Bench of anisotropic problems
SUSHI: a scheme using stabilization and hybrid interfaces

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Construction of a discrete gradient

Discretization $\mathcal{D}$:

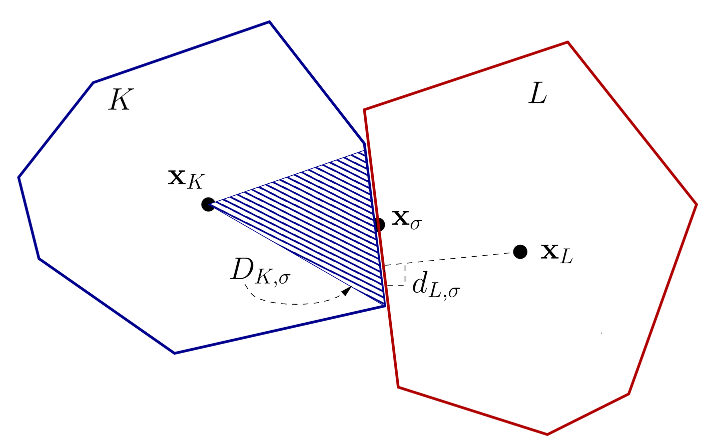
- general polygonal mesh $\mathcal{M}$
- planar edges: $\mathcal{E}$, $\mathcal{E} = \mathcal{E}_{\text{int}} \cup \mathcal{E}_{\text{ext}}$, $\mathcal{E}_{\text{int}} = \mathcal{B} \cup \mathcal{H}$, $\mathcal{H} = \mathcal{E}_{\text{int}} \setminus \mathcal{B}$
- cell points $(\mathbf{x}_K)_{K\in\mathcal{M}}$

Discrete unknowns: $u = ((u_K)_{K\in\mathcal{M}}, (u_\sigma)_{\sigma\in\mathcal{E}}) \in X_{\mathcal{D}}$

$$\nabla_{K,\sigma} u = \nabla_K u + R_{K,\sigma} u \, \mathbf{n}_{K,\sigma}$$

$$R_{K,\sigma} u = \alpha \frac{\sqrt{d}}{d_{K,\sigma}} (u_\sigma - u_K - \nabla_K u \cdot (\mathbf{x}_\sigma - \mathbf{x}_K))$$

$$\nabla_{\mathcal{D}} u(\mathbf{x}) = \nabla_{K,\sigma} u \text{ for a.e. } \mathbf{x} \in D_{K,\sigma}$$



Description of the scheme

$$X_{\mathcal{D},\mathcal{B}} = \{v \in X_{\mathcal{D}}, v_\sigma = 0 \forall \sigma \in \mathcal{E}_{\text{ext}}, v_\sigma = \sum_{K\in\mathcal{M}} \beta_\sigma^K v_K \forall \sigma \in \mathcal{B}\}$$

with $\sum_{K\in\mathcal{M}} \beta_\sigma^K = 1$ and $\mathbf{x}_\sigma = \sum_{K\in\mathcal{M}} \beta_\sigma^K \mathbf{x}_K$.

Discrete unknowns: $u = ((u_K)_{K\in\mathcal{M}}, (u_\sigma)_{\sigma\in\mathcal{H}})$

$$\begin{cases} \text{Find } u \in X_{\mathcal{D},\mathcal{B}} \text{ such that:} \\ \displaystyle\int_\Omega \Lambda \nabla_{\mathcal{D}} u \cdot \nabla_{\mathcal{D}} v \mathrm{d}\mathbf{x} = \int_\Omega f \;\; \Pi_{\mathcal{M}} v \mathrm{d}\mathbf{x}, \; \forall v \in X_{\mathcal{D},\mathcal{B}}. \end{cases}$$

where $\Pi_{\mathcal{M}} v = v_K$ on K.

card($\mathcal{M}$)) + card($\mathcal{H}$) equations and unknowns

If $\Lambda = Id$, with triangles or rectangles $\rightsquigarrow$ classical FV scheme.

Two 鮓 schemes: non-parametric (np) and parametric (p)

鮓-np $\alpha = 1$, $\mathcal{H} = \mathcal{E} \setminus \mathcal{B}$ at the discontinuities of $\mathbf{K}$

鮓-p choose $\alpha > 0$, choose $\mathcal{H} \subset \mathcal{E}_{\text{int}}$.

鮓-p $\mathcal{H} = \mathcal{E}_{\text{int}}$: full hybrid finite volume scheme.

鮓-p with $\mathcal{B} = \mathcal{E}_{\text{int}}$: full barycentric scheme (no edge unknowns).

Results for `Test 1`

`umin`= 0.0, `umax`=$1 + \sin(1)$.

- `Test 1.1` `mesh1` (triangles) $\rightsquigarrow$ **ocvl2**= 2.00, **ocvgradl2**= 1.99
- `Test 1.1` `mesh4_j_i` (distorted quadrangles)

grid	nunkw	nnmat	sumflux	erl2	ergrad
C	289	4419	9.77E-15	8.94E-02	1.31E-01
F	1089	17489	-1.11E-13	2.77E-02	4.77E-02

grid	erflx0	erflx1	erfly0	erfly1	umin	umax
C	2.51E-02	3.81E-02	2.56E-02	3.79E-02	7.64E-03	8.88E-01
F	7.03E-03	1.13E-02	8.00E-03	1.04E-02	2.33E-03	9.61E-01

- `Test 1.2` `mesh1` (triangles) $\rightsquigarrow$**ocvl2**= 2.00, **ocvgradl2**= .98

i	nunkw	nnmat	sumflux	erl2	ergrad	ratiol2	ratiog
1	56	752	-2.22E-15	1.26E-02	1.07E-01	-	-
3	896	14380	9.86E-14	7.87E-04	2.26E-02	1.99	1.05
5	14336	238828	1.48E-11	4.92E-05	5.59E-03	2.00	1.00
7	229376	3854396	1.54E-09	3.14E-06	1.42E-03	1.99	0.99

i	erflx0	erflx1	erfly0	erfly1	umin	umax
1	2.72E-03	3.44E-03	8.90E-03	5.25E-03	4.40E-03	1.36E+00
3	1.50E-04	3.22E-04	5.17E-04	2.57E-04	2.81E-04	1.71E+00
5	6.04E-06	2.99E-05	3.14E-05	9.75E-06	1.76E-05	1.81E+00
7	4.23E-07	1.40E-06	2.53E-06	2.01E-06	1.10E-06	1.83E+00

- `Test 1.2` `mesh3` (loc. ref.) $\rightsquigarrow$ **ocvl2**= 2.02, **ocvgradl2**= 1.46

i	nunkw	nnmat	sumflux	erl2	ergrad	ratiol2	ratiog
1	40	540	2.55E-15	2.92E-02	4.27E-02	-	-
3	640	8484	-2.13E-14	1.73E-03	5.86E-03	2.04	1.44
5	10240	133844	6.30E-12	1.05E-04	7.62E-04	2.02	1.46

i	erflx0	erflx1	erfly0	erfly1	umin	umax
1	1.47E-03	1.29E-02	1.36E-02	8.96E-04	9.03E-03	1.66E+00
3	5.45E-04	4.98E-04	1.09E-03	1.75E-03	6.44E-04	1.79E+00
5	6.44E-05	2.61E-05	7.52E-05	2.38E-04	4.06E-05	1.83E+00

- **Comments** $\mathcal{B} = \mathcal{E}_{\text{int}}$ (full barycentric scheme)

  Maximum principle OK for all tests.

Results for `Test 2` Numerical locking $\delta = 10^6$

`mesh1` (triangles) `umin`= -1, `umax`= 1.

- 鮓-np $\rightsquigarrow$ **ocvl2**= 0.046, **ocvgradl2**= 1.98

i	nunkw	nnmat	sumflux	erl2	ergrad	ratiol2	ratiog	umin	umax
1	72	928	-5.57E-10	7.87E-01	6.46E+00	-	-	-3.13E-01	3.14E-01
2	256	3756	5.42E-11	9.48E-01	2.70E+01	-2.94E-01	-2.26	-7.88E-02	7.88E-02
3	960	15084	4.26E-09	9.87E-01	1.07E+02	-6.09E-02	-2.09	-2.03E-02	2.01E-02
4	3712	60396	6.09E-09	9.96E-01	3.19E+02	-1.35E-02	-1.61	-6.71E-03	6.62E-03
5	14592	241644	2.51E-08	9.96E-01	2.89E+02	-6.68E-04	0.14	-6.48E-03	6.42E-03
6	57856	965692	2.01E-07	9.89E-01	8.99E+01	1.09E-02	1.70	-1.38E-02	1.42E-02
7	230400	3865660	3.48E-06	9.58E-01	2.28E+01	4.60E-02	1.99	-4.39E-02	4.61E-02

- 鮓-p, $\alpha = 1.$, $\mathcal{B} = \emptyset \rightsquigarrow$ **ocvl2**= 2.53, **ocvgradl2**= 1.23.

i	nunkw	nnmat	sumflux	erl2	ergrad	ratiol2	ratiog	umin	umax
1	148	820	2.98E-10	2.61E-01	3.56E-01	-	-	-1.07	1.07
2	576	3264	7.28E-08	1.13E+01	1.00E+00	-5.54	-1.52E+00	-18.6	16.3
3	2272	13024	1.68E-07	2.06E+00	9.98E-01	2.48	1.92E-03	-6.59	6.08
4	9024	52032	3.91E-07	3.16E-01	9.87E-01	2.72	1.63E-02	-1.84	1.75
5	35968	208000	5.61E-08	5.12E-02	9.07E-01	2.63	1.22E-01	-1.06	1.06
6	143616	831744	-3.72E-06	8.66E-03	6.04E-01	2.57	5.89E-01	-1.00	1.00
7	573952	3326464	7.65E-06	1.50E-03	2.58E-01	2.53	1.23E+00	-1.00	1.00

- **Comments** 鮓-np does not perform well: maximum principle always satisfied, but bad approximation. 鮓-p with $\mathcal{B} = \emptyset$, i.e. fully hybrid scheme, with $\alpha = 1$.

  Maximum principle is violated on coarser grids, but convergence of the solution obtained.

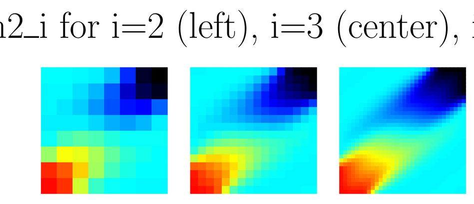
Results for `Test 3` : Oblique flow

- `mesh2` (uniform rectangles) `umin`= 0.0, `umax`= 1.0.

i	nunkw	nnmat	sumflux	umin	umax
1	16	132	-5.27E-16	.06	.940
3	256	3012	-1.95E-14	.011	.989
5	4096	51972	-3.35E-13	.003	.997
7	65536	846852	1.99E-11	.0008	.999

i	flux0	flux1	fluy0	fluy1	ener1	ener2	eren
1	-1.80E-01	1.80E-01	-1.14E-01	1.14E-01	2.25E-01	3.23E-01	3.01E-01
3	-1.87E-01	1.87E-01	-1.06E-01	1.06E-01	2.53E-01	3.01E-01	1.57E-01
5	-1.94E-01	1.94E-01	-9.89E-02	9.89E-02	2.50E-01	2.66E-01	6.15E-02
7	-1.93E-01	1.93E-01	-9.85E-02	9.85E-02	2.43E-01	2.46E-01	1.28E-02

- Solution on mesh2_i for i=2 (left), i=3 (center), i=4 (right)



Solutions for the oblique flow on mesh2_i for i=2 (left), i=3 (center), i=4 (right) white = maximum, black = minimum

- **Comments** $\mathcal{B} = \mathcal{E}_{\text{int}}$ (full barycentric). Maximum principle OK

$$\texttt{ener1} = \sum_{K\in\mathcal{M}} (\int_K \mathbf{K}(\mathbf{x})\mathrm{d}\mathbf{x}) \nabla_{\mathbf{K}} \mathbf{u} \cdot \nabla_{\mathbf{K}} \mathbf{u}$$

$$\texttt{ener2} = -\sum_{\sigma\in\mathcal{E}_{\text{ext}}} F_{K,\sigma}(u) u_\sigma$$

$$\texttt{ener2} - \texttt{ener1} = \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} \mathrm{m}(D_{K,\sigma}) R_{K,\sigma}^2(u) \mathbf{K}_K \mathbf{n}_{K,\sigma} \cdot \mathbf{n}_{K,\sigma} \to 0$$

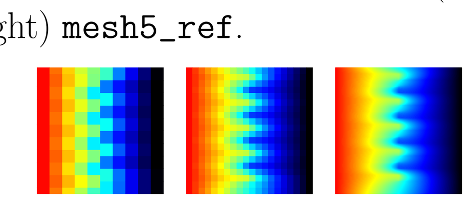
Results for `Test 4` : Vertical fault

- `mesh5` (non conforming rectangles) `umin`= 0.0, `umax`= 1.0

i	nunkw	nnmat	sumflux	umin	umax
1	304	2270	1.82E-13	4.15E-02	9.61E-01
reg	1160	8528	1.79E-13	1.99E-02	9.82E-01
ref	105424	1347242	4.53E-10	1.32E-03	9.99E-01

i	flux0	flux1	fluy0	fluy1	ener1	ener2	eren
1	-40.9	43.1	-2.21	6.94E-04	39.1	41.9	6.67E-02
reg	-39.9	42.6	-2.68	8.01E-04	39.1	41.0	4.83E-02
ref	-42.1	44.4	-2.33	7.97E-04	43.1	43.2	8.88E-04

- Solution for the vertical fault on the meshes: (Left) `mesh5` (center) `mesh5_reg`. (Right) `mesh5_ref`.



- **Comments**

  Maximum principle is respected.

  `ener2` and `ener1` converge

  `ener2` and `ener1` satisfy: `ener2` $\geq$ `ener1`.

Results for `Test 5` : Heterogeneous rotating anisotropy

- `mesh5` (non conforming rectangles) `umin`= 0.0, `umax`= 1.0.

  $\rightsquigarrow$ **ocvl2**= 1.84, **ocvgradl2**= 1.30.

i	nunkw	nnmat	sumflux	erl2	ergrad	ratiol2	ratiog
1	16	132	-1.22E-15	1.47E-01	2.12E-01	1.38	1.12
2	64	676	7.77E-16	4.94E-02	9.72E-02	1.58	1.13
3	256	3012	-5.22E-15	1.64E-02	3.97E-02	1.59	1.29
4	1024	12676	4.44E-15	4.90E-03	1.61E-02	1.75	1.30
5	4096	51972	1.10E-13	1.36E-03	6.55E-03	1.85	1.30

i	erflx0	erflx1	erfly0	erfly1	umin	umax
1	-6.62E-02	9.35E-02	-6.62E-02	9.35E-02	7.76E-02	.921
2	-4.03E-02	4.68E-02	-4.03E-02	4.68E-02	1.64E-02	.970
3	-1.51E-02	1.66E-02	-1.51E-02	1.66E-02	3.50E-03	.992
4	-4.58E-03	4.97E-03	-4.58E-03	4.97E-03	7.79E-04	.998
5	-1.27E-03	1.36E-03	-1.27E-03	1.36E-03	1.80E-04	.999

- **Comments**

  $\mathcal{B} = \mathcal{E}_{\text{int}}$ (full barycentric scheme)

  Maximum principle again OK

  Asymptotic rates of convergence not reached for $i = 5$

Results for `Test 6` and `Test 7`

- `Test 6` Oblique drain, min $= -1.2$, max $= 0$, coarse (C) and fine (F) oblique meshes, `mesh6` and `mesh7`

grid	nunkw	nnmat	sumflux	erl2	ergrad
C	239	2583	-1.16E-13	1.74E-15	1.35E-14
F	319	3627	8.17E-14	3.67E-15	4.24E-14

grid	erflx0	erflx1	erfly0	erfly1	erflm	umin	umax
C	7.61E-15	1.21E-14	8.47E-15	3.05E-15	1.68E-12	-1.15	-5.39E-02
F	1.67E-14	2.99E-15	7.49E-15	2.91E-15	5.38E-12	-1.15	-5.39E-02

- `Test 7` Oblique barrier, min $= -5.575$, max $= 0.575$, coarse oblique mesh `mesh6`

nunkw	nnmat	sumflux	erl2	ergrad
239	2583	-8.97E-14	1.30E-15	3.78E-15

erflx0	erflx1	erfly0	erfly1	erflm	umin	umax
1.28E-13	-1.43E-13	-3.89E-15	3.16E-14	1.77E-13	-5.54	.537

- **Comments**

  As for tests 1, 3, 4, 5, we use 鮓-np.

  For 1, 3, 4, 5, 鮓-np = full barycentric.

  Here 鮓-np partially barycentric: $\mathcal{H} \neq \emptyset$,
  $\mathcal{H}$ corresponds to the interfaces between the subdomains.

  Exact fluxes thanks to the design of the 鮓-np scheme

Results for `Test 8` and `Test 9`

- `Test 8` `mesh8` (perturbed parallelograms), `umin`= 0.0, `umax`= 1.0

	nunkw	nnmat	sumflux	umin	umax	flux0	flux1	fluy0	fluy1
SUSHI-np	121	2011	0.00E+00	-1.19E-03	5.65E-02	7.35E-04	1.29E-04	4.99E-01	5.00E-01
SUSHI-p	121	2011	0.00E+00	3.26E-06	6.77E-03	-4.21E-02	-3.29E-02	5.38E-01	5.37E-01

  Comments

  $\mathcal{B} = \mathcal{E}_{\text{int}}$ (full barycentric scheme)

  鮓-np slight violation of the maximum principle.

  鮓-p scheme, $\mathcal{B} = \mathcal{E}_{\text{int}}$, $\alpha \geq 85$: maximum principle OK, but solution damped as the center.

- `Test 9` Anisotropy & wells `mesh9` (squares) `umin`= 0.0, `umax`= 1.0

	nunkw	nnmat	sumflux	umin	umax
SUSHI-np	165	1745	-3.65E-16	-1.00E+00	2.00E+00
SUSHI-p	385	2805	-8.47E-16	0.00E+00	1.00E+00

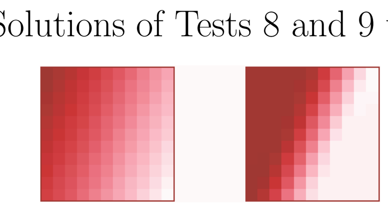
  Comments

  $\mathcal{B} = \mathcal{E}_{\text{int}}$ (full barycentric scheme)

  鮓-np $\rightsquigarrow$ violation of the maximum principle.

  Max principle obtained with 鮓-p scheme, $\mathcal{B} = \emptyset$, $\alpha = .3$ or 鮓-p scheme, $\mathcal{B} = \mathcal{E}_{\text{int}}$ $\alpha = .005$, but larger inner oscillations in this latter case.

- Solutions of Tests 8 and 9 with 鮓-np (left) and 鮓-p (right)



Test 8

Test 9