmatrix} \xi \\ \phi \end{pmatrix} := \begin{pmatrix} 1/2 + \mathcal{K} & -\mathcal{V} \\ -\mathcal{W} & 1/2 - \mathcal{K}^* \end{pmatrix} \begin{pmatrix} \xi \\ \phi \end{pmatrix} + \begin{pmatrix} a \\ 0 \end{pmatrix}$$

$\mathcal{V}$ single layer op., $\mathcal{K}$ double layer op.,
$\mathcal{K}^*$ adjoint double layer op., $\mathcal{W}$ hypersingular op.

Linear System of Equation

The discrete problem reads:

Find $u_h \in \mathcal{P}_0(\mathcal{T})$, $u_{h,\Gamma_C} \in \mathcal{S}_1(\mathcal{E}_C)$, $\xi_h \in \mathcal{S}_1(\mathcal{E}_C)$, $\phi_h \in \mathcal{S}_0(\mathcal{E}_C)$ and $\lambda \in \mathbb{R}$ such that

$$
\begin{aligned}
-\sum_{E \in \mathcal{E}_T \setminus \Gamma_C} \sigma_{T,E} F_E^D(u_h) - \int_{\partial T \cap \Gamma_C} \phi_h \, ds &= \int_T f \, dx + \int_{\partial T \cap \Gamma_C} t_0 \, ds \qquad && \forall T \in \mathcal{T}, \\
\overline{u}_{h,a} - u_{h,\Gamma_C}(a) + \overline{\varsigma}_{a,\phi_h} &= \overline{\varsigma}_{a,t_0} && \forall a \in \mathcal{N}_C, \\
-\langle u_{h,\Gamma_C}, \psi_h \rangle - \langle \mathcal{V}\phi_h, \psi_h \rangle + \langle (1/2 + \mathcal{K})\xi_h, \psi_h \rangle &= -\langle u_0, \psi_h \rangle && \forall \psi_h \in \mathcal{S}_0(\mathcal{E}_C), \\
\langle (1/2 + \mathcal{K}^*)\phi_h, \theta_h \rangle + \langle \mathcal{W}\xi_h, \theta_h \rangle + \langle \lambda, \theta_h \rangle &= 0 && \forall \theta_h \in S_1(\mathcal{E}_C), \\
\langle \xi_h, \mu \rangle &= 0 && \forall \mu \in \mathbb{R}.
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-\langle u_{h,\Gamma_C}, \psi_h\rangle - \langle \mathcal{V}\phi_h, \psi_h\rangle + \langle (1/2+\mathcal{K})\xi_h, \psi_h\rangle &= -\langle u_0, \psi_h\rangle && \forall \psi_h \in \mathcal{S}_0(\mathcal{E}_C), \\
\langle (1/2+\mathcal{K}^*)\phi_h, \theta_h\rangle + \langle \mathcal{W}\xi_h, \theta_h\rangle + \langle \lambda, \theta_h\rangle &= 0 && \forall \theta_h \in S_1(\mathcal{E}_C), \\
\langle \xi_h, \mu\rangle &= 0 && \forall \mu \in \mathbb{R}.
\end{aligned}
$$

Quantities in A Posteriori Error Estimate

Let $\mathbf{n}_E$ and $\mathbf{t}_E$ denote the normal and tangential vector of an edge E. Define

$$R_1^2 := h_T^2 \|f - f_{\mathcal{T}}\|^2_{L^2(T)}$$

$$R_2^2 := \sum_{E \in \mathcal{E}_T} h_E \|[\![\nabla_{\mathcal{T}}(\mathcal{I}u_h) \cdot \mathbf{n}_E]\!]\|^2_{L^2(E)}\, ds$$

$$R_3^2 := \sum_{E \in \mathcal{E}_T} h_E \|[\![\nabla_{\mathcal{T}}(\mathcal{I}u_h) \cdot \mathbf{t}_E]\!]\|^2_{L^2(E)}\, ds$$

$$R_4^2 := \sum_{E \in \mathcal{E}_T \cap \Gamma_C} h_E \|\mathcal{W}\xi_h + (1/2 + \mathcal{K}^*)\phi_h\|^2_{L^2(E)}$$

where

$$[\![\nabla_{\mathcal{T}}(\mathcal{I}u_h) \cdot \mathbf{n}_E]\!] := \begin{cases} \big(\nabla(\mathcal{I}u_h)|_{T'} - \nabla(\mathcal{I}u_h)|_T\big) \cdot \mathbf{n}_E, & \text{if } E \subset \Omega, \\ \nabla(\mathcal{I}u_h)|_T \cdot \mathbf{n}_E - \phi_h - t_0, & \text{if } E \subset \Gamma_C, \end{cases}$$

$$[\![\nabla_{\mathcal{T}}(\mathcal{I}u_h) \cdot \mathbf{t}_E]\!] := \begin{cases} \big(\nabla(\mathcal{I}u_h)|_{T'} - \nabla(\mathcal{I}u_h)|_T\big) \cdot \mathbf{t}_E, & \text{if } E \subset \Omega, \\ \nabla(\mathcal{I}u_h)|_T\big) \cdot \mathbf{t}_E - \frac{d}{ds}\big(u_0 + (1/2 + \mathcal{K})\xi_h - \mathcal{V}\phi_h\big), & \text{if } E \subset \Gamma_C, \end{cases}$$

Error Estimator

Refinement indicator for all $T \in \mathcal{T}$

$$\eta_T^2 := R_1^2 + R_2^2 + R_3^2 + R_4^2$$

and Error Estimator

$$\eta := \Big(\sum_{T \in \mathcal{T}} \eta_T^2 \Big)^{1/2}$$

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Theorem (in work)

There holds

$$\|\nabla_{\mathcal{T}}(u - \mathcal{I}u_h)\|_{L^2(\Omega)} \leq C_{rel}^{-1}\eta$$

and

$$C_{eff}^{-1}\eta \leq \|\nabla_{\mathcal{T}}(u - \mathcal{I}u_h)\|_{L^2(\Omega)} + \|h(f - f_{\mathcal{T}})\|_{L^2(\Omega)},$$

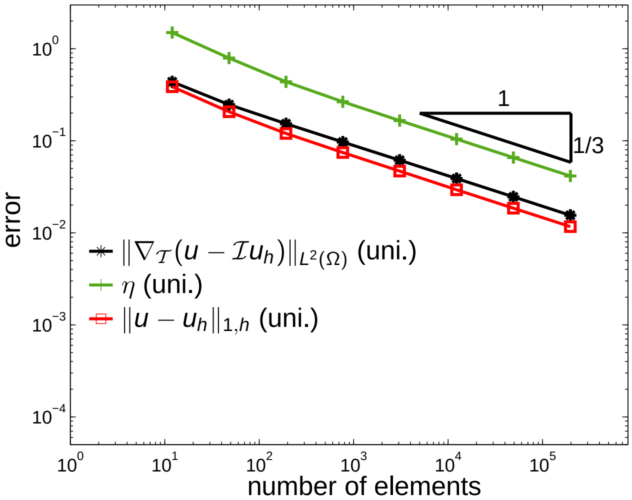
where $C_{rel}, C_{eff} > 0$ only depend on the shape of the elements in $\mathcal{T}$.

Proof: Ideas from above and [Carstensen,Funken, Computing 1999].

Error Lshape (Triangle)

Choose $a = 0$, $b = 1$, the jumps u_0 and t_0 appropiate to

$$u = r^{2/3} \sin\left(2\varphi/3\right), \qquad v = \log\left(\left|(x+0.5, y-0.5)\right|\right), \qquad f = 0.$$



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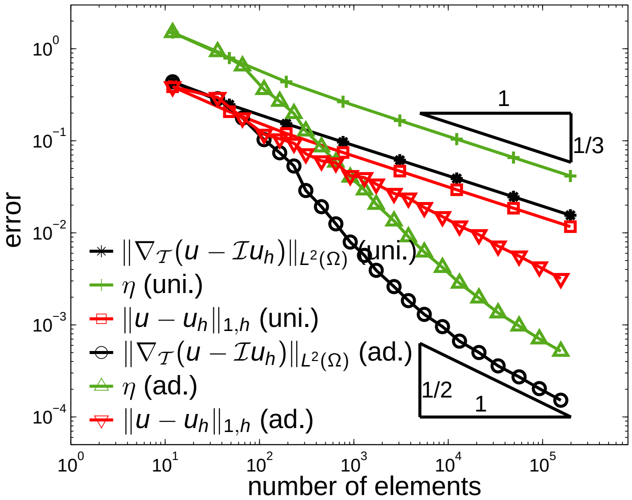

Further Questions?

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