# AN ADAPTIVE FINITE VOLUME METHOD FOR NONSTATIONARY PROBLEMS


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# EULER EQUATIONS

$$\frac{\partial \boldsymbol{w}}{\partial t} + \sum_{s=1}^{N} \frac{\partial \boldsymbol{f}_s(\boldsymbol{w})}{\partial x_s} = 0 \quad \text{in } Q_T = \Omega \times (0,T)$$

$$\boldsymbol{w}(x,0) = \boldsymbol{w}^0(x), \qquad x \in \Omega,$$

$$B(\boldsymbol{w}(x,t)) = 0 \quad \text{for } (x,t) \in \partial\Omega \times (0,T).$$

# NAVIER–STOKES EQUATIONS

$$\frac{\partial \boldsymbol{w}}{\partial t} + \sum_{i=1}^{N} \frac{\partial \boldsymbol{f}_i(\boldsymbol{w})}{\partial x_i} = \boldsymbol{F}(\boldsymbol{w}) + \sum_{i=1}^{N} \frac{\partial \boldsymbol{R}_i(\boldsymbol{w}, \nabla \boldsymbol{w})}{\partial x_i}$$

a) Triangular mesh

b) Quadrilateral mesh

c) Dual mesh over a triangular grid

d) Barycentric mesh over a triangular grid

# FINITE VOLUME SCHEME

$0 = t_0 < t_1 < \ldots$ partition of $[0,T]$, $\qquad \tau_k = t_{k+1} - t_k$

$$\frac{\partial \boldsymbol{w}}{\partial t} + \sum_{s=1}^{N} \frac{\partial \boldsymbol{f}_s(\boldsymbol{w})}{\partial x_s} = 0 \qquad \int_{D_i}\int_{t_k}^{t_k+1}, \text{ Green}$$

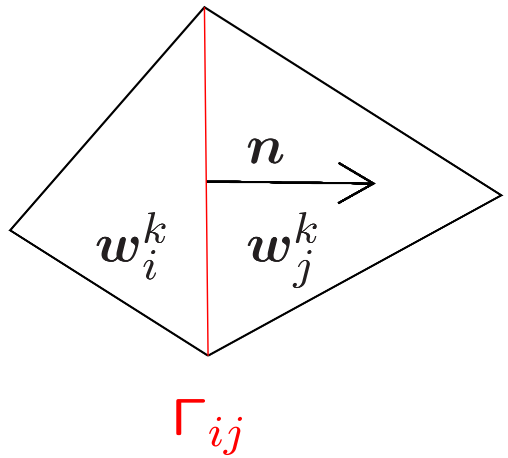
$$\boldsymbol{w}(\cdot, t_k)|_{D_i} \approx \boldsymbol{w}_i^k \in I\!R^{N+2}$$

$$\frac{\boldsymbol{w}_i^{k+1} - \boldsymbol{w}_i^k}{\tau_k} + \frac{1}{|D_i|}\sum_j \boldsymbol{H}(\boldsymbol{w}_i^k, \boldsymbol{w}_j^k, \boldsymbol{n}_{ij})|\Gamma_{ij}| = 0$$

FLUX $\qquad$ NUMERICAL FLUX

$$\sum_{s=1}^{N} \boldsymbol{n}_s \boldsymbol{f}_s(\boldsymbol{w}(\cdot, t_k)|_{\Gamma_{ij}} \quad \approx \quad \boldsymbol{H}(\boldsymbol{w}_i^k, \boldsymbol{w}_j^k, \boldsymbol{n}_{ij})$$

# SECOND ORDER - ADER




$$\int_{t_k}^{t_{k+1}} \int_{\Gamma_{ij}} \underbrace{\sum_{s=1}^{N} n_s \boldsymbol{f}_s(\boldsymbol{w})}_{\mathbb{Q}^{-1}\boldsymbol{f}_1(\boldsymbol{q})} \, dS dt \approx \tau_k \, \widehat{\boldsymbol{H}}(\boldsymbol{w}_i^k, \boldsymbol{w}_j^k, \boldsymbol{n}_{ij}) \, |\Gamma_{ij}|$$

$$\frac{\partial \boldsymbol{q}}{\partial t} + \frac{\partial \boldsymbol{f}_1(\boldsymbol{q})}{\partial \tilde{x}_1} = 0$$

$$\boldsymbol{q}(\tilde{x}_1, 0) = \begin{cases} \widehat{\boldsymbol{q}}_L(\tilde{x}_1), & \tilde{x}_1 < 0 \\ \widehat{\boldsymbol{q}}_R(\tilde{x}_1), & \tilde{x}_1 > 0 \end{cases}$$

## INVISCID BURGERS EQUATION

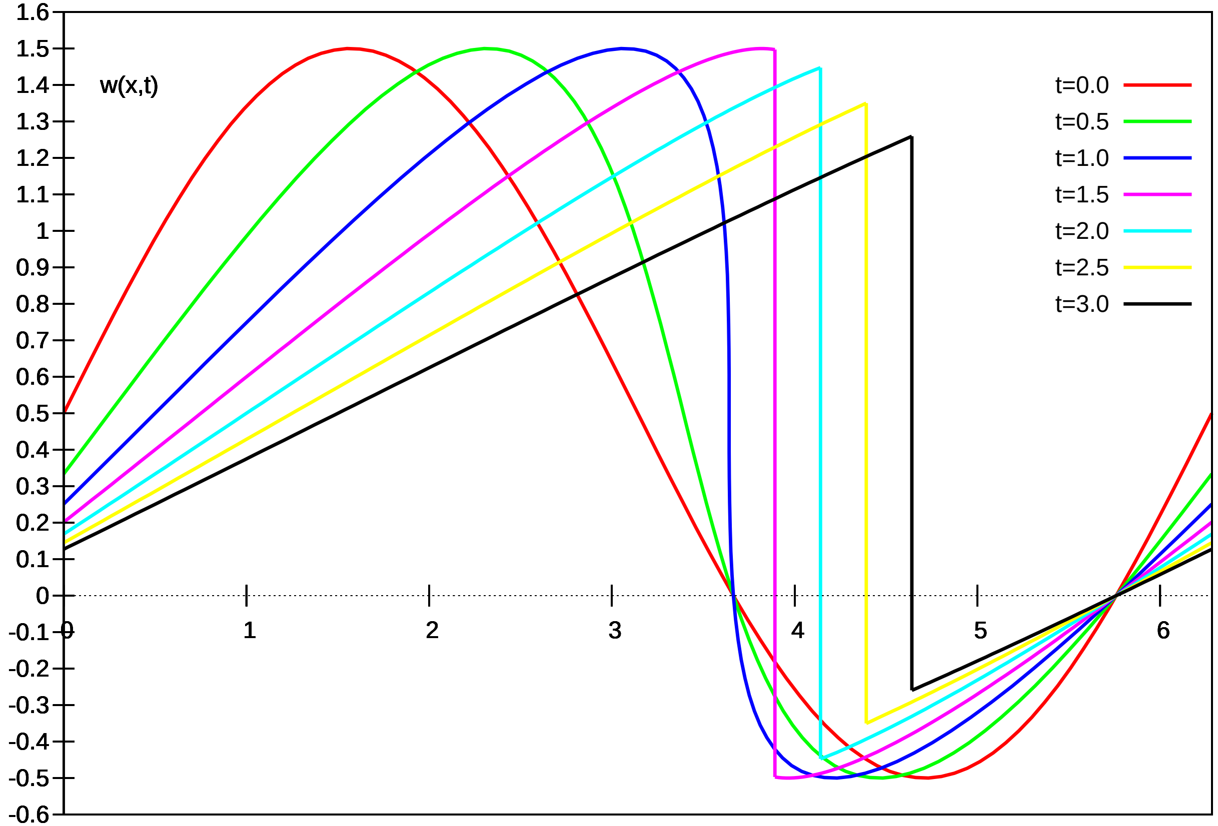
$$\frac{\partial w}{\partial t} + \frac{\partial f(w)}{\partial x} = 0 \quad \text{in } (a,b) \times (0,T) \qquad f(w) = \frac{1}{2} w^2$$

$$w(x,0) = w^0(x), \quad x \in (a,b),$$

$$w(a,t) = w^a(t), w(b,t) = w^b(t) \quad t \in (0,T)$$

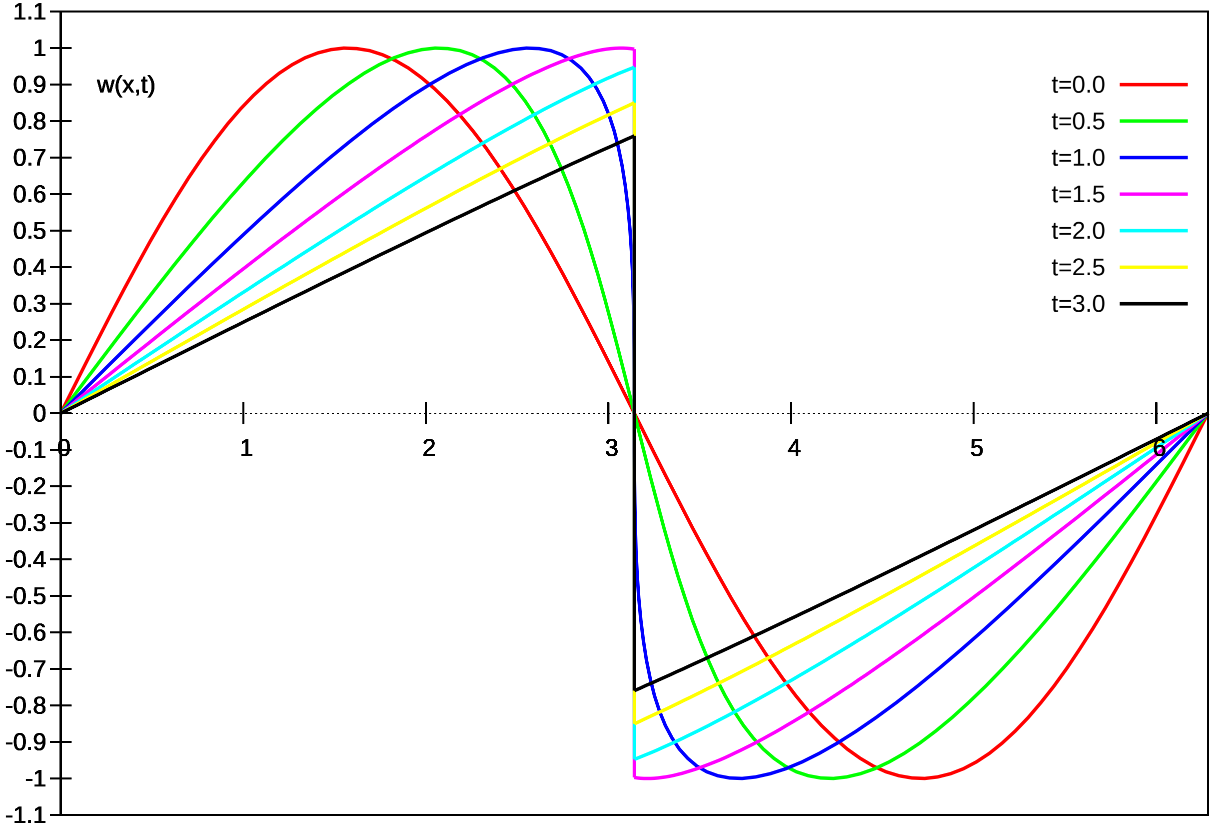
Momentum equation for the 1D isothermal Euler equations, where density variations are neglected

# INITIAL CONDITION $w^0(x) = 0.5 + \sin(x)$



INITIAL CONDITION $w^0(x) = \sin(x)$

zero speed shock



# IMPROVEMENT OF THE FV SCHEME

$$\boldsymbol{w}_i^{k+1} = \boldsymbol{w}_i^k - \frac{\tau_k}{|D_i|} \sum_j \boldsymbol{H}(\boldsymbol{w}_i^k, \boldsymbol{w}_j^k, \boldsymbol{n}_{ij}) |\Gamma_{ij}|$$

| | |
|---|---|
| numerical flux $H(\boldsymbol{w}_i, \boldsymbol{w}_j, \boldsymbol{n}_{ij})$ | OSHER–SOLOMON<br>ROE<br>HLL<br>HLLC (C = contact)<br>flux schemes (Ghidaglia)<br>VFRoe-ncv (Gallouët) |
| adaptivity | refinement<br>coarsening<br>alignment |
| higher order | recovery strategy<br>Discontinuous Galerkin<br>ADER |

# MESH ADAPTATION FOR NONSTATIONARY PROBLEMS

## AMA - anisotropic, RES - residual based

- PREDICTION PART

  1. Prediction: $\mathbf{w}^{k+1}_{\mathcal{D}^k} := \mathrm{FVsol}\left(\mathbf{w}^k_{\mathcal{D}^k}, \mathcal{D}^k\right)$

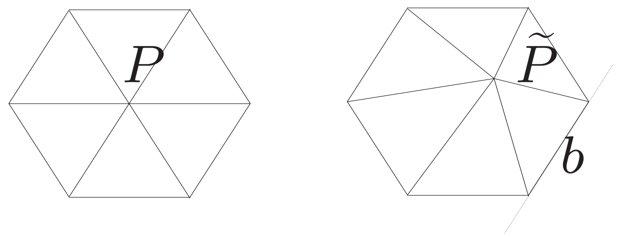
  2. Adaptation: $\mathcal{D}^{k+1} := \mathrm{MeshAdapt}\left(\mathcal{D}^k, \mathbf{w}^{k+1}_{\mathcal{D}^k}\right)$ (AMA)
     $:= \mathrm{MeshAdapt}\left(\mathcal{D}^k, \mathbf{w}^{k+1}_{\mathcal{D}^k}, \mathbf{w}^k_{\mathcal{D}^k}\right)$ (RES)

  3. Recovery: $\tilde{\mathbf{w}}^k_{\mathcal{D}^{k+1}} := \mathrm{SolRecovery}\left(\mathbf{w}^k_{\mathcal{D}^k}, \mathcal{D}^k, \mathcal{D}^{k+1}\right)$ (GMCL)

- PDE EVOLUTION PART

  4. Update: $\mathbf{w}^{k+1}_{\mathcal{D}^{k+1}} := \mathrm{FVsol}\left(\tilde{\mathbf{w}}^k_{\mathcal{D}^{k+1}}, \mathcal{D}^{k+1}\right)$ (Finite volume method)

# MOVING A VERTEX




Admissible set for the vertex motion $\mathcal{K}_P = \bigcup\limits_{\substack{D \in \mathcal{D}^k \\ D \ni P}} D$

Quality parameter of the vertex $P$

$$Q_P(x) \longrightarrow \text{min}$$

$\mathcal{E}_P$ - the set of local edges $\ni P$

# ANISOTROPIC VERTEX QUALITY PARAMETER

http://www.karlin.mff.cuni.cz/~dolejsi/angen/angen3.1.htm

$\sigma^k = \{P_i; i \in I\}$ set of all vertices

$\mathbb{M}(P_i)$ approximate Hessian matrix related to each vertex $P_i$

$$\mathbb{M}(P_i) = \overline{c}\left[\mathbb{I} + \overline{\alpha}\left(\|\overline{\mathbb{H}}(P_i)\|\right)\overline{\mathbb{H}}(P_i)\right].$$

1. $\mathbf{w}^{k+1}_{\mathcal{D}^k}$ is used for the construction of $\overline{\mathbb{H}}(P_i)$

2. $\overline{\mathbb{H}}(P_i)$ is modified to control

   - the number of new tetrahedra (parameter $\overline{c}$)
   - the ratio edge$_{\text{max}}$/edge$_{\text{min}}$ (parameter $\overline{\alpha}$)
   - the transition coarse $\longrightarrow$ fine (parameter $\overline{\alpha}$)

# RIEMANN NORM of an edge $e$

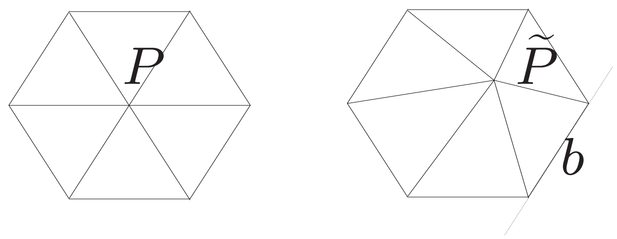
$$\|e\|_{\mathbf{w}^{k+1}_{\mathcal{D}^k}} := \boldsymbol{e}^{\mathsf{T}} \mathbb{M}_e \boldsymbol{e}$$

$$\mathbb{M}_e := (\mathbb{M}(P_i) + \mathbb{M}(P_j))/2$$

## Vertex quality parameter

$$Q_P(x) = \sum_{e \in \mathcal{E}_P} \left( \|e\|_{\mathbf{w}^{k+1}_{\mathcal{D}^k}} - c_N \right)^2, \qquad c_N = \sqrt{3} \text{ for } N = 2$$

$\mathcal{E}_P$ - the set of local edges $\ni P$



# RESIDUAL VERTEX QUALITY PARAMETER

$$\|\boldsymbol{r}^k\|_{\boldsymbol{H}^{-1}(\Omega)} \approx \left( \sum_{D_i \in \mathcal{D}^k} (g_i^k)^2 \right)^{1/2}, \; g_i^k = \left( \sum_{\ell} (g_i^k)_\ell^2 \right)^{1/2}, \; (g_i^k)_\ell = \max_{j \in s(i)} (g_{ij}^k)_\ell$$

and

$$(g_{ij}^k)_\ell = \frac{1}{|\varphi_{ij}|_{H_0^1(\Omega)}} \Bigg| (\boldsymbol{w}_{i\ell}^{k+1} - \boldsymbol{w}_{i\ell}^k) \frac{1}{4} |D_i|$$
$$+ (\boldsymbol{w}_{j\ell}^{k+1} - \boldsymbol{w}_{j\ell}^k) \frac{1}{4} |D_j| - \frac{\tau_k |\Gamma_{ij}|}{2} \sum_{s=1}^{2} (n_{ij})_s \left( f_{s\ell}(\boldsymbol{w}_i^k) - f_{s\ell}(\boldsymbol{w}_j^k) \right) \Bigg| .$$

$$R_i := \sqrt{1 + \tilde{\alpha} (g_i^k)^2}$$

$$Q_P(x) = \textstyle\sum_{i \in \mathcal{K}_P} \left( |D_i| - \frac{\overline{R}}{R_i} |\overline{D}| \right)^2$$

$$\overline{R} = (1/\#\mathcal{D}^k) \textstyle\sum_{D_i \in \mathcal{D}^k} R_i, \qquad \overline{D} = |\Omega| / \#\mathcal{D}^k$$

## RESIDUAL ERROR INDICATOR FOR THE NONSTATIONARY EULER EQUATIONS

$$\frac{\partial \boldsymbol{w}}{\partial t} + \sum_{s=1}^{N} \frac{\partial \boldsymbol{f}_s(\boldsymbol{w})}{\partial x_s} = 0$$

FV solution - $\mathbf{w}_h$ - piecewise constant vector valued function at time $t_k$

$$\boldsymbol{w}(\cdot, t_k)|_{D_i} \approx \boldsymbol{w}_i^k, \qquad \mathbf{w}_h^k|_{D_i} = \boldsymbol{w}_i^k$$

Residual = ?

Weak formulation $/ \cdot \boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_{N+2})^\mathsf{T} \in H_0^1(\Omega)^{N+2}$

$$\int_{t_k}^{t_{k+1}} \qquad \int_\Omega \qquad \text{Green}$$

$$\begin{aligned} 0 &= \int_\Omega (\boldsymbol{w}_\ell(x, t_{k+1}) - \boldsymbol{w}_\ell(x, t_k)) \boldsymbol{\varphi}_\ell \, dx \\ &- \int_{t_k}^{t_{k+1}} \int_\Omega \sum_{s=1}^{N} \boldsymbol{f}_{s\ell}(\boldsymbol{w}) \frac{\partial \boldsymbol{\varphi}_\ell}{\partial x_s} \, dx \, dt \\ &=: \boldsymbol{a}_\ell^k(\boldsymbol{w}, \boldsymbol{\varphi}). \end{aligned}$$

Component-wise

$$a_\ell^k(\boldsymbol{w}, \boldsymbol{\varphi}) = 0, \qquad \ell = 1, \dots, N+2.$$

Riesz theorem ($\ell, \boldsymbol{w}$ fixed)

$$H_0^1(\Omega) \ni \varphi_\ell \longrightarrow a_\ell^k(\boldsymbol{w}, \boldsymbol{\varphi})$$

continuous linear functional

$$a_\ell^k(\boldsymbol{w}, \boldsymbol{\varphi}) = 0 \iff \left(\mathcal{A}_\ell^k(\boldsymbol{w}), \varphi_\ell\right) = 0$$

Nonstationary Euler equations $\qquad \boldsymbol{\mathcal{A}}^k(\boldsymbol{w}) = 0$

Residual $\qquad \boldsymbol{\mathcal{A}}^k(\mathbf{w}_h^k) = \boldsymbol{r}_h^k$

$$\mathbf{w}_h(\cdot, t)|_{D_i} = \boldsymbol{w}_i^k, \; t \in [t_k, t_{k+1})$$

## The norm of the residual

$$\|\boldsymbol{r}_h^k\|_{\boldsymbol{H}^{-1}(\Omega)} = \left( \sum_{\ell=1}^{m} \|r_{h\ell}^k\|_{H^{-1}(\Omega)}^2 \right)^{1/2}$$

$$\|r_{h\ell}^k\|_{H^{-1}(\Omega)} = \sup_{\substack{\varphi \neq 0 \\ \varphi \in H_0^1(\Omega)}} \frac{\left|(r_{h\ell}^k, \varphi)\right|}{\|\varphi\|_{H_0^1(\Omega)}}.$$

## The aim

$$\|\boldsymbol{r}_h^k\|_{\boldsymbol{H}^{-1}(\Omega)} \approx \left( \sum_{D_i \in \mathcal{D}_h} (g_i^k)^2 \right)^{1/2}$$

$g_i^k$ - residual error indicator

$$g_i^k = h_i * \text{something}$$

# Approximation of the norm of the residual

$$
\begin{aligned}
\|r_{h\ell}^k\|_{H^{-1}(\Omega)} &\approx \max_{\varphi_{ij}} \frac{\left|(r_{h\ell}^k, \varphi_{ij})\right|}{\|\varphi_{ij}\|_{H_0^1(\Omega)}} \\
&= \max_{D_i \in \mathcal{D}_h} \underbrace{\max_{j \in s(i)} (g_{ij}^k)_\ell}_{(g_i^k)_\ell} \\
&\le \left( \sum_{D_i \in \mathcal{D}_h} (g_i^k)_\ell^2 \right)^{1/2}
\end{aligned}
$$

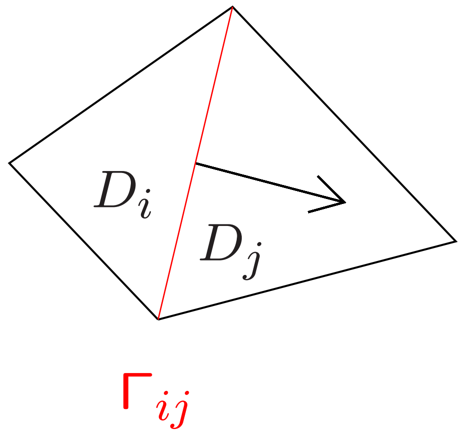
## Conclusion

$$\begin{aligned}
\|\boldsymbol{r}_h^k\|^2 &= \sum_\ell \|r_{h\ell}^k\|^2_{H^{-1}(\Omega)} \\
&\approx \sum_\ell \sum_{D_i \in \mathcal{D}_h} (g_i^k)_\ell^2 \\
&\approx \sum_{D_i \in \mathcal{D}_h} \underbrace{\sum_\ell (g_i^k)_\ell^2} \\
&\approx \sum_{D_i \in \mathcal{D}_h} (g_i^k)^2
\end{aligned}$$

$g_i^k$ - residual error indicator

$$g_i^k = \left( \sum_\ell (g_i^k)_\ell^2 \right)^{1/2}$$

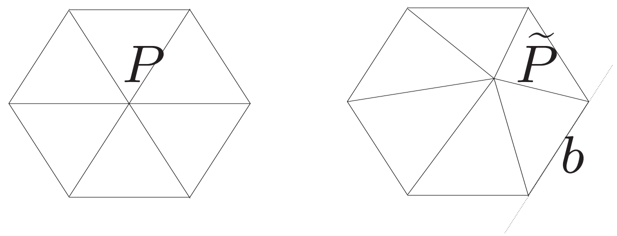
$$(g_i^k)_\ell = \max_{j \in s(i)} (g_{ij}^k)_\ell$$




Two dimensional case

$$
\begin{aligned}
(g_{ij}^k)_\ell &= \frac{1}{|\varphi_{ij}|_{H_0^1(\Omega)}} \bigg| (\boldsymbol{w}_{i\ell}^{k+1} - \boldsymbol{w}_{i\ell}^{k})\frac{1}{4}|D_i| + (\boldsymbol{w}_{j\ell}^{k+1} - \boldsymbol{w}_{j\ell}^{k})\frac{1}{4}|D_j| \\
&- \frac{\tau_k|\Gamma_{ij}|}{2}\sum_{s=1}^{2}(n_{ij})_s\,(f_{s\ell}(\boldsymbol{w}_i^k) - f_{s\ell}(\boldsymbol{w}_j^k))\bigg|
\end{aligned}
$$

# MOVING A VERTEX




Admissible set for the vertex motion $\mathcal{K}_P = \bigcup_{\substack{D \in \mathcal{D}^k \\ D \ni P}} D$

Anisotropic vertex quality parameter of the vertex $P$

$$Q_P(x) = \sum_{e \in \mathcal{E}_P} \left( \|e\|_{\mathbf{w}^{k+1}_{\mathcal{D}^k}} - c_N \right)^2, \qquad c_N = \sqrt{3} \text{ for } N = 2$$

Residual vertex quality parameter of the vertex $P$

$$Q_P(x) = \sum_{i \in \mathcal{K}_P} \left( |D_i| - \frac{\overline{R}}{R_i} |\overline{D}| \right)^2$$

## REMARK

$$\|r_{h\ell}^k\|_{H^{-1}(\Omega)} \approx \max_{\varphi_{ij}} \frac{\left|(r_{h\ell}^k, \varphi_{ij})\right|}{\|\varphi_{ij}\|_{H_0^1(\Omega)}}$$

$$= \max_{D_i \in \mathcal{D}_h} \max_{j \in s(i)} (g_{ij}^k)_\ell$$

$$Q_P(x) = \sum_{\Gamma_{ij} \ni P} g_{ij}^k$$

$$g_{ij}^k = \left( \sum_\ell (g_{ij}^k)_\ell^2 \right)^{1/2}$$

## MINIMIZATION OF THE VERTEX QUALITY PARAMETER $Q_P(x)$

## BARRIER FUNCTION METHOD

$$\Phi_P(x,\alpha) := \alpha Q_P(x) + B_P(x) \longrightarrow \text{min}$$

$\alpha > 0$ - weighting parameter

- $B_P(x)$ - barrier function
  - $\lim\limits_{x \to y \in \partial \mathcal{K}_P} B_P(x) \to \infty$
  - $B_P(x) := \sum_{b \subset \partial \mathcal{K}_P} \frac{1}{\text{dist}(x,b)}$

1. Set $\widetilde{P} := P$ and $\alpha := 1$ or $\alpha := B_P(P)/Q_P(P)$

2. Find $\widetilde{P} = \arg \min_{x \in \mathcal{K}_P} \Phi_P(x, \alpha)$

   (BFGS quasi-Newton method)

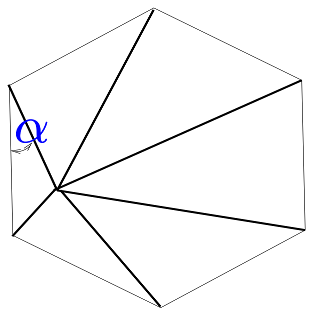
Step 2 is repeated with the increasing parameter $\alpha := \alpha\beta, \beta = 2$,

- Stopping criterion
  - prescribed number of repetitions (10)
  - decrease of $Q_P(\widetilde{P})$

# ANGLE CONTROL METHOD

## Angle control

- Function minimization: $\Phi_P(x,\alpha,\alpha_{min}) := \frac{Q_P(x)}{\Psi(\alpha,\alpha_{min})}$
- Prescribed minimal angle : $\alpha_{min}$
- Minimal angle in $D_i$ associated with $P$ : $\alpha = \min_{D_i \in \mathcal{K}_P} \min_{\beta \in D_i} (\beta)$



## Properties of $\Psi(\alpha, \alpha_{min})$

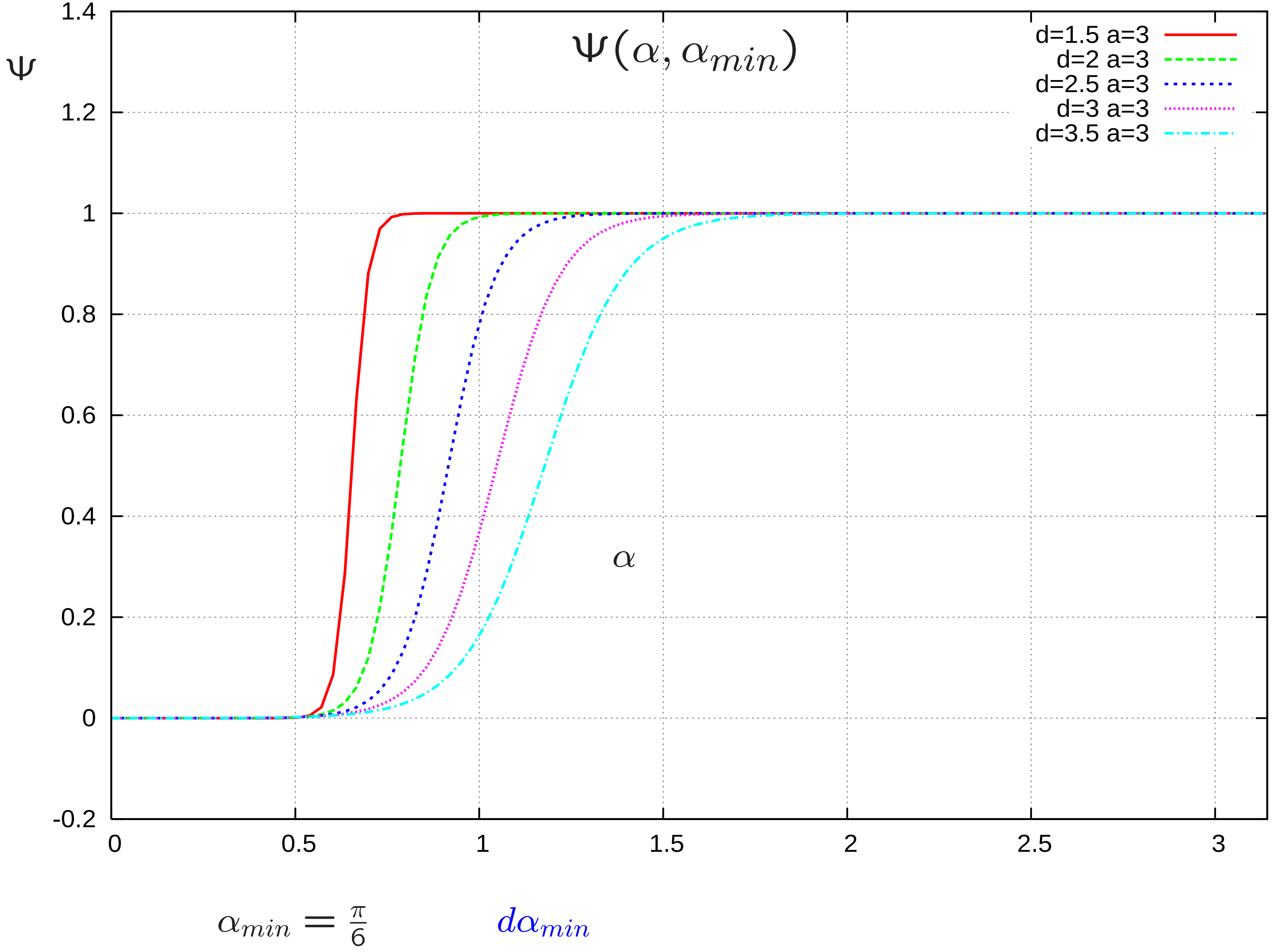
- $\Psi(\alpha, \alpha_{min}) \longrightarrow 0$ if $\alpha \longrightarrow \alpha_{min}$

- $\Psi(\alpha, \alpha_{min}) \longrightarrow 1$ if $\alpha \longrightarrow d\alpha_{min}$ where $d > 1$

Use of $tanh(x)$: $\Psi(\alpha, \alpha_{min}) := \frac{tanh(\frac{a}{(d-1)*\alpha_{min}}(2\alpha - \alpha_{min}(d+1))) + 1}{2}$

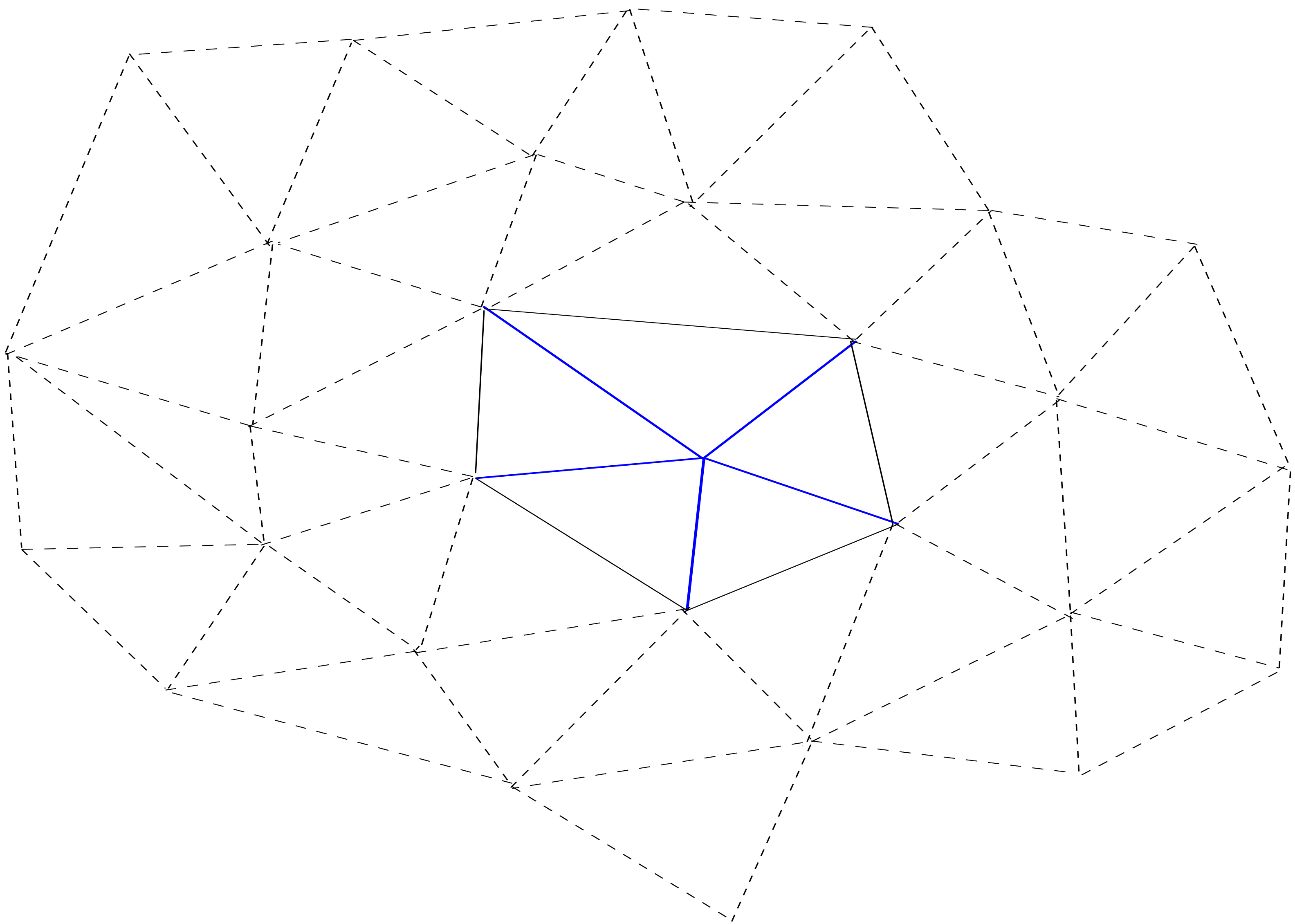
- d - transition between 0 and 1

- a - influences value $\Psi(\alpha_{min}, \alpha_{min})$


$\Psi$
$\Psi(\alpha, \alpha_{min})$
d=1.5 a=3
d=2 a=3
d=2.5 a=3
d=3 a=3
d=3.5 a=3
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
0
0.5
1
1.5
2
2.5
3
$\alpha$
$\alpha_{min} = \frac{\pi}{6}$
$d\alpha_{min}$

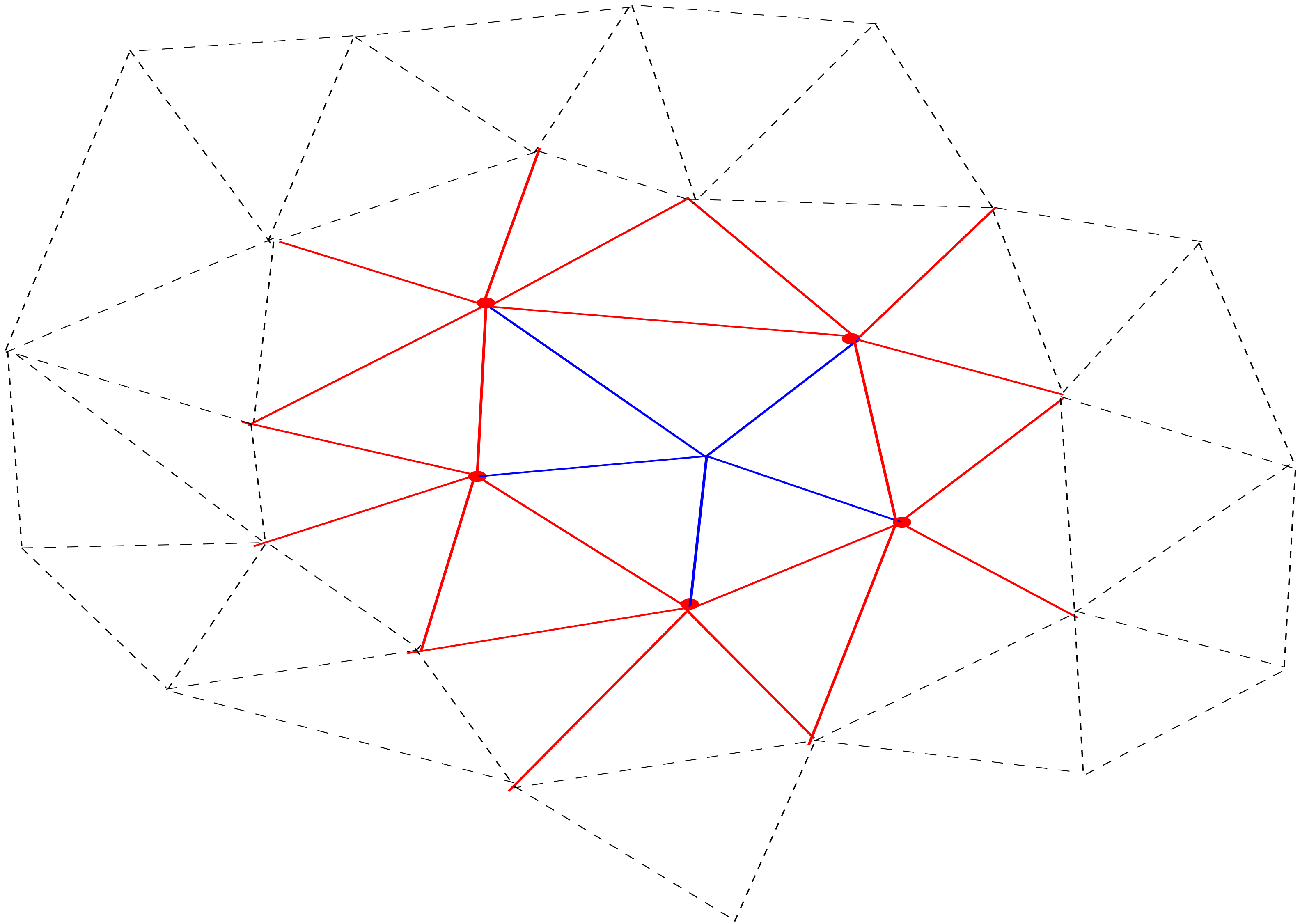
Level 0 Quality Parameter of a Vertex $P$:

$$LQ(P, 0)(x) = Q_P(x)$$



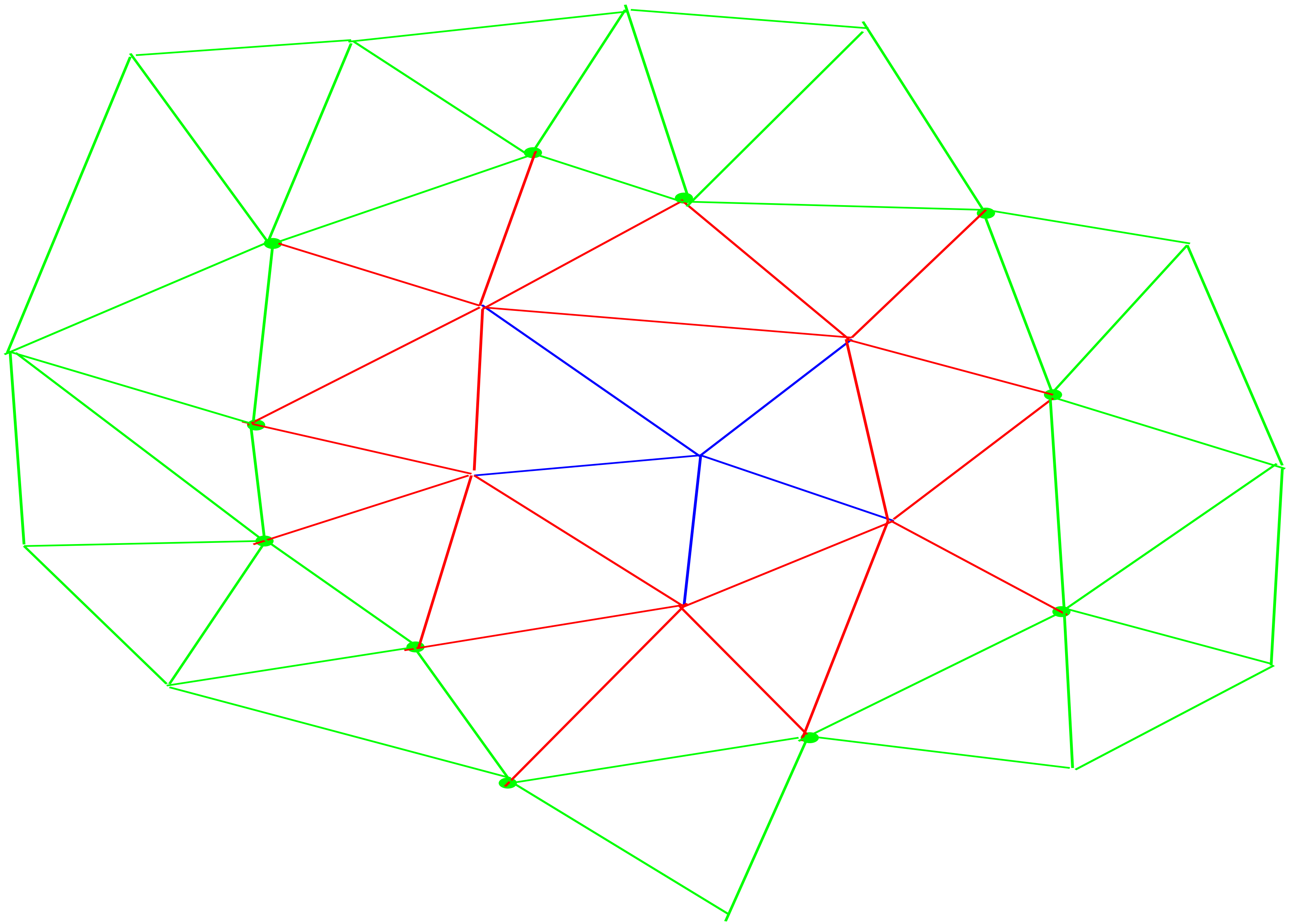
Level 1 Quality Parameter of a Vertex $P_i$:

$$LQ(P_i, 1)(x) = LQ(P_i, 0)(x) + \sum Q_{P_j}(x)$$



Level 2 Quality Parameter of a Vertex $P_i$:

$$LQ(P_i, 2)(x) = LQ(P_i, 1)(x) + \sum Q_{P_j}(x)$$



# Angle control algorithm

1. Compute average $\overline{LQ} = \frac{\sum_i LQ(P_i,1)(x)}{npoin}$

2. IF $LQ(P_i, 1)(x) > \overline{LQ}$ find $\widetilde{P} =$ arg $\min_{x \in \mathcal{K}_P} \Phi_P(x, \alpha, \alpha_{min})$

   - Do STEP 2 for red and green neighbours of $P_i$

- Stopping criterion
  - prescribed number of repetitions $\approx 5$
  - decrease of $Q_{\mathcal{D}_h}$

## GEOMETRIC MASS CONSERVATION LAW

Recompute the solution $\mathbf{w}^k_{\mathcal{D}^k}$ to its recovery $\tilde{\mathbf{w}}^k_{\mathcal{D}^{k+1}}$

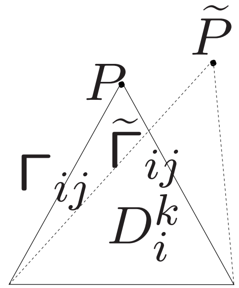
Define $\tilde{\mathbf{w}}^k_{\mathcal{D}^{k+1}}$

$$\sum_i \left| D_i^{k+1} \right| \widetilde{\boldsymbol{w}}_i^k = \sum_i \left| D_i^k \right| \boldsymbol{w}_i^k,$$

$\widetilde{\boldsymbol{w}}_i^k = \tilde{\mathbf{w}}^k_{\mathcal{D}^{k+1}}\Big|_{\overset{\circ}{D_i^{k+1}}}$. ($\circ$ denotes the interior of $D_i^{k+1}$.)

Moving vertex adaptation - perturbation method

# PERTURBATION METHOD




Mapping $D_i^k \longrightarrow D_i^{k+1}$

$$x \longrightarrow \widetilde{x} := x - \boldsymbol{c}(x)$$

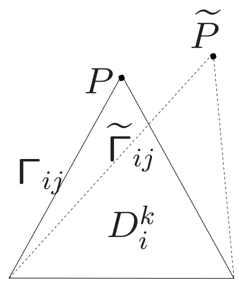
$\boldsymbol{c}$ linear, given by the displacement of vertices of $D_i^k$, "small"

$$\int_{D_i^{k+1}} \boldsymbol{w}(\widetilde{x})\, d\widetilde{x} \approx \int_{D_i^k} \boldsymbol{w}\, dx - \int_{\partial D_i^k} \boldsymbol{w} c_n\, dS,$$

(Higher order terms neglected)

Passage to volume averages and the approximation of $\int_{\partial D_i^k}$

$$\left|D_i^{k+1}\right|\widetilde{\boldsymbol{w}}_i^k = \left|D_i^k\right|\boldsymbol{w}_i^k - \sum_{j\in s(i)}\left|\Gamma_{ij}\right|\left(c_{n_{ij}}^{+}\boldsymbol{w}_i^k + c_{n_{ij}}^{-}\boldsymbol{w}_j^k\right).$$




$$\int_{\Gamma_{ij}} \boldsymbol{w} c_n \, dS \approx |\Gamma_{ij}| c_{n_{ij}} \boldsymbol{w}_i^k.$$

Equivalently

$$\int_{\Gamma_{ij}} \boldsymbol{w} c_n \, dS \approx \begin{cases} |\Gamma_{ij}| c_{n_{ij}} \boldsymbol{w}_i^k & \text{if} \quad \widetilde{e}_{ij} \in D_i^k, \\ |\Gamma_{ij}| c_{n_{ij}} \boldsymbol{w}_j^k & \text{if} \quad \widetilde{e}_{ij} \in D_j^k, \end{cases}$$

$\widetilde{e}_{ij}$ - the center of gravity of $\widetilde{\Gamma}_{ij}$

# PERTURBATION METHOD (details)

$$\tilde{x} = x - \boldsymbol{c}(x)$$

with the Jacobian

$$J(x) = \det\frac{D\tilde{x}}{Dx} = \det\begin{pmatrix} 1 - \dfrac{\partial c_1}{\partial x_1}, & \dfrac{\partial c_1}{\partial x_2} \\ \dfrac{\partial c_2}{\partial x_1}, & 1 - \dfrac{\partial c_2}{\partial x_2} \end{pmatrix}$$

Supposed that the displacement $\boldsymbol{c}$ is small we can write

$$\begin{aligned}
\int_{D_i^{k+1}} \boldsymbol{w}(\tilde{x})\, d\tilde{x} &= \int_{D_i^k} \boldsymbol{w}(x - \boldsymbol{c}(x)) J(x)\, dx \\
&= \int_{D_i^k} \boldsymbol{w}(x - \boldsymbol{c}(x))(1 - \mathrm{div}\boldsymbol{c}(x) + \mathcal{O})\, dx \\
&= \int_{D_i^k} (\boldsymbol{w} - \nabla\boldsymbol{w} \cdot \boldsymbol{c} + \mathcal{O})(1 - \mathrm{div}\boldsymbol{c}(x) + \mathcal{O})\, dx \\
&= \int_{D_i^k} (\boldsymbol{w} - \boldsymbol{w}\mathrm{div}\boldsymbol{c} - \nabla\boldsymbol{w} \cdot \boldsymbol{c} + (\nabla\boldsymbol{w} \cdot \boldsymbol{c})\mathrm{div}\, \boldsymbol{c} + \mathcal{O})\, dx \\
&\approx \int_{D_i^k} (\boldsymbol{w} - \mathrm{div}(\boldsymbol{w}\boldsymbol{c}))\, dx
\end{aligned}$$

$$\int_{D_i^{k+1}} \boldsymbol{w}(\widetilde{x})\, d\widetilde{x} \approx \int_{D_i^k} \boldsymbol{w}\, dx - \int_{\partial D_i^k} \boldsymbol{w} c_n\, dS$$

$$\left|D_i^{k+1}\right| \widetilde{\boldsymbol{w}}_i^k = \left|D_i^k\right| \boldsymbol{w}_i^k - \sum_{j \in s(i)} \left|\Gamma_{ij}\right| \left(c_{n_{ij}}^+ \boldsymbol{w}_i^k + c_{n_{ij}}^- \boldsymbol{w}_j^k\right).$$

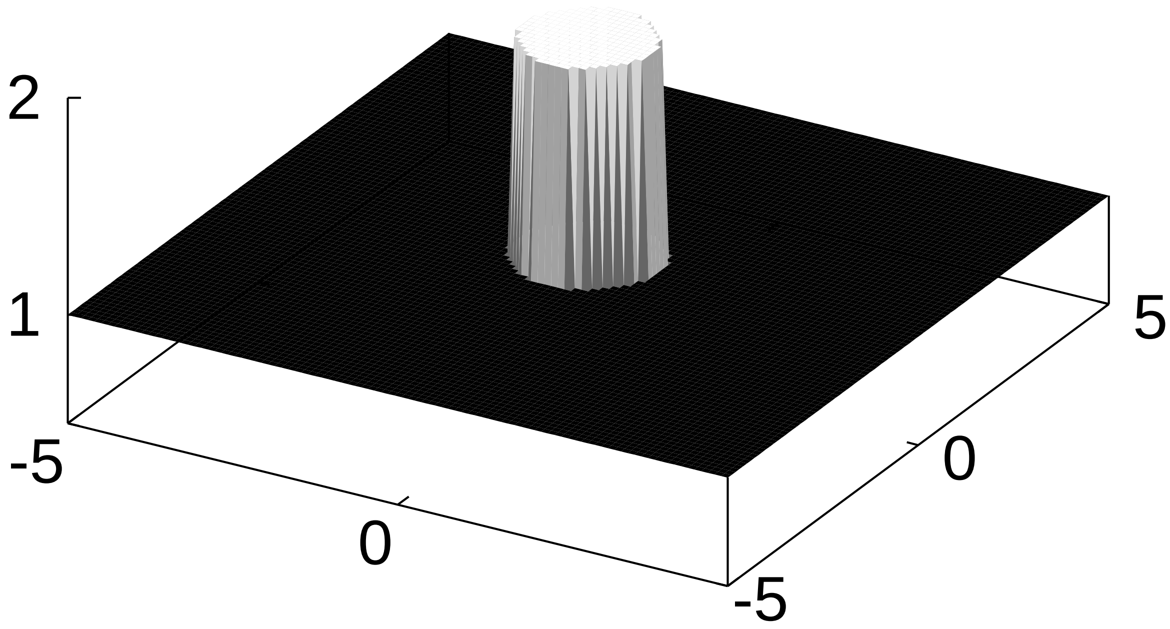
# EXAMPLE

Euler equations in $\Omega_h = (-5, 5) \times (-5, 5)$

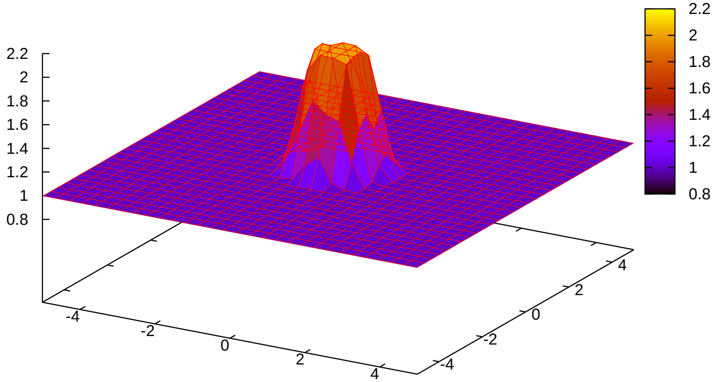
BC: channel flow (left to right)

$$\text{IC: } \boldsymbol{w}^0(x) = \begin{cases} (2, 0, 0, 1)^{\mathsf{T}} & \text{if } x_1^2 + x_2^2 \leq 1, \\ (1, 0, 0, .5)^{\mathsf{T}} & \text{if } x_1^2 + x_2^2 > 1. \end{cases}$$

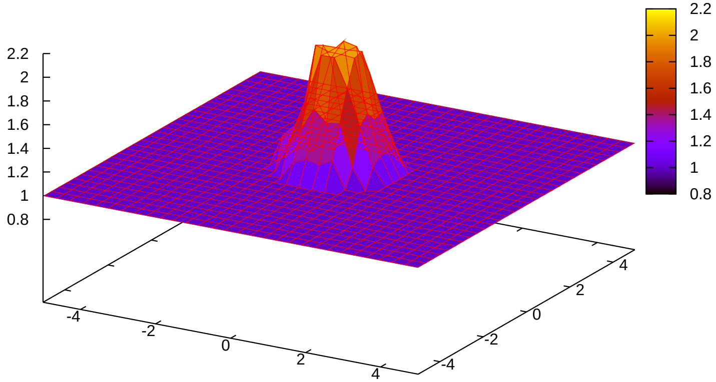
$$\boldsymbol{w} = (\rho, \rho v_1, \ldots, \rho v_N, E)^{\mathsf{T}}$$



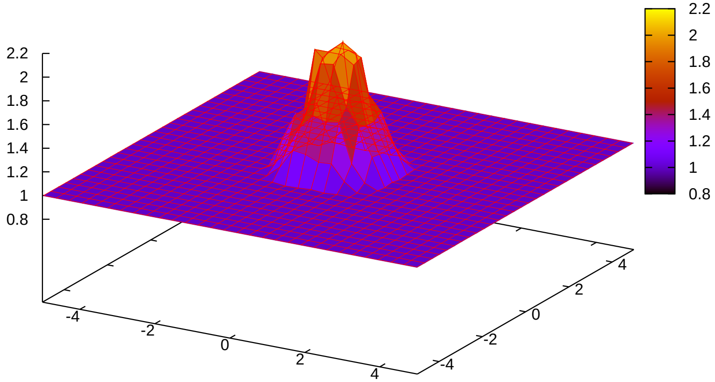

Initial condition $w^0(x)$ in $(-5,5)\times(-5,5)$



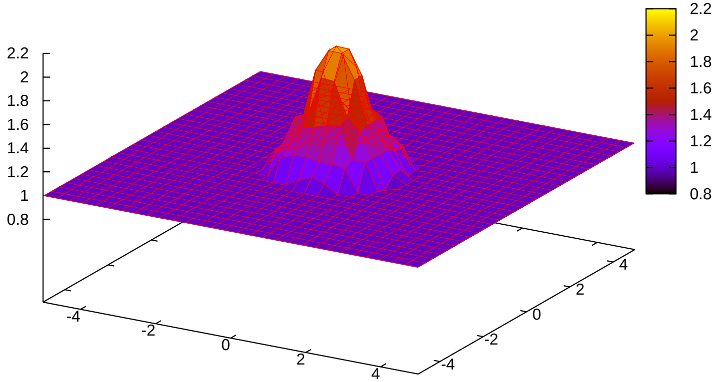

Density at time t=0.1



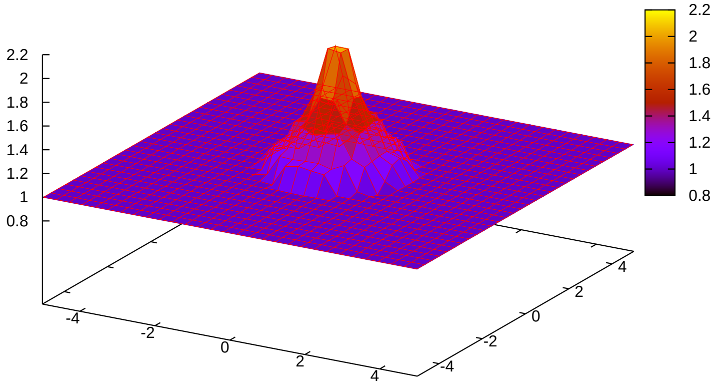

Density at time t=0.2



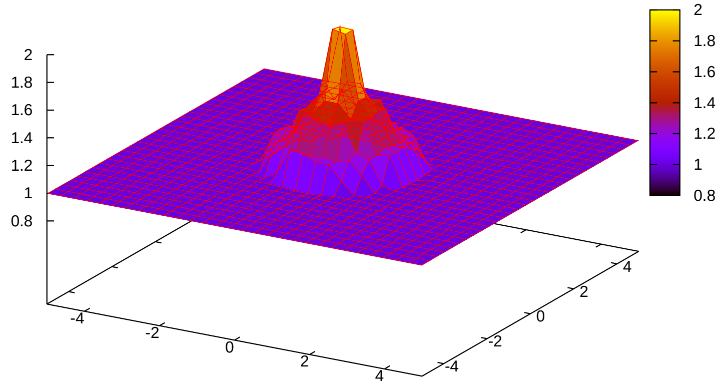

Density at time t=0.3



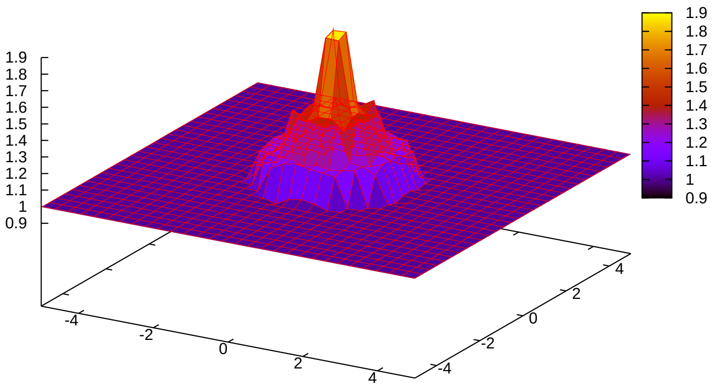

Density at time t=0.4



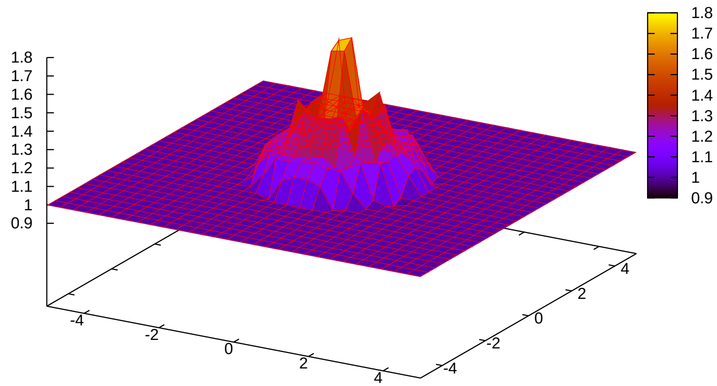

Density at time t=0.5



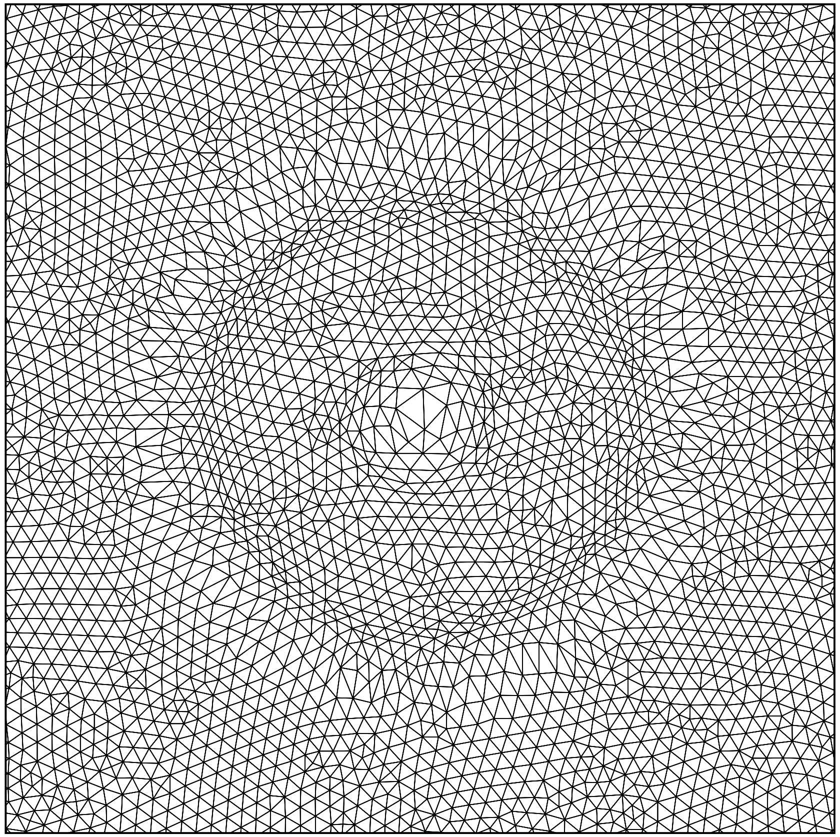

Density at time t=0.6



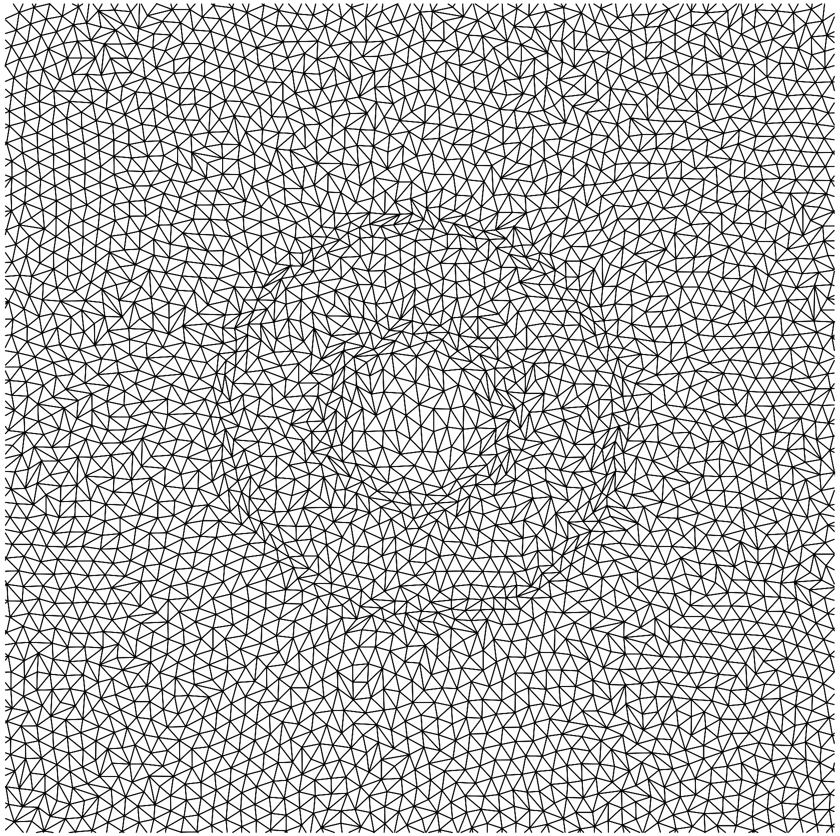

Density at time t=0.7



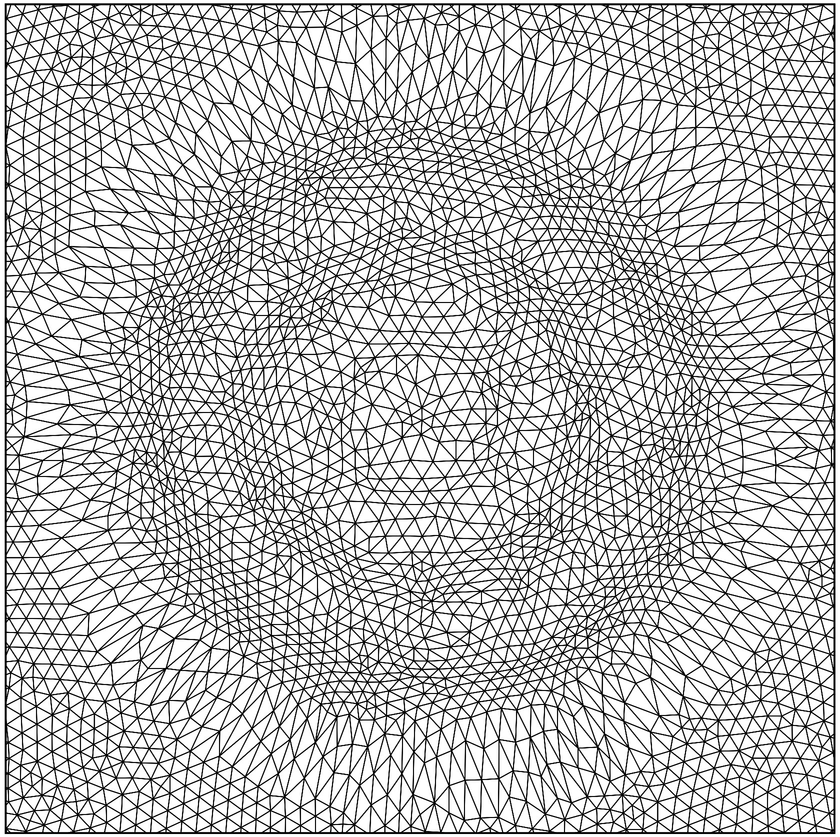

Density at time t=0.8



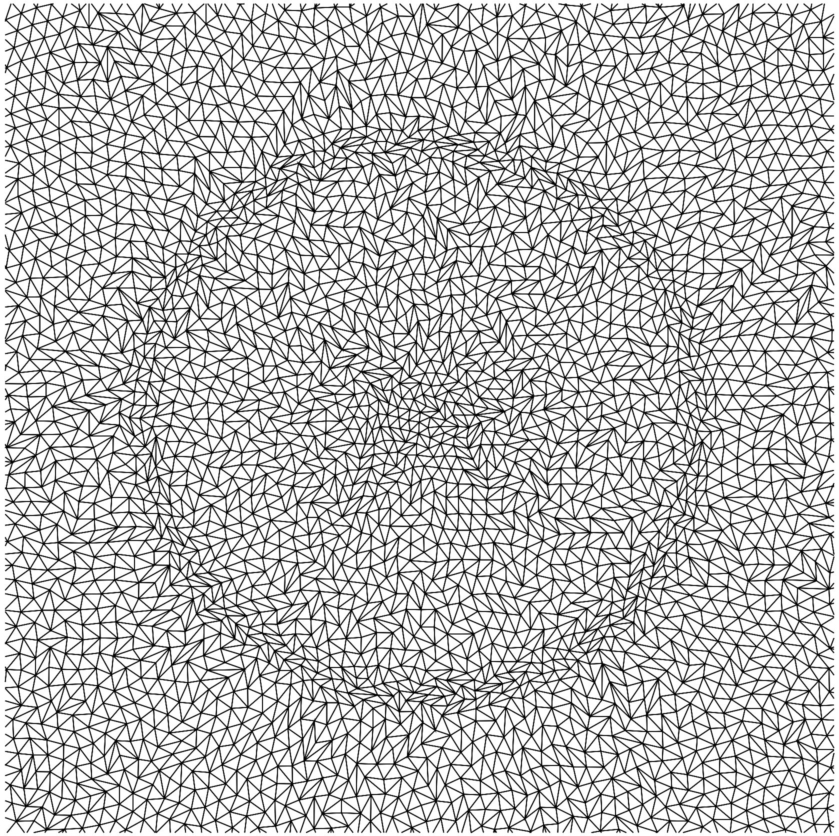
Anisotropic vertex parameter mesh refinement at time t=0.2



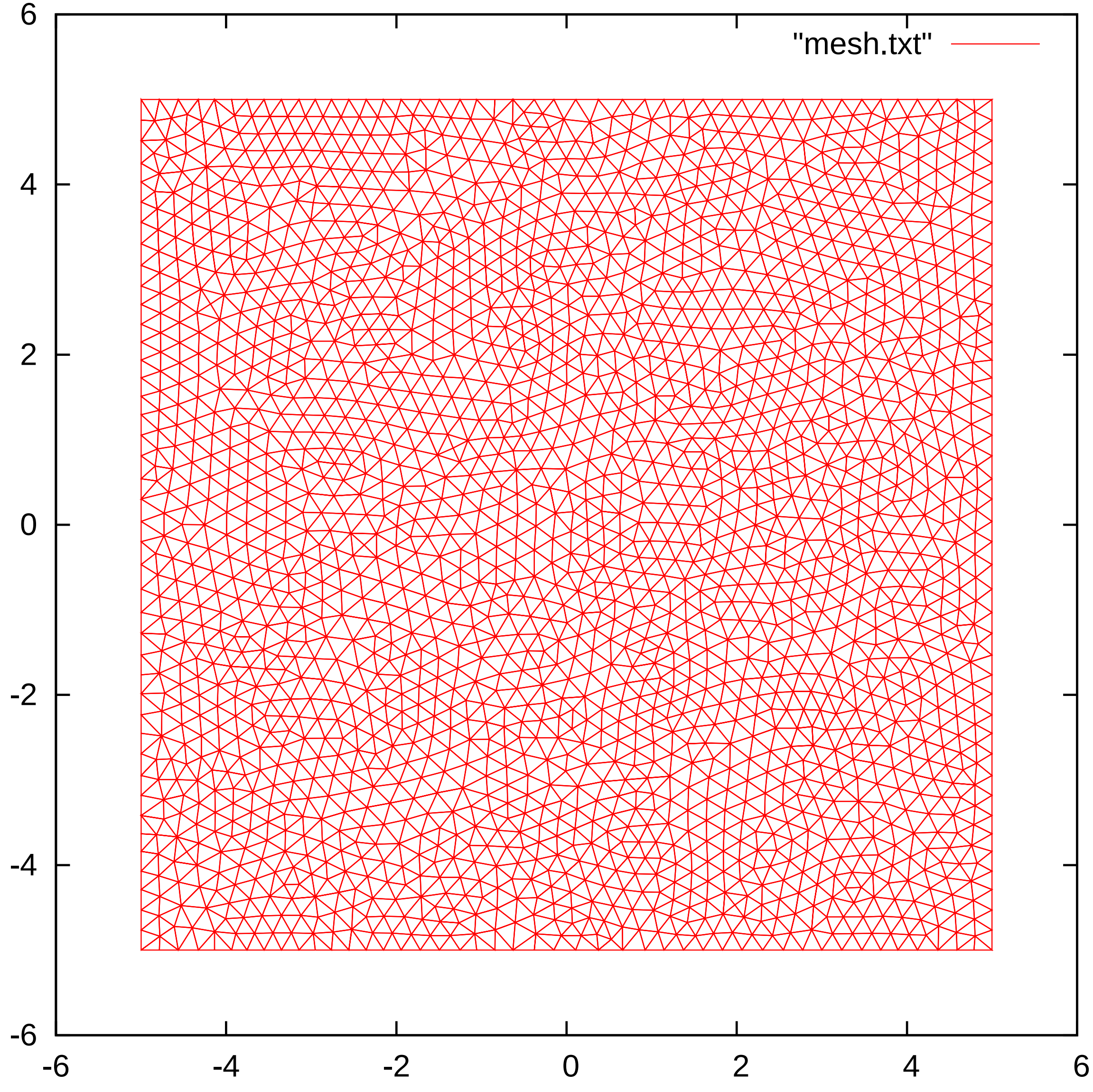
Residual vertex parameter mesh refinement at time t=0.2



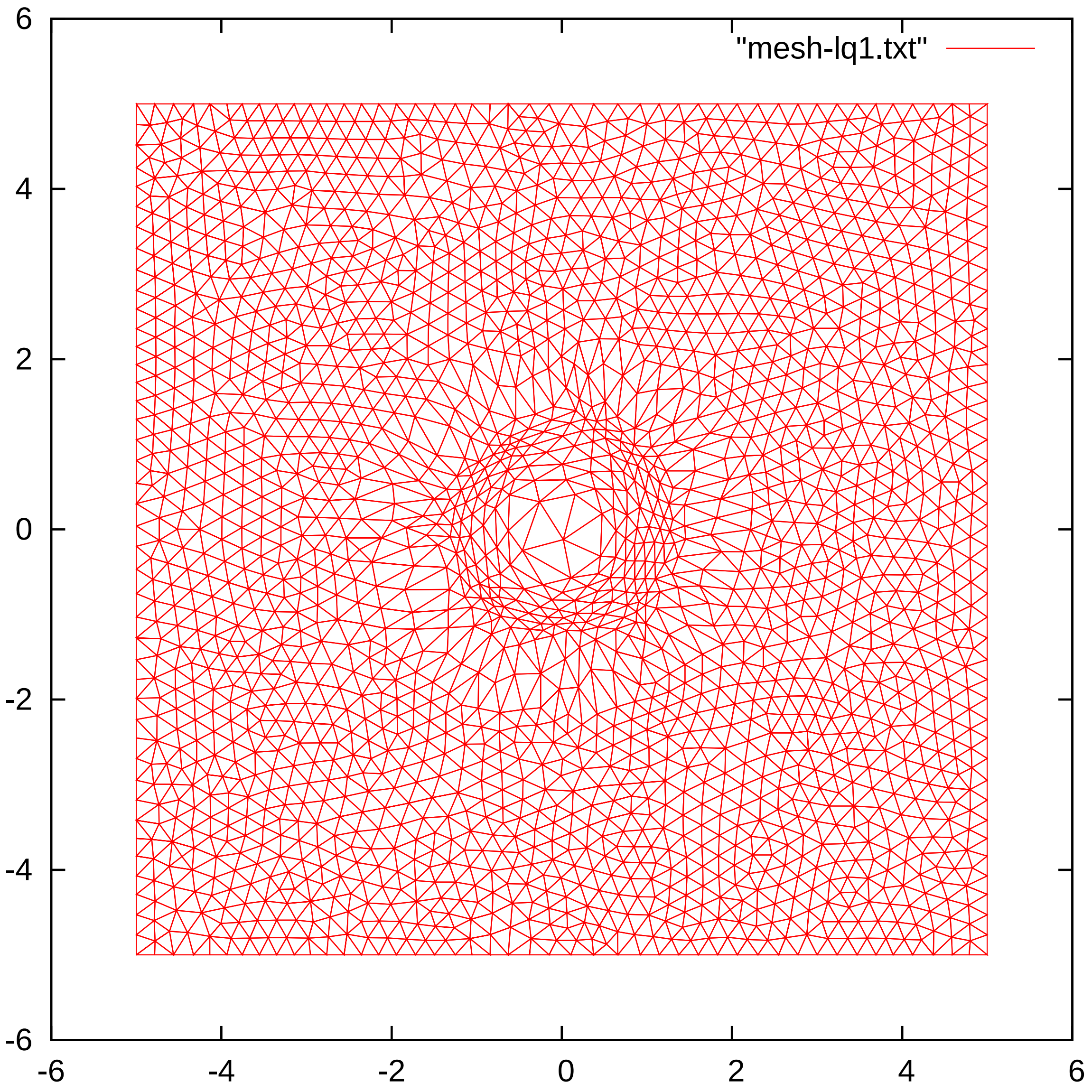
Anisotropic vertex parameter mesh refinement at time t=0.8



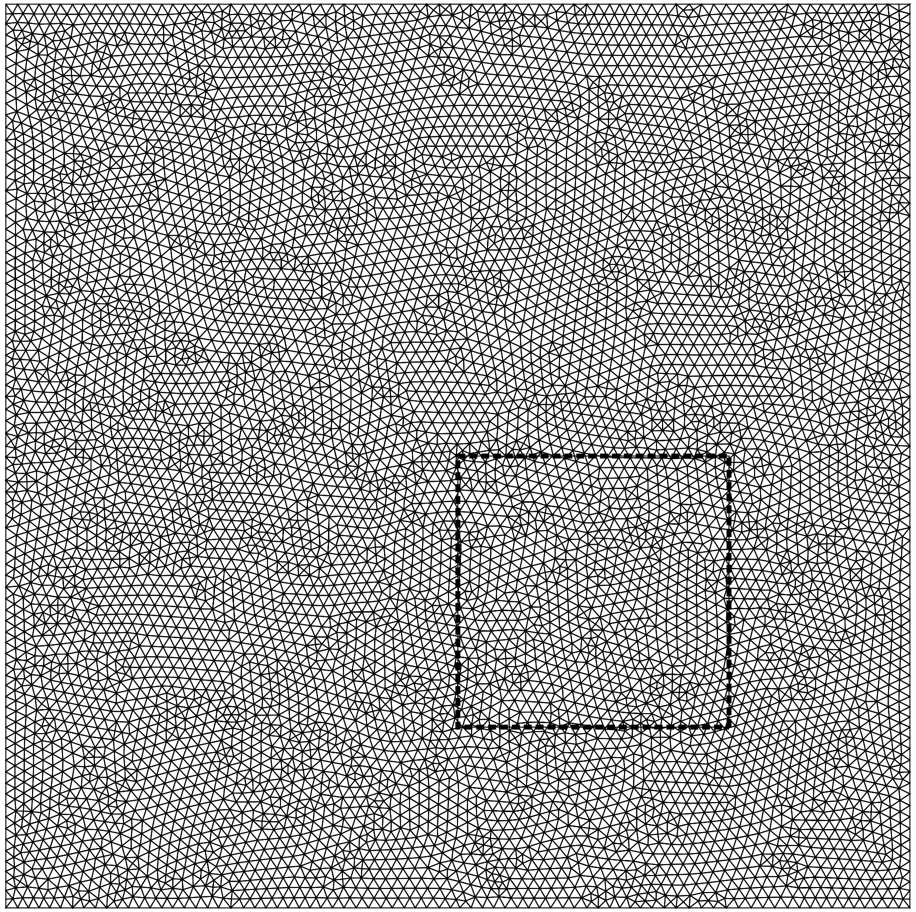
Residual vertex parameter mesh refinement at time t=0.8



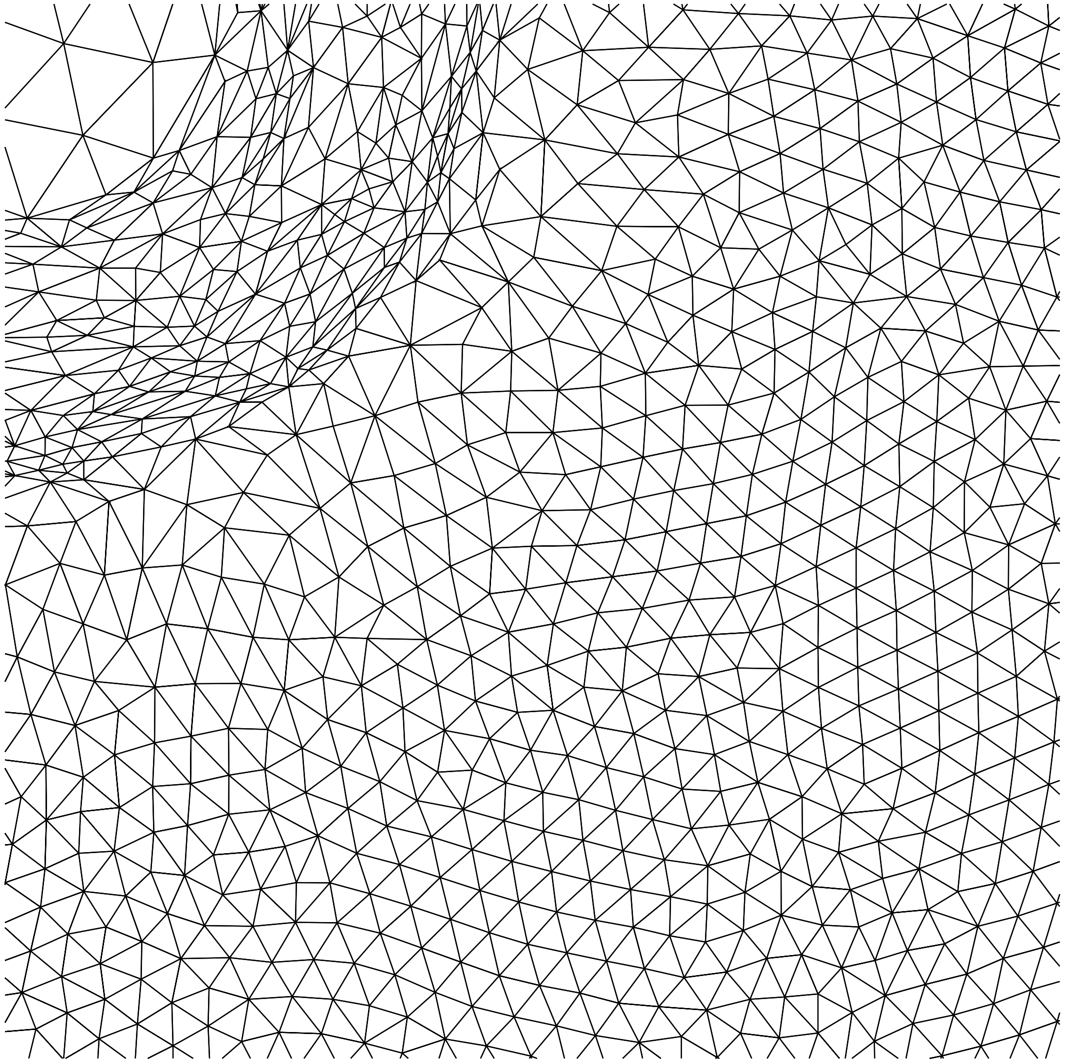

Starting finite volume mesh with 4096 volumes



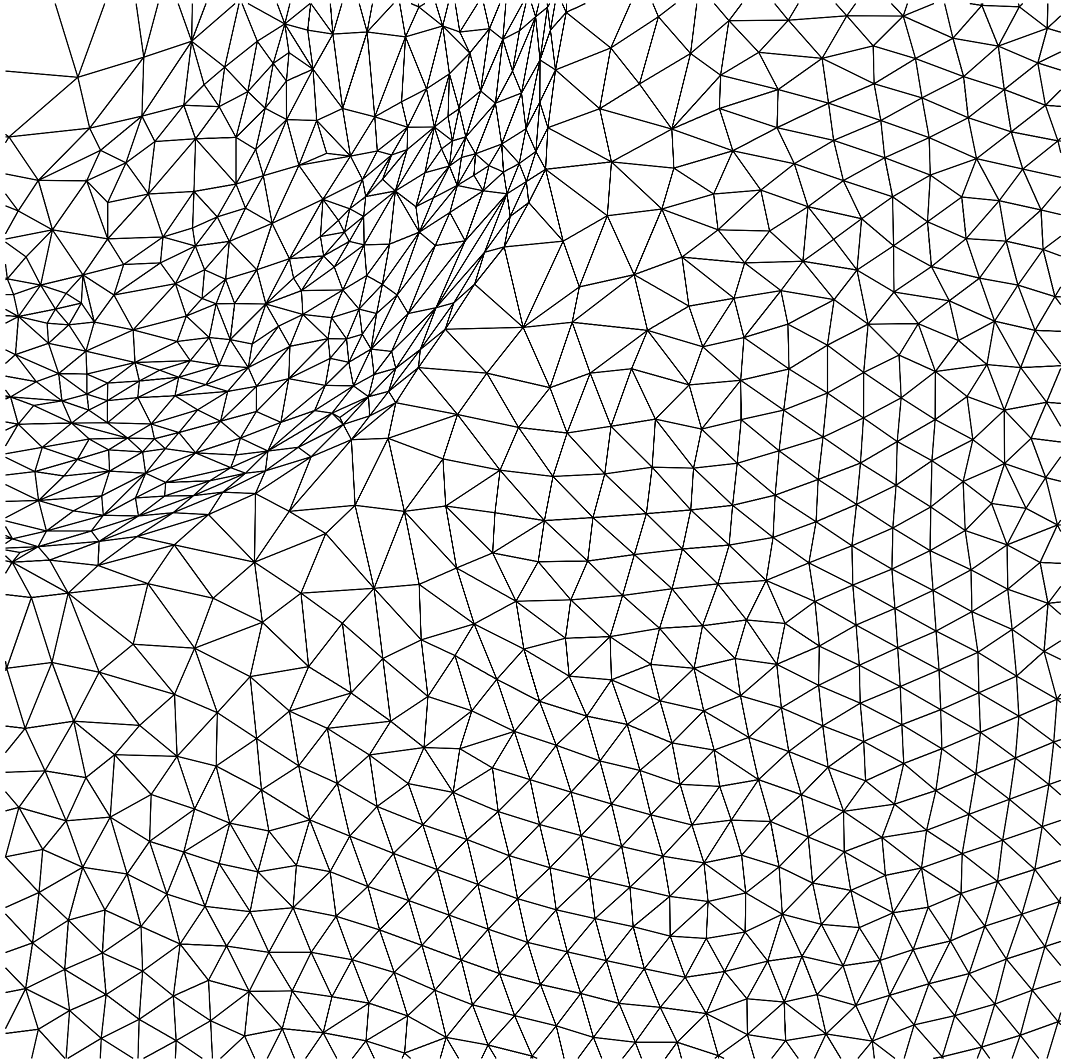

Adapted mesh (LQ(P,1)) for the initial condition $\boldsymbol{w}^0(x)$



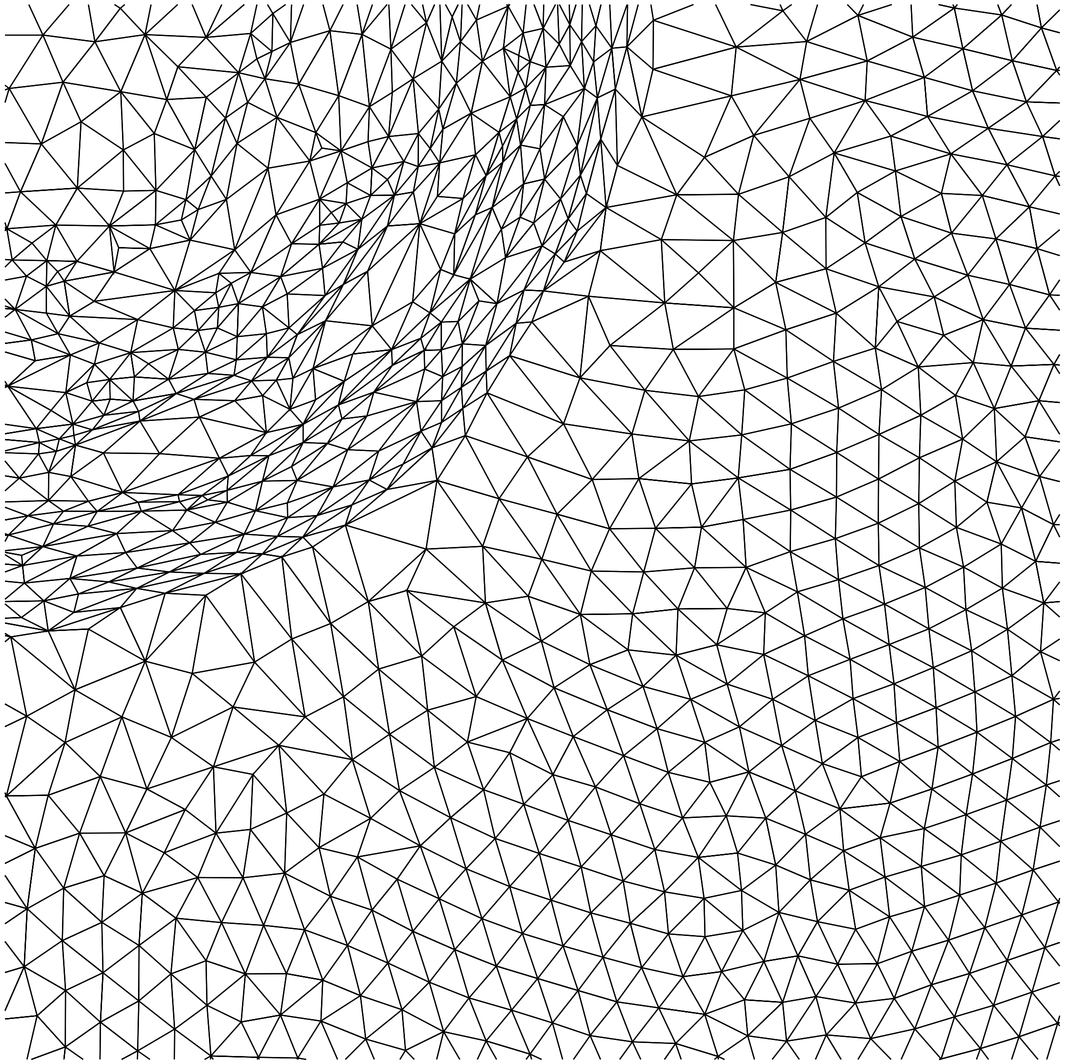
Starting computational mesh with 16384 triangles



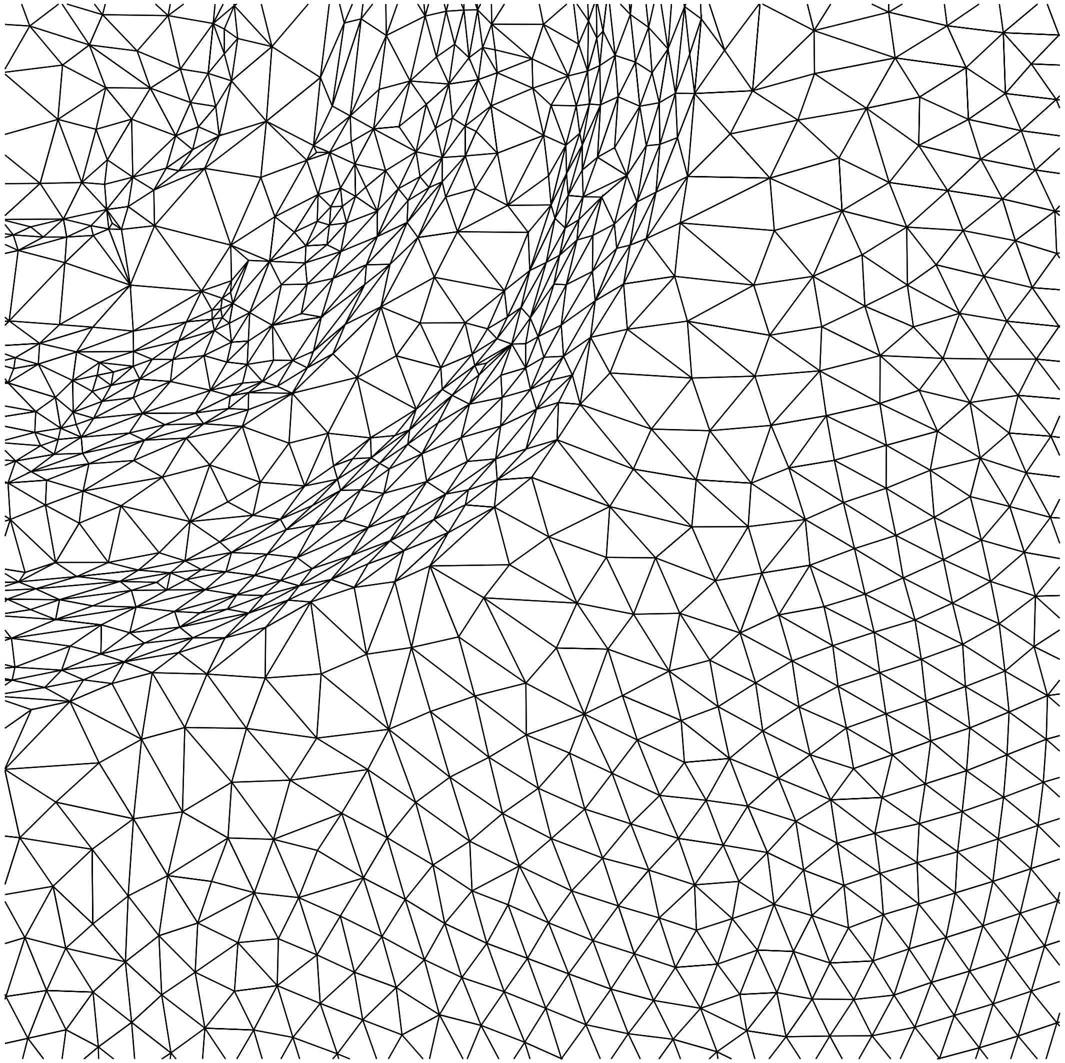
Mesh adaptation at time $t = 0.2$.



Mesh adaptation at time $t = 0.4$.



Mesh adaptation at time $t = 0.6$.



Mesh adaptation at time $t = 0.8$.