Flux-based level set method for two-phase flow problem

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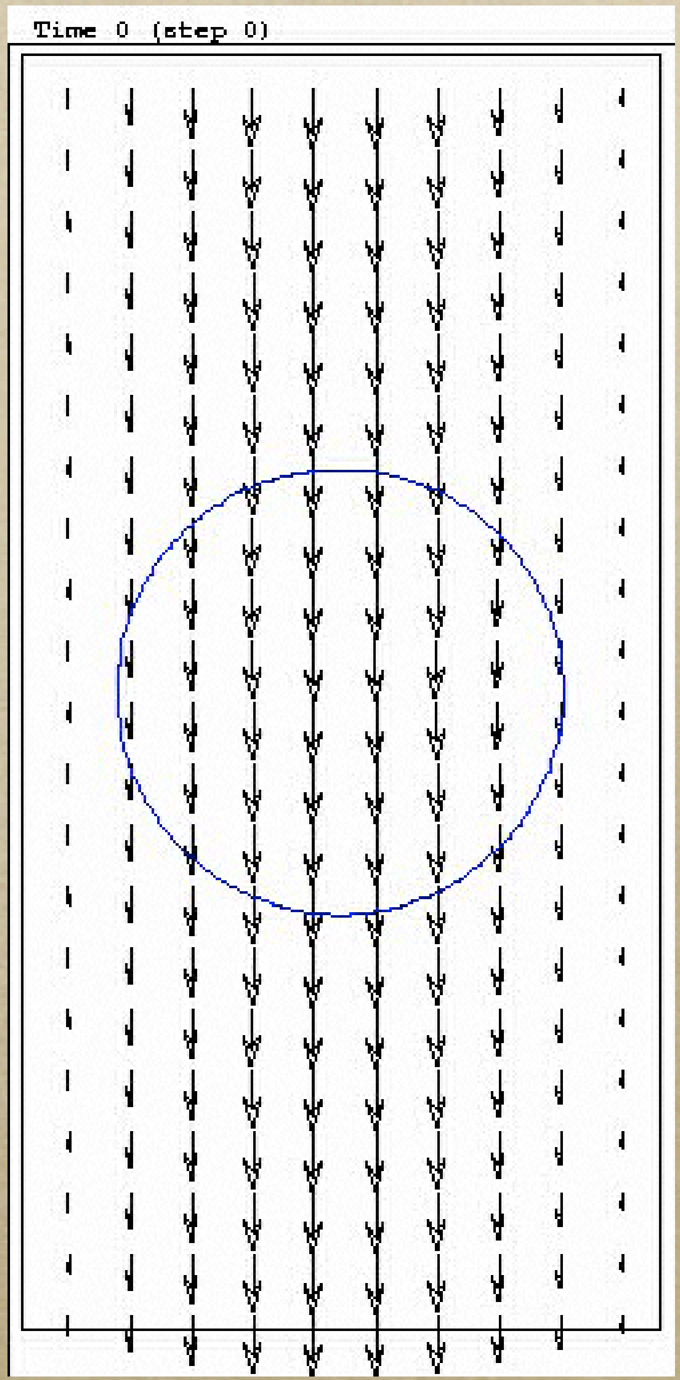
Outline

- *Level set methods*
  - *application to two-phase flow problem*
- *Numerical difficulties*
  - *“mass conservation”*
- *Flux-based level set method*
- *Benchmarks*

Level set methods

Two-phase flow

- *interface between material discontinuities*
  - *density, viscosity*
- *interface condition (here simplified):*
  - *pressure jump in normal direction is proportional to curvature*
- *dynamic interface*
  - *continuous nonzero velocity at the interface*



Naegele, Wittum: *... methods for the incompressible Navier-Stokes equations,* JCP 2007
Frolkovič, Logashenko, Wittum: *... level set method for two-phase flow,* FVCA 2008

Level set methods

Two-phase flow

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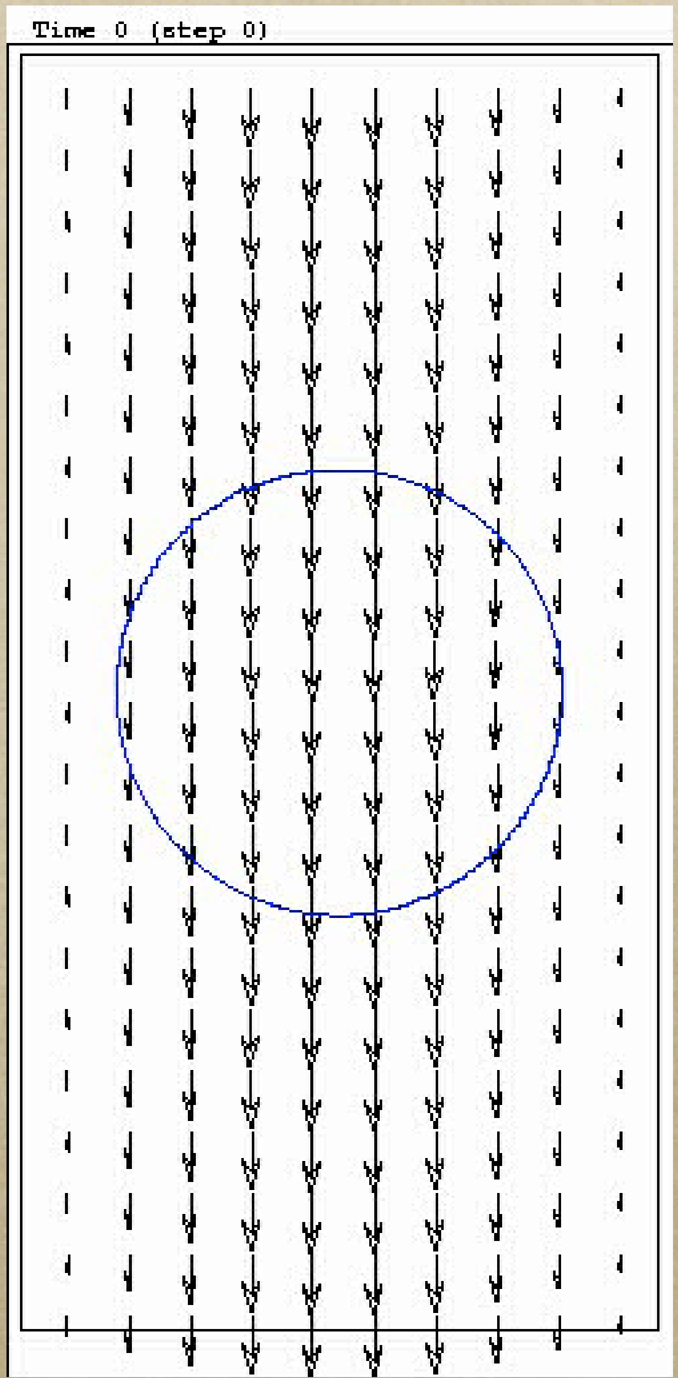


Naegele, Wittum: ... *methods for the incompressible Navier-Stokes equations,* JCP 2007
Frolkovič, Logashenko, Wittum: ... *level set method for two-phase flow,* FVCA 2008

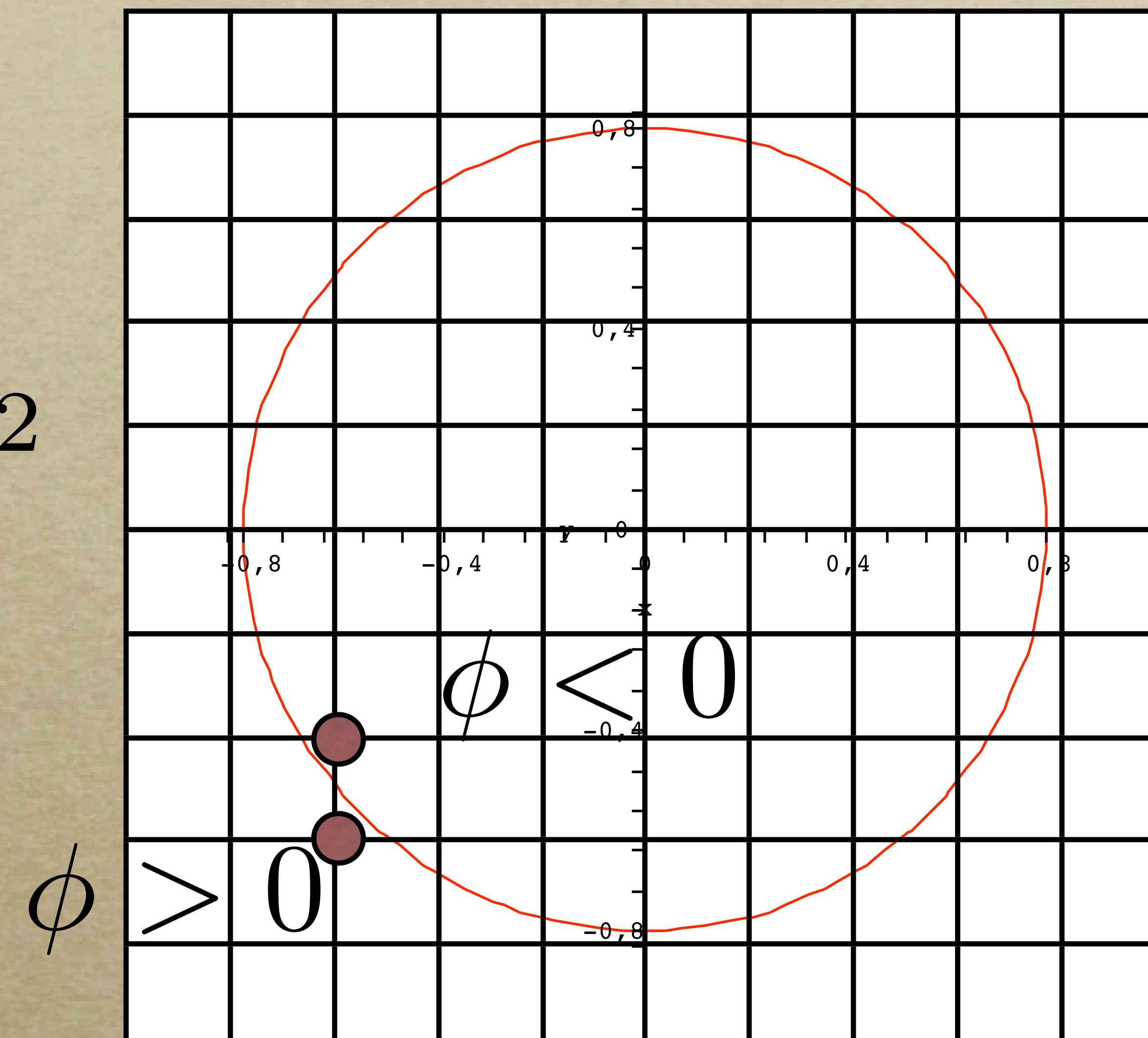
Level set methods

Level set function

- *the interface is given implicitly using a smooth function*
- *example of a circular interface:*
  - *choose a level set function:*

$$\phi(x, y) := x^2 + y^2 - 0.8^2$$

  - *indicate points inside*
  - *indicate points outside*
  - *capture interface* $\phi = 0$

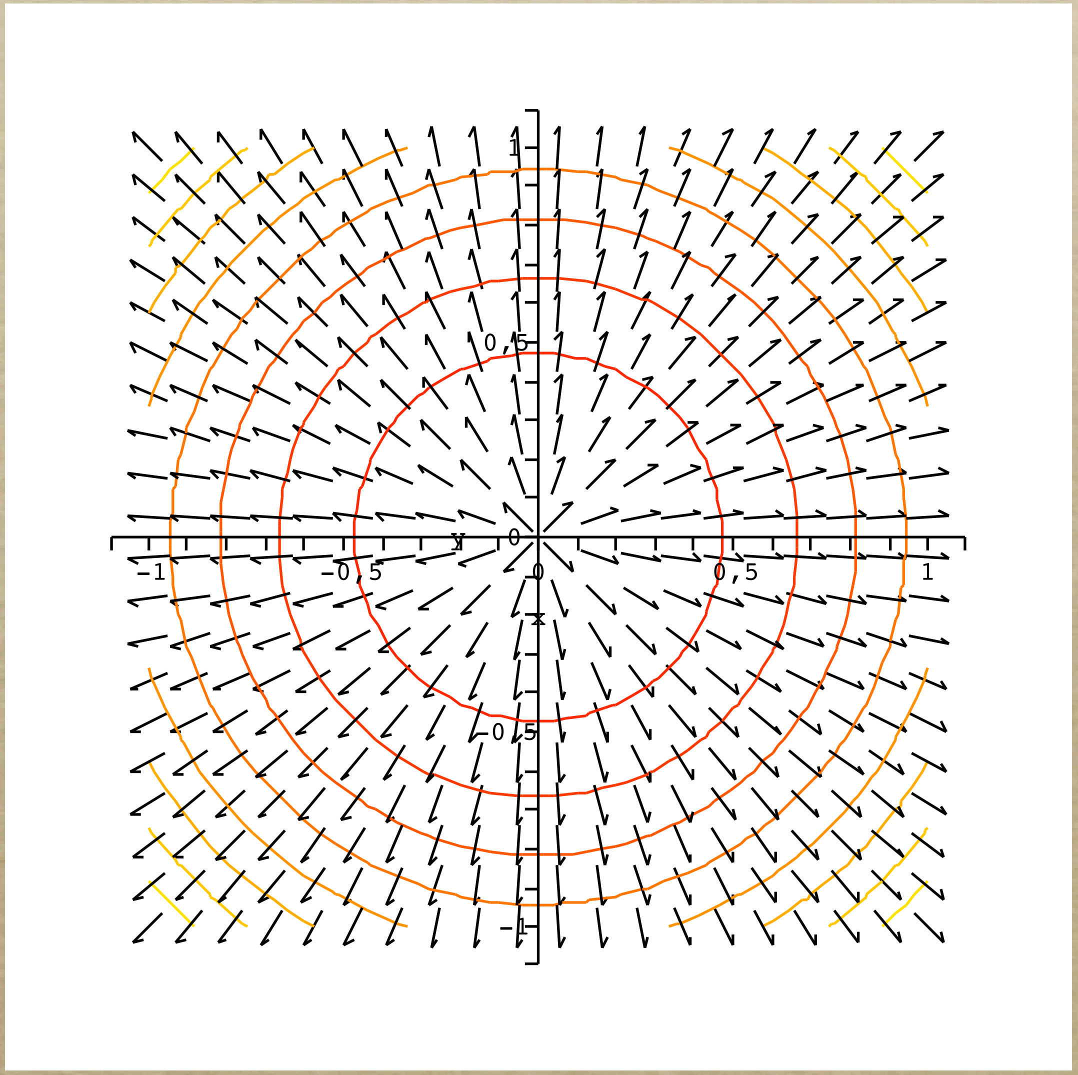


Level set methods

Level set function

- *the interface is given implicitly using a smooth function*
- *the interface condition*
  - *normal unit vectors*

$$\vec{N} = \frac{\nabla \phi}{|\nabla \phi|}$$



Level set methods

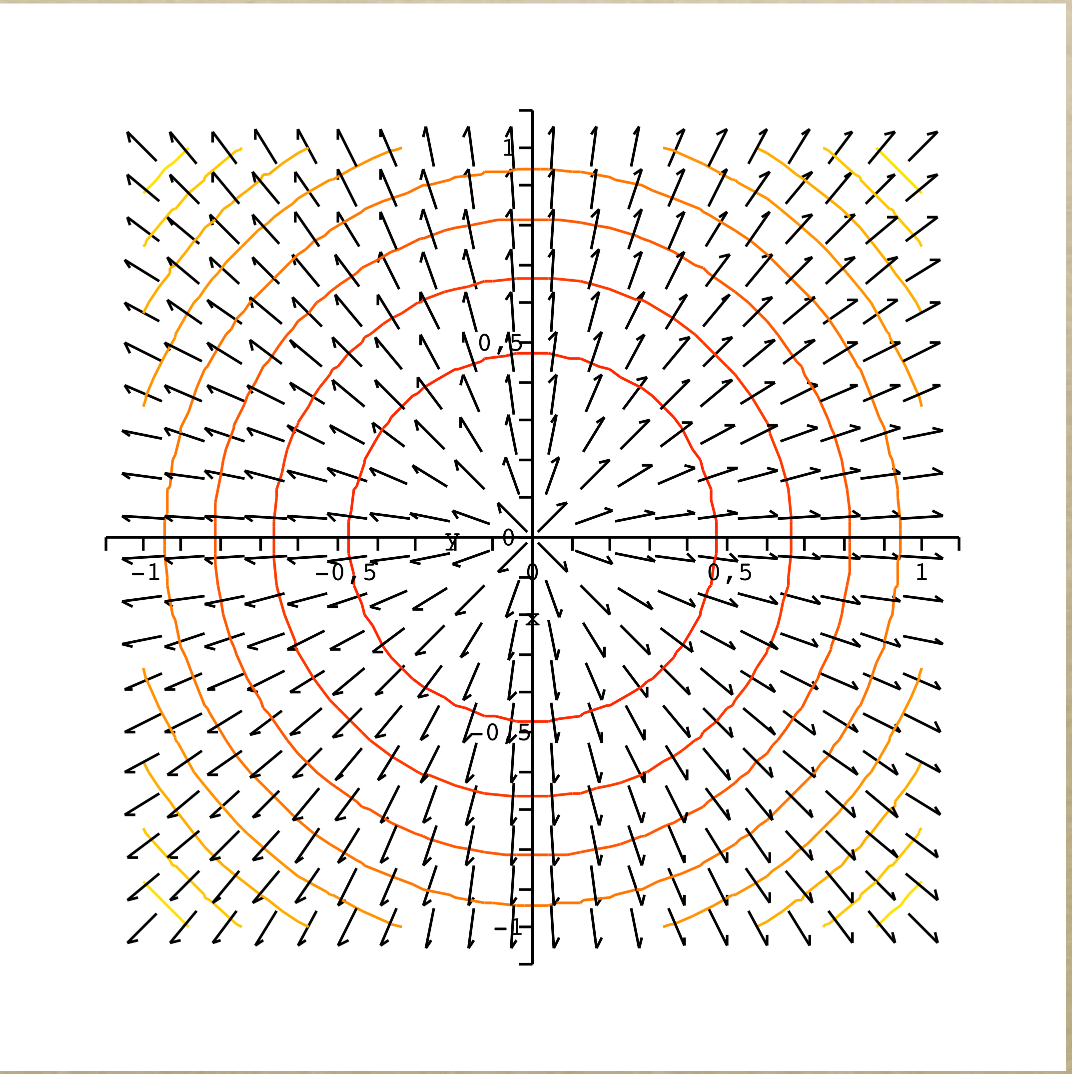
Level set function

- *the interface is given implicitly using a smooth function*
- *the interface condition*
  - *normal unit vectors*

$$\vec{N} = \frac{\nabla \phi}{|\nabla \phi|}$$

  - *mean curvature*

$$\kappa = \nabla \cdot \vec{N}$$



Frolkovič, Logashenko, Wittum: ... *level set method for two-phase flow,* FVCA 2008

Level set methods

Level set equation

- *the dynamic interface according to advection equation*

$$0 = \partial_t \phi + \vec{V} \cdot \nabla \phi = \partial_t \phi + v_1 \, \partial_x \phi + v_2 \, \partial_y \phi$$

- *the level set function is constant along characteristic curves*

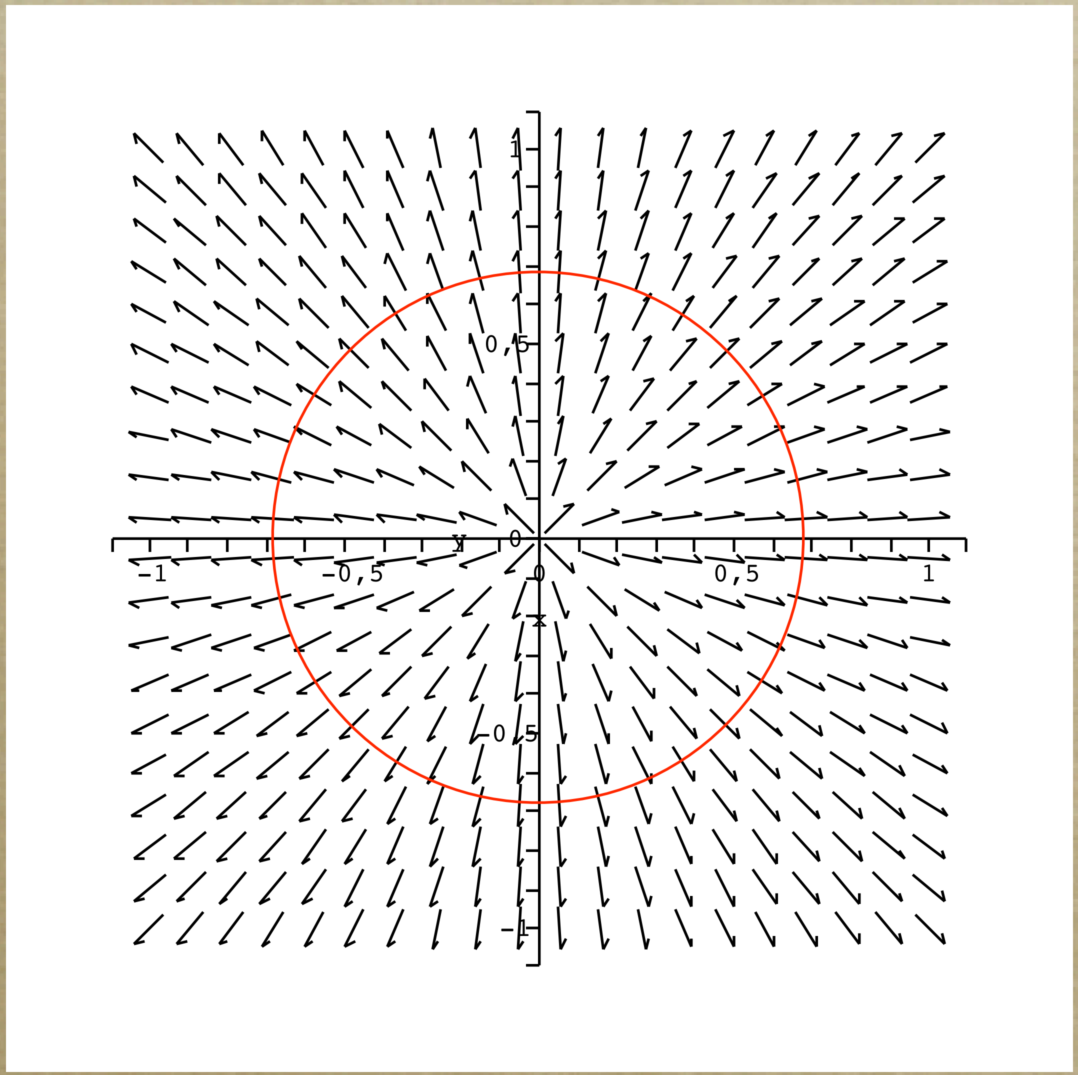
$$\frac{dx(t)}{dt} = v_1 \, , \quad \frac{dy(t)}{dt} = v_2$$

$$\phi(t, x(t), y(t)) \equiv const$$

Level set methods

Nonlinear level set equation

- *e.g., expanding of the circle:* $\partial_t \phi + \frac{\nabla \phi}{|\nabla \phi|} \cdot \nabla \phi = 0$



Level set methods

Nonlinear level set equation

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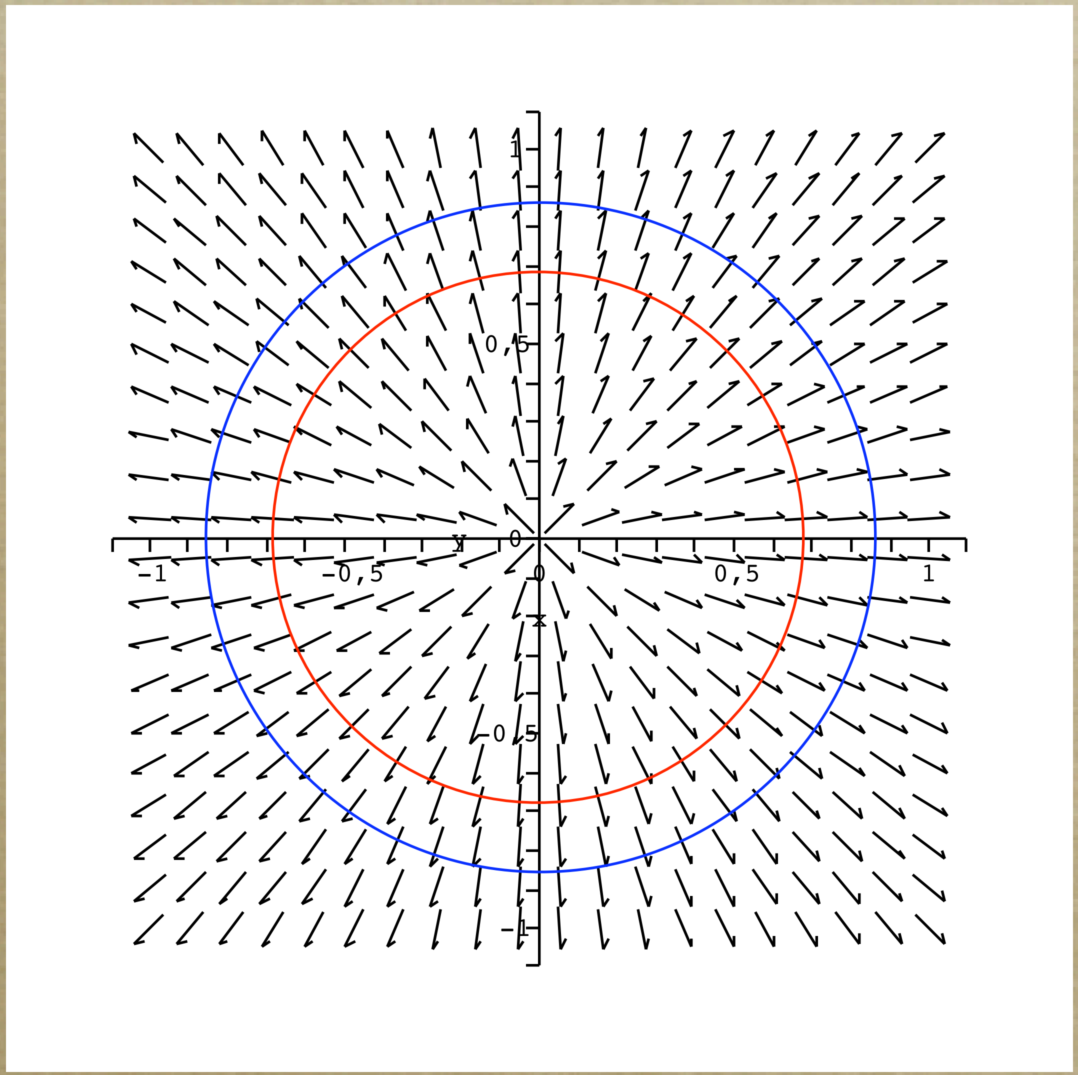
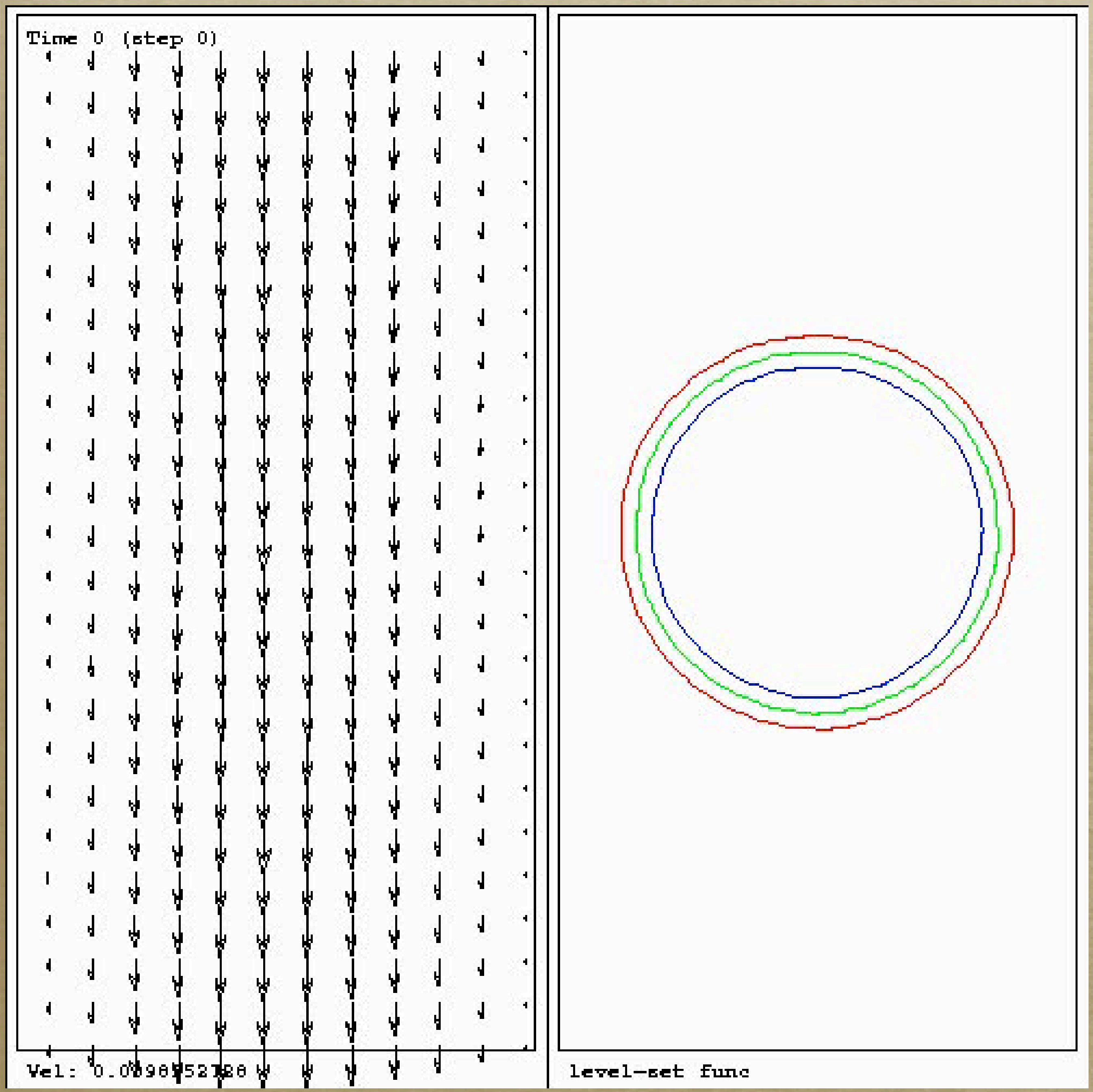


Level set methods

Two-phase flow - reinitialization of the level set function

- *without and with reinitialization:*



Level set methods

Two-phase flow - reinitialization of the level set function

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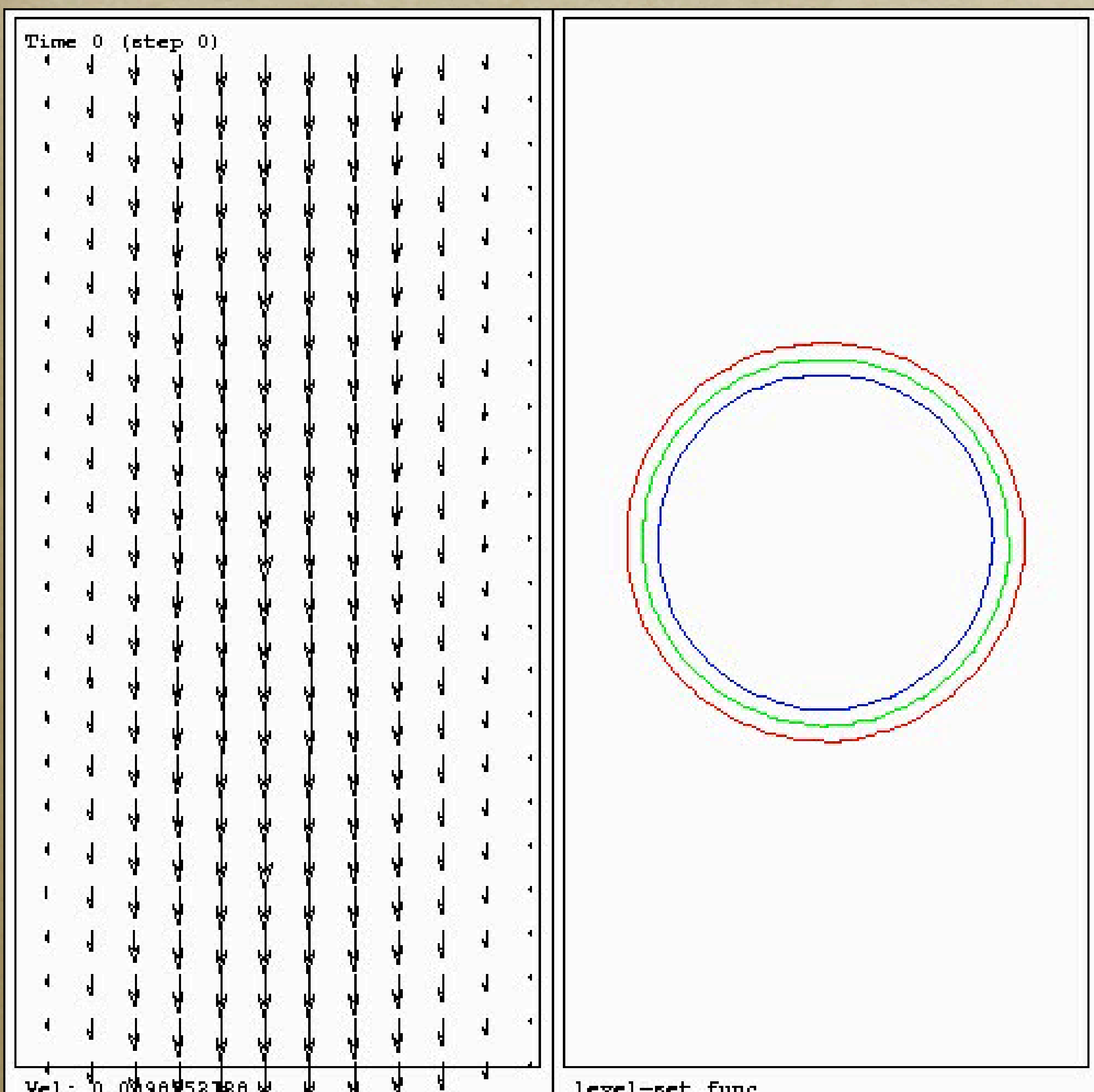
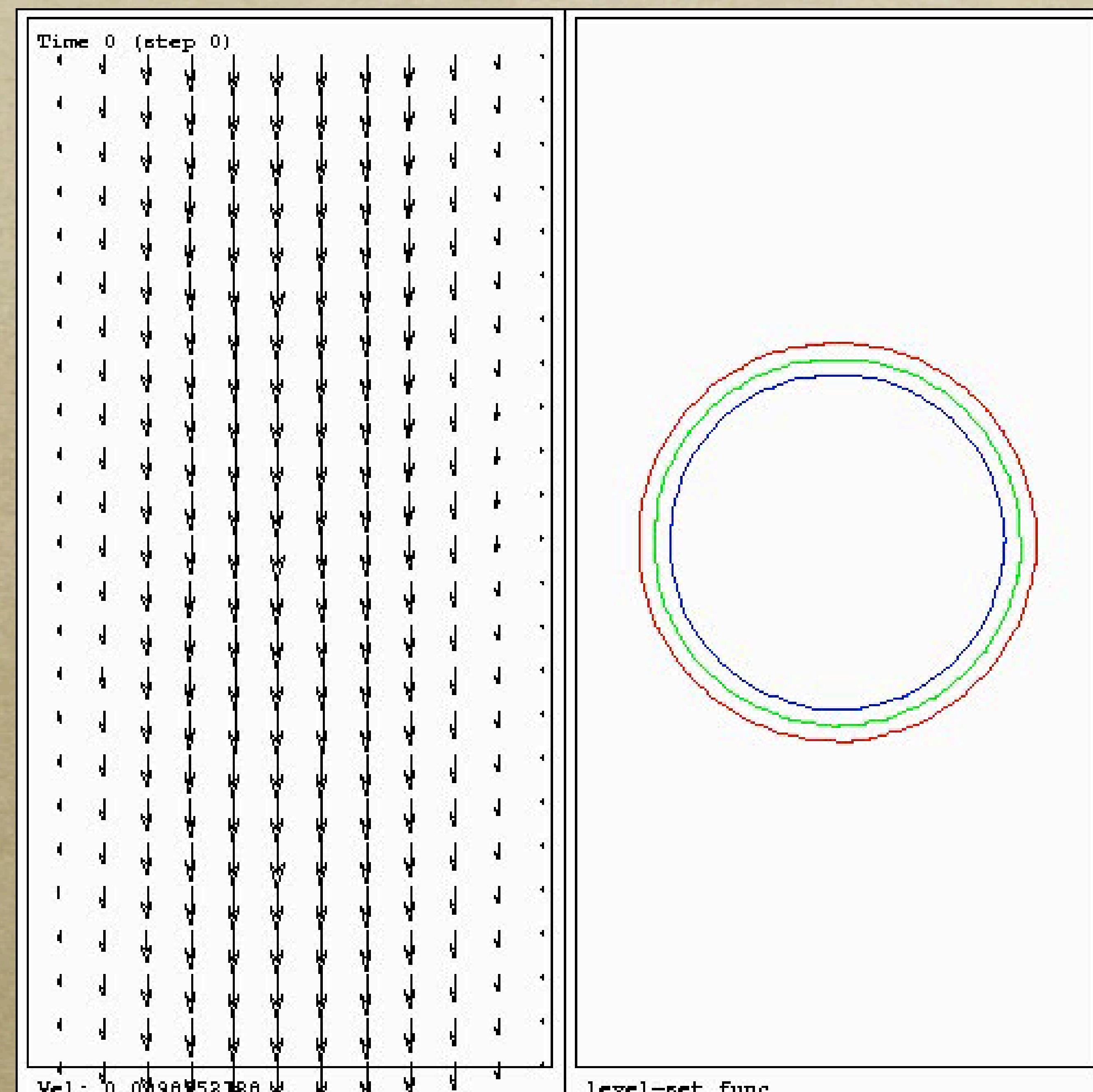




Numerical difficulties

Two-phase flow

Sussman, Smereka, Osher: *A level set approach for computing solutions to incompressible two-phase flow*; 1994 (second order ENO scheme, ...)

- *Matlab Levelset Toolbox* [Mitchell 2004]

$$\partial_t \phi + \vec{V} \cdot \nabla \phi = 0$$

- *rotation of a circle*
- *2nd order ENO*
- *grid 20x20*

Applications and numerical difficulties

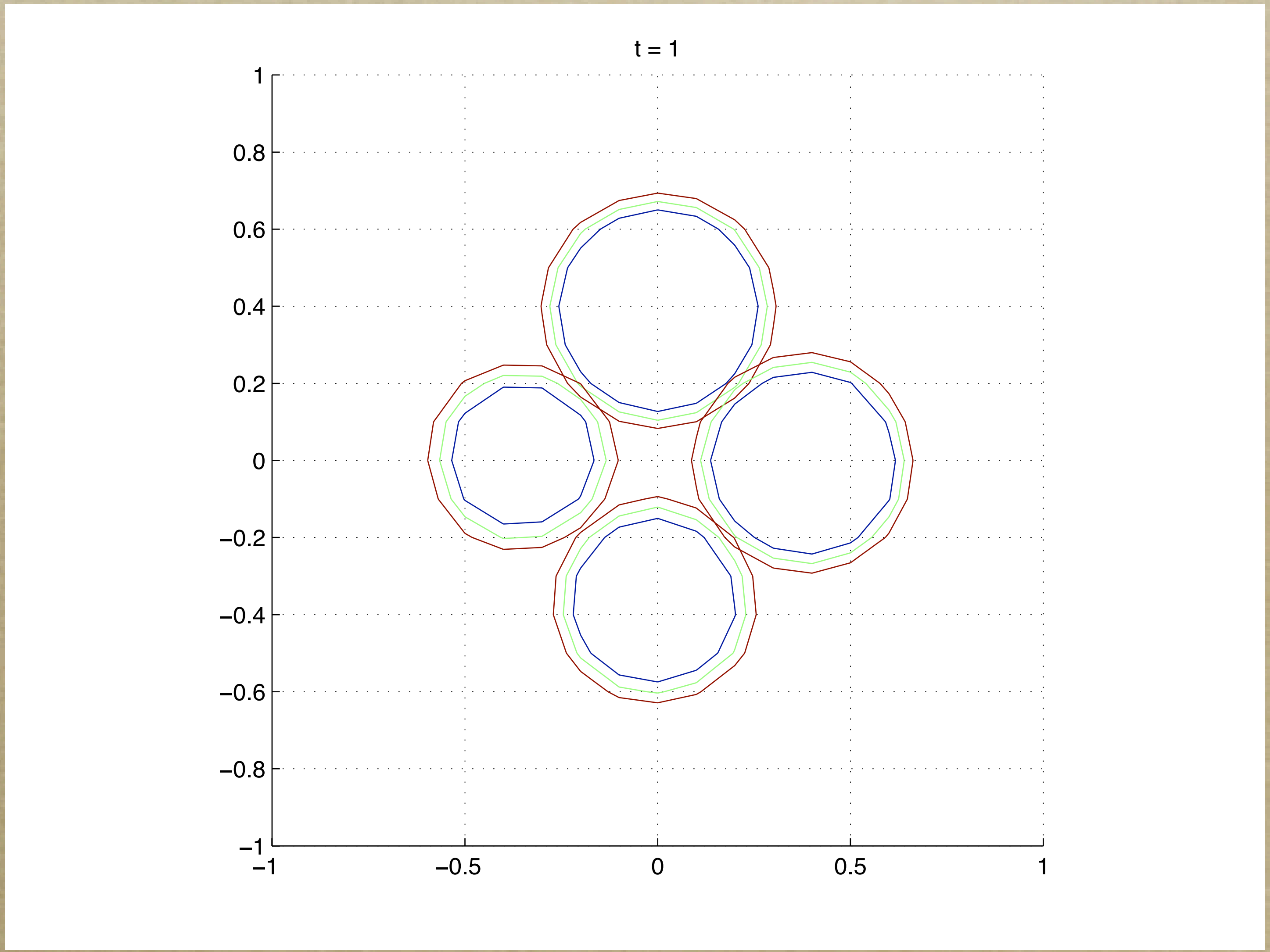
Two-phase flow

Sussman, Smereka, Osher: *A level set approach for computing solutions to incompressible two-phase flow*; 1994 (second order ENO scheme, ...)

- *Matlab Levelset Toolbox* [Mitchell 2004]

$$\partial_t \phi + \vec{V} \cdot \nabla \phi = 0$$

- *rotation of a circle*
- *2nd order ENO*
- *grid 20x20*
  - *loss of the area!*



Applications and numerical difficulties

Two-phase flow

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$$\partial_t \phi + \vec{V} \cdot \nabla \phi = 0$$

- *rotation of a circle*
- *2nd order ENO*
- *grid 20x20*
  - *loss of the area!*

Losasso, Fedkiw, Osher (2006):
"Interestingly, this work also demonstrated the largest weakness of the level set method, i.e. mass or information loss characteristic of most Eulerian capturing techniques."

Numerical difficulties

Two-phase flow - improvements

Sussman, Fatemi, Smereka, Osher: *An improved level set method for incompressible two-phase flow*; 1995 (higher order ENO scheme, ...)

Sussman, Fatemi: *An efficient interface-preserving level set redistancing algorithm ..., 1999*

Sussman, Puckett: *A coupled level set and volume of fluid method ...*, 2000

Enright, Fedkiw, Ferziger, Mitchell: *A hybrid particle level set method ...*, 2002

Hieber, Koumoutsakos: *A Lagrangian particle level set method,* 2005

Losasso, Fedkiw, Osher: *Spatially Adaptive Techniques for Level Set Methods and Incompressible Flow,* 2006 - review

$$\partial_t \phi + \vec{V} \cdot \nabla \phi = 0$$

Numerical difficulties

Conservative schemes

Frolkovic, Mikula: *Flux-based level set method: a finite volume method for evolving interfaces*, Preprint 2003, Applied Numerical Mathematics 2007 (not for two phase flow)

Olsson, Kreiss: *A conservative level set method for two phase flow*, JCP 2005 (finite volume method with limiter and artificial compressibility)

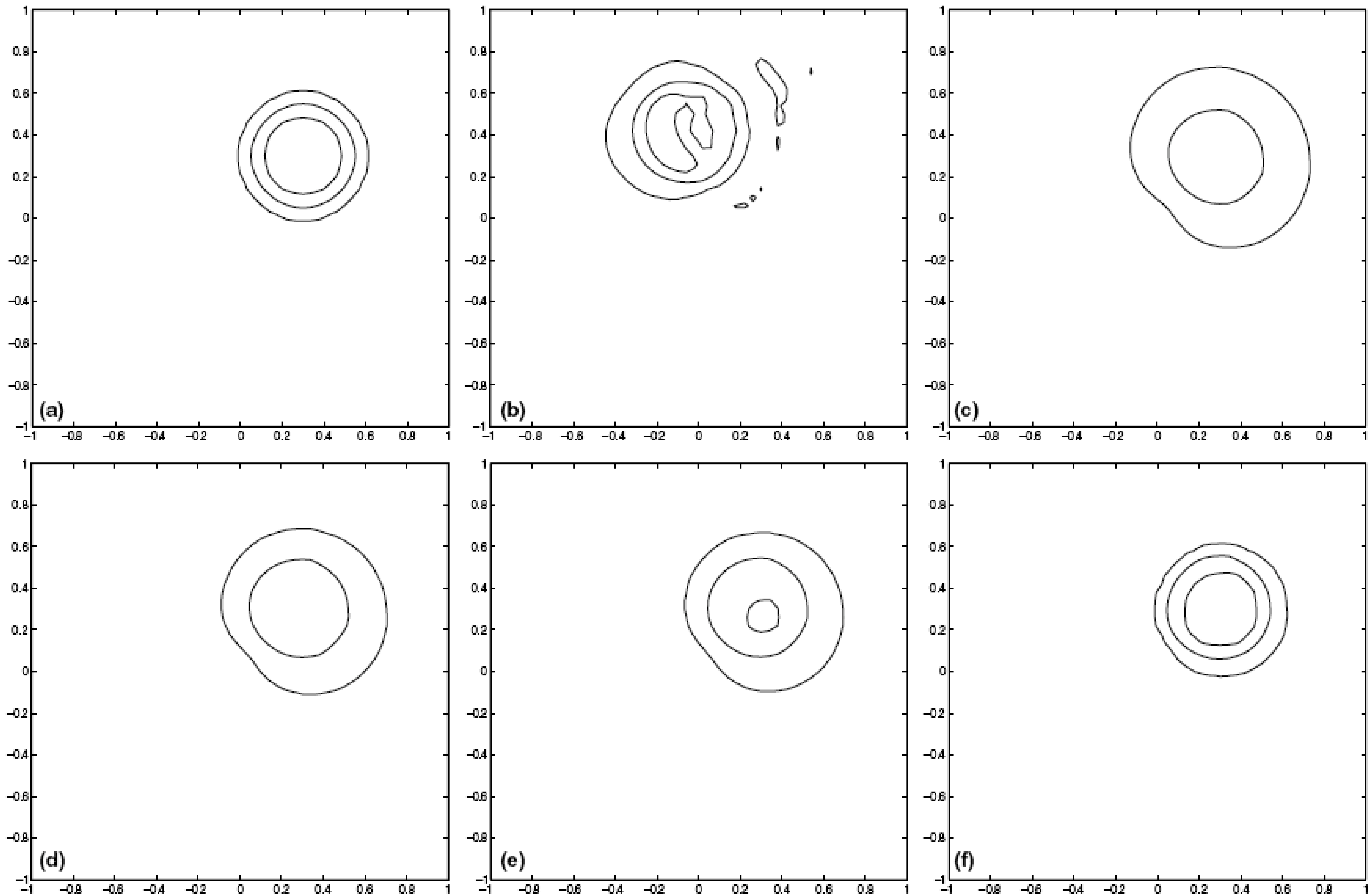
Marchandise, Remacle, Chevaugeon: *A quadrature-free discontinuous Galerkin method for the level set equation*, JCP 2005

Di Pietro, Lo Forte, Parolini: *Mass preserving finite element implementations of the level set method*, ANM 2006

Zheng, Lowengrub, Anderson, Cristini: *Adaptive unstructured volume remeshing...*, JCP 2005 (discontinuous Galerkin)

$$\partial_t \phi + \nabla \cdot \left(\phi \vec{V} \right) = \phi \nabla \cdot \vec{V}$$

Numerical difficulties



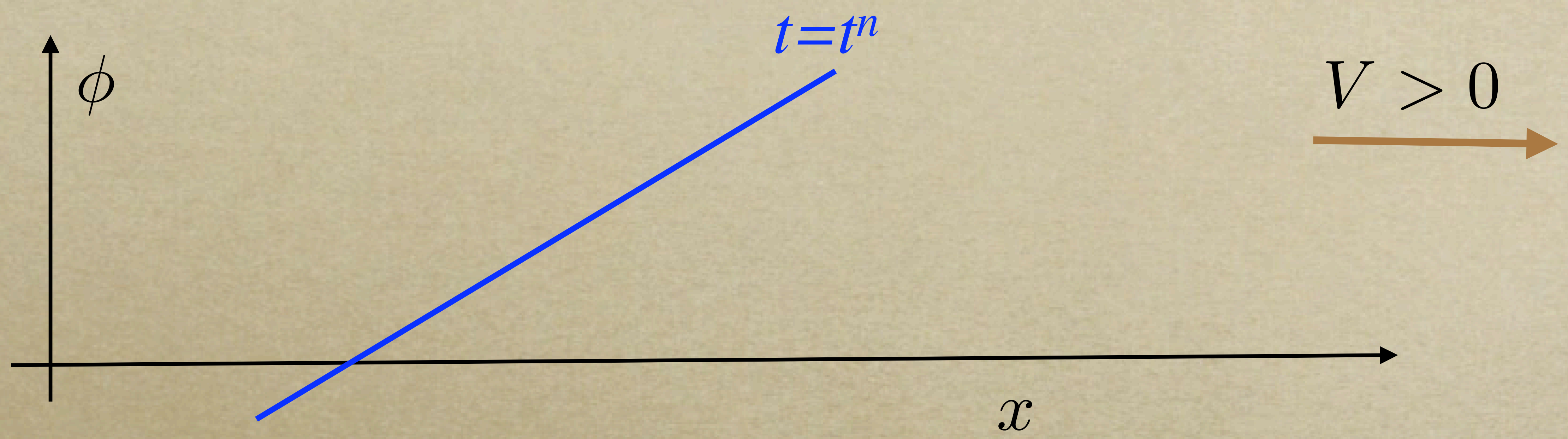
E. Olsson, G. Kreiss / Journal of Computational Physics 210 (2005) 225–246

Fig. 1. 0.05, 0.5 and 0.95 Contours of Φ initially and after one revolution using different numerical methods. (c)–(f) are second order TVD methods, (b) is second order but not TVD. (a) Initial state; (b) centered differences; (c) upwind with Minmod; (d) upwind with Van Albada; (e) upwind with Van Leer; (f) upwind with Superbee.

Olsson, Kreiss: *A conservative level set method for two phase flow*, JCP 2005

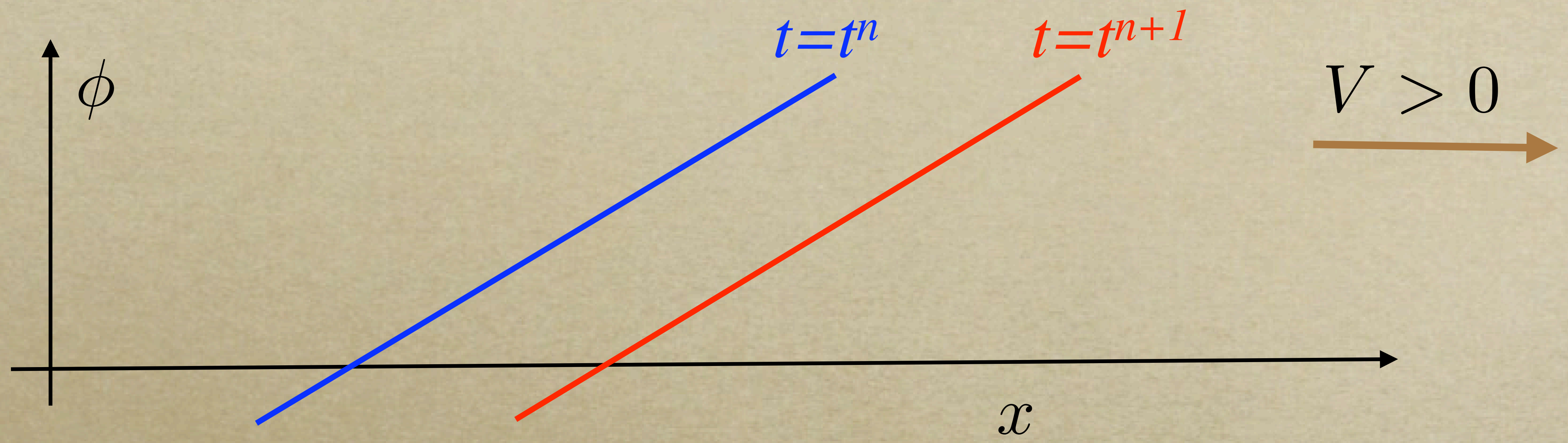
Flux-based level set method

1D illustration of the idea



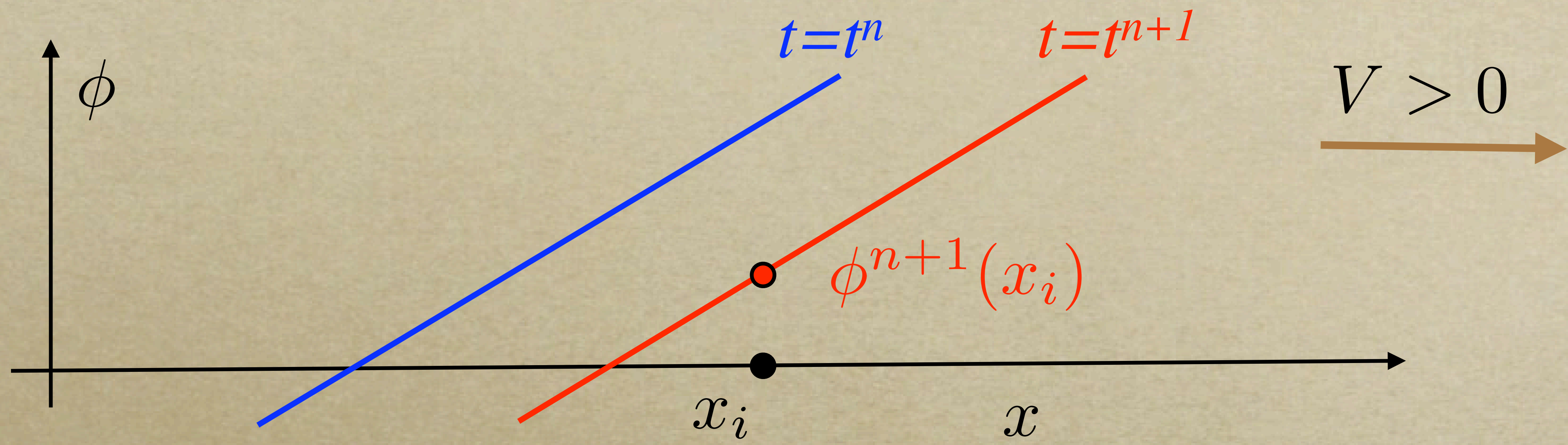
Flux-based level set method

1D illustration of the idea



Flux-based level set method

1D illustration of the idea

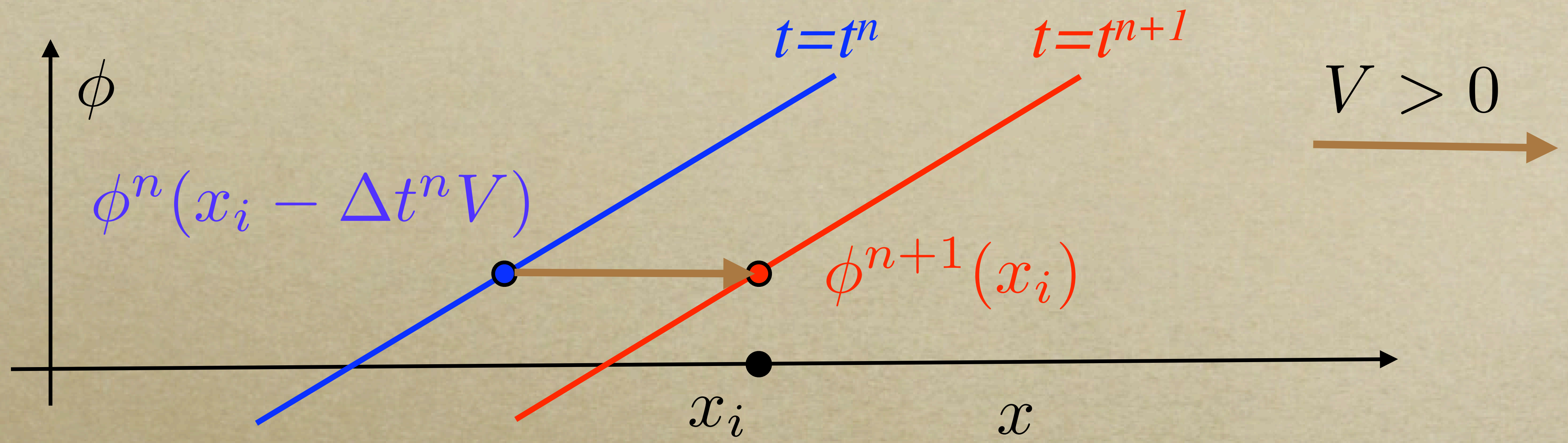


- *standard upwind approach:*

$$\phi^{n+1}(x_i) = ?$$

Flux-based level set method

1D illustration of the idea

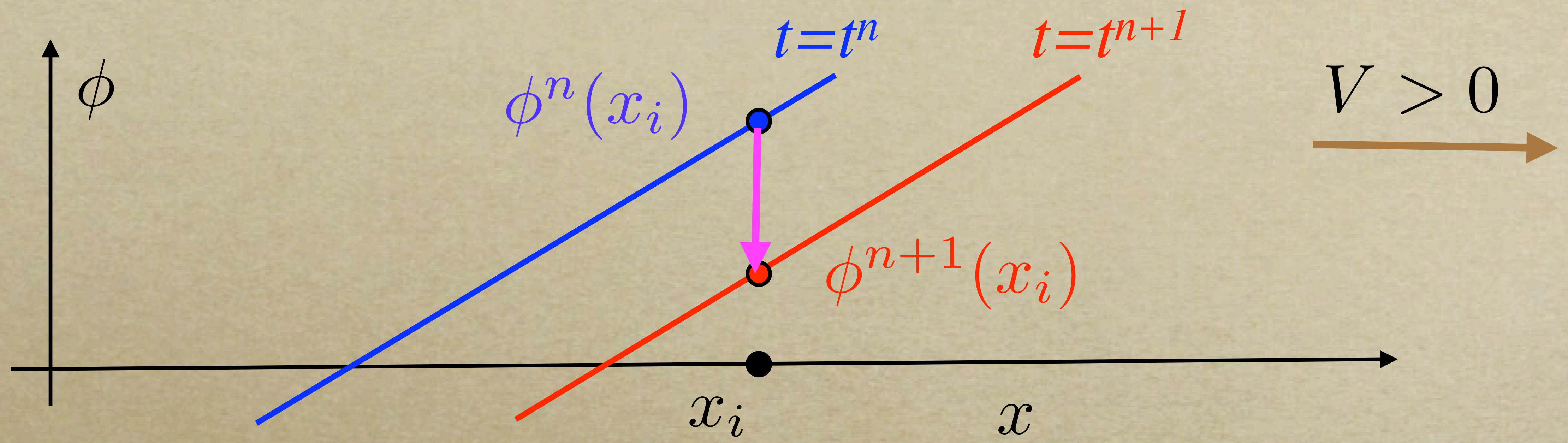


- *standard upwind approach* (*does not require the gradient of solution!*):

$$\phi^{n+1}(x_i) = \phi^n(x_i - \Delta t^n V)$$

Flux-based level set method

1D illustration of the idea



- *standard upwind approach:*

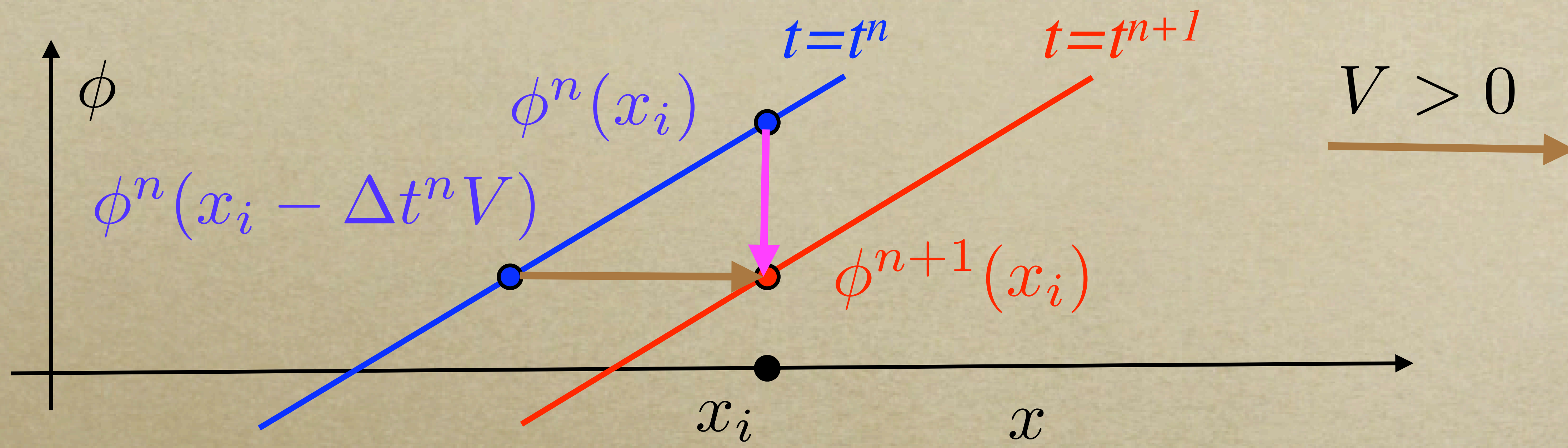
$$\phi^{n+1}(x_i) = \phi^n(x_i - \Delta t^n V)$$

- *the level set approach (does require the gradient of solution!):*

$$\phi^{n+1}(x_i) = \phi^n(x_i) - \Delta t^n V \partial_x \phi$$

Flux-based level set method

1D illustration of the idea



- *standard upwind approach and Taylor expansion*

$$\phi^n(x_i - \Delta t^n V) = \phi^n(x_i) - \Delta t^n V \partial_x \phi$$

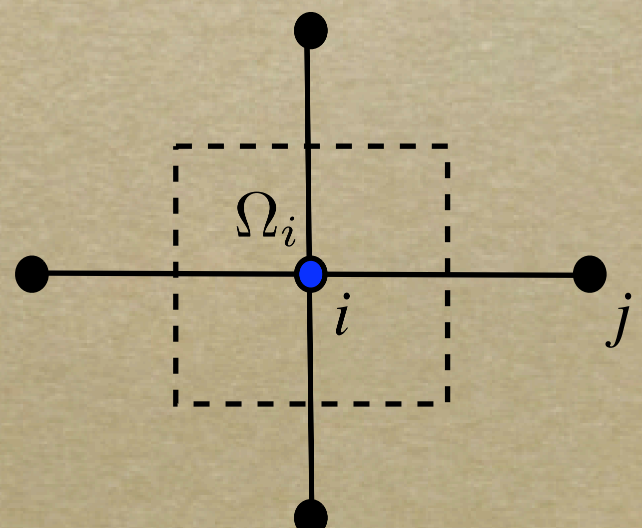
- *the level set approach:*

$$\phi^{n+1}(x_i) = \phi^n(x_i) - \Delta t^n V \partial_x \phi$$

Flux-based level set method

Finite volume method?

$$\nabla \cdot \vec{V} = 0 \qquad \partial_t \phi + \vec{V} \cdot \nabla \phi = 0$$

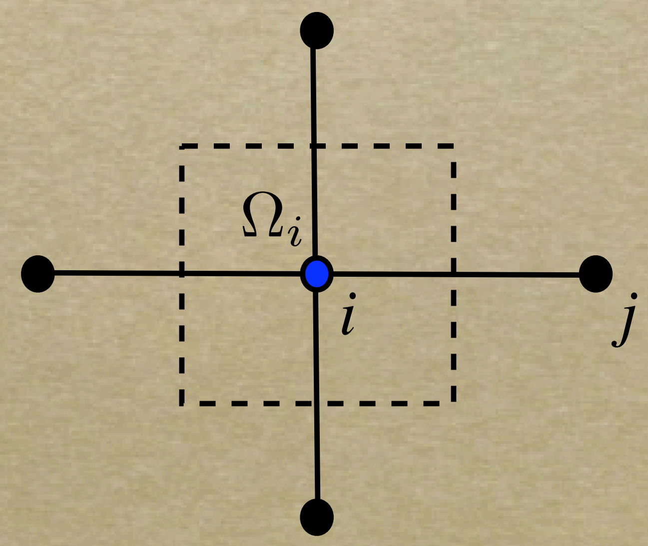


Flux-based level set method

Finite volume method!

$$\nabla \cdot \vec{V} = 0$$

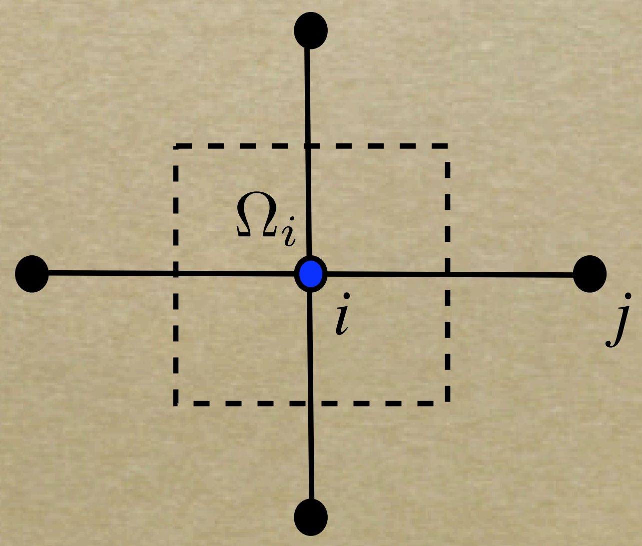
$$\partial_t \phi + \nabla \cdot (\phi \vec{V}) = 0$$



Flux-based level set method

Finite volume method

$$\int_{t^n}^{t^{n+1}} \int_{\Omega_i} \left(\partial_t \phi + \nabla \cdot (\phi \vec{V}) \right) \, dx \, dt = 0$$



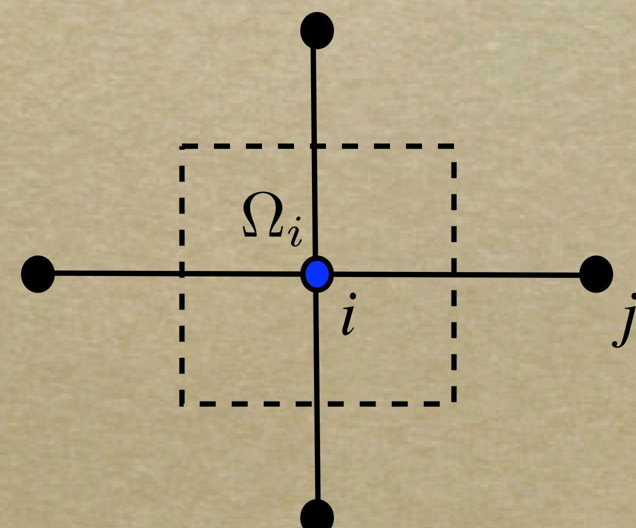
Flux-based level set method

Finite volume method - analytical formulation

$$\int_{\Omega_i} \phi(t^{n+1}) = \int_{\Omega_i} \phi(t^n) - \int_{t^n}^{t^{n+1}} \int_{\partial\Omega_i} \vec{n}_i \cdot \vec{V}\, \phi$$

Finite volume method - discrete formulation

$$\phi_i^{n+1}|\Omega_i| = \phi_i^n|\Omega_i| - \Delta t^n \sum_j V_{ij}\phi_{ij}^{n+1/2}$$



Flux-based level set method

Finite volume method - analytical formulation

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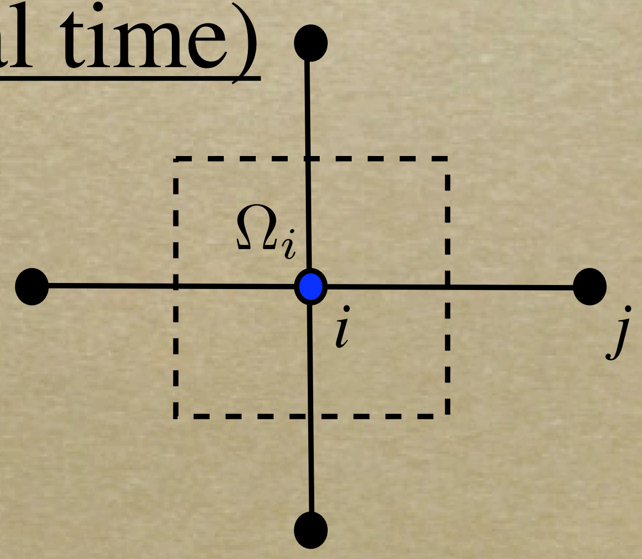
Finite volume method - discrete formulation

$$\phi_i^{n+1}|\Omega_i| = \phi_i^n|\Omega_i| - \Delta t^n \sum_j V_{ij}\phi_{ij}^{n+1/2}$$

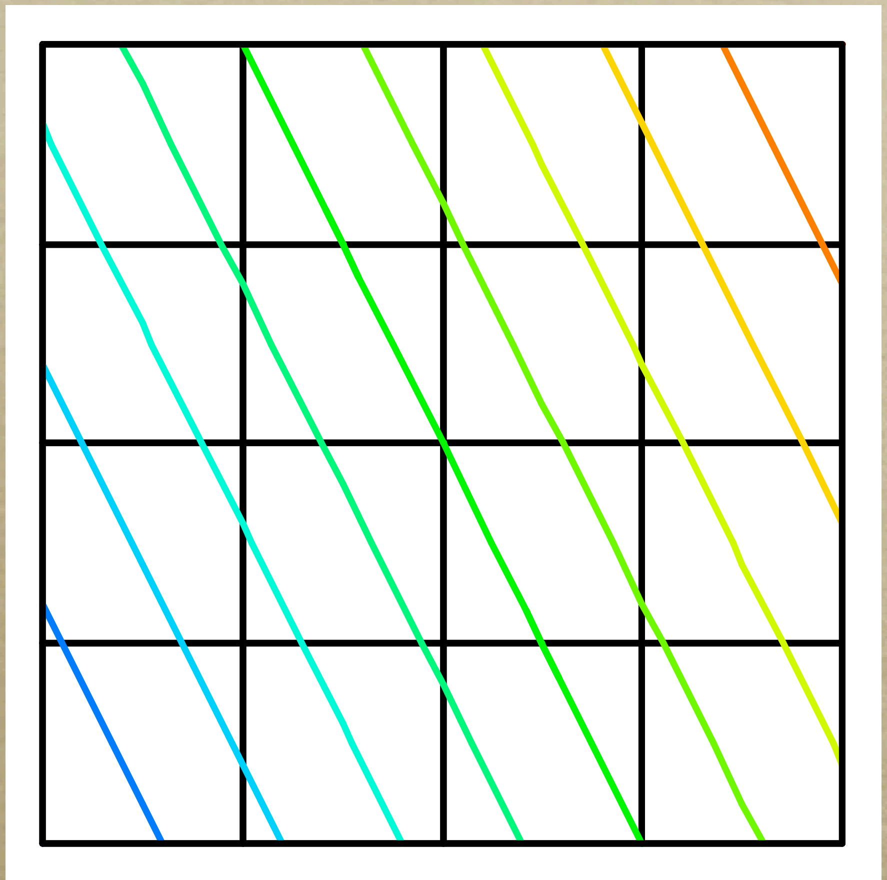
Definition of values in nodes (e.g., for initial time)

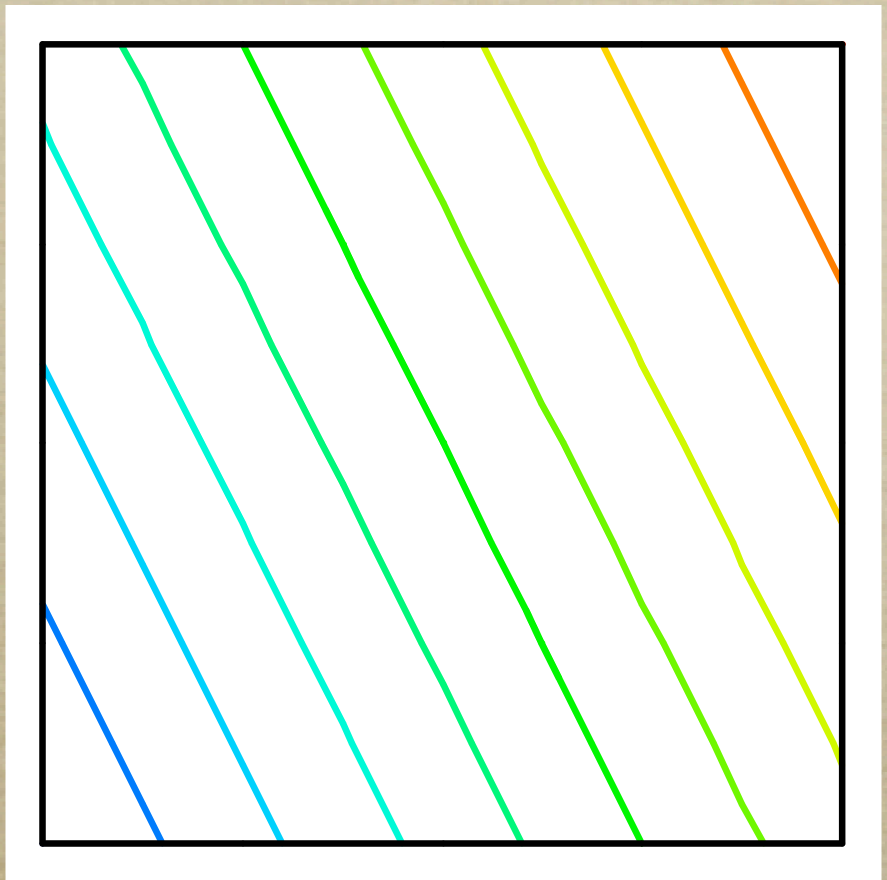
$$\phi_i^0 := \phi^0(x_i)$$



Flux-based level set method

Consistency of finite volume scheme on a structured grid:



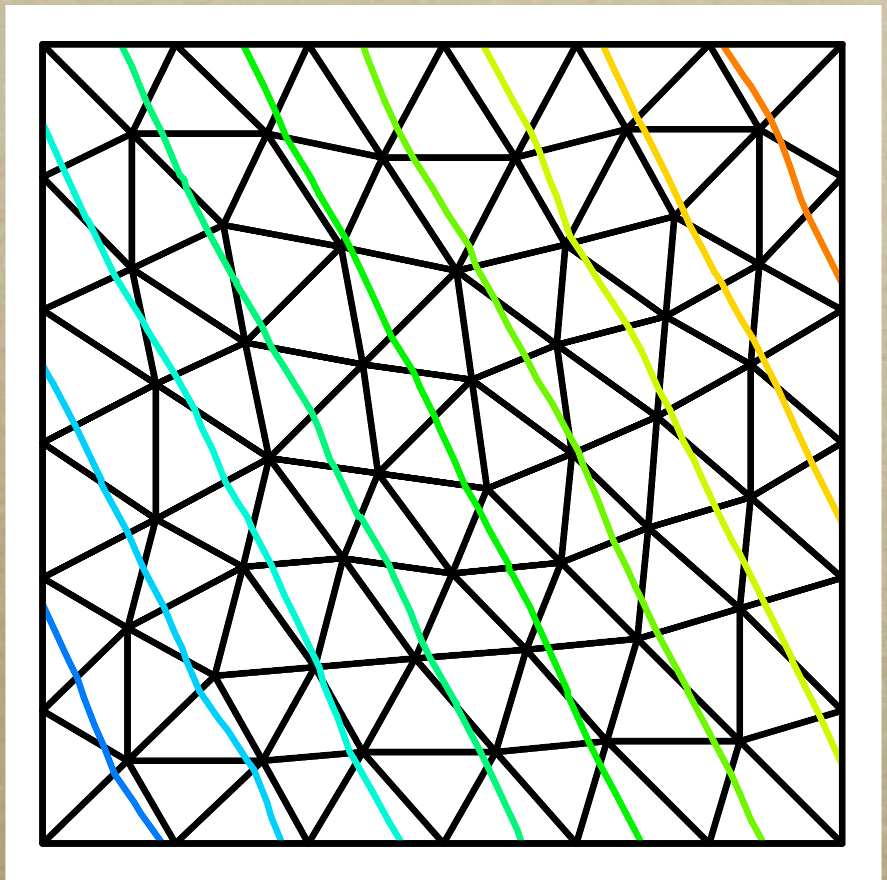


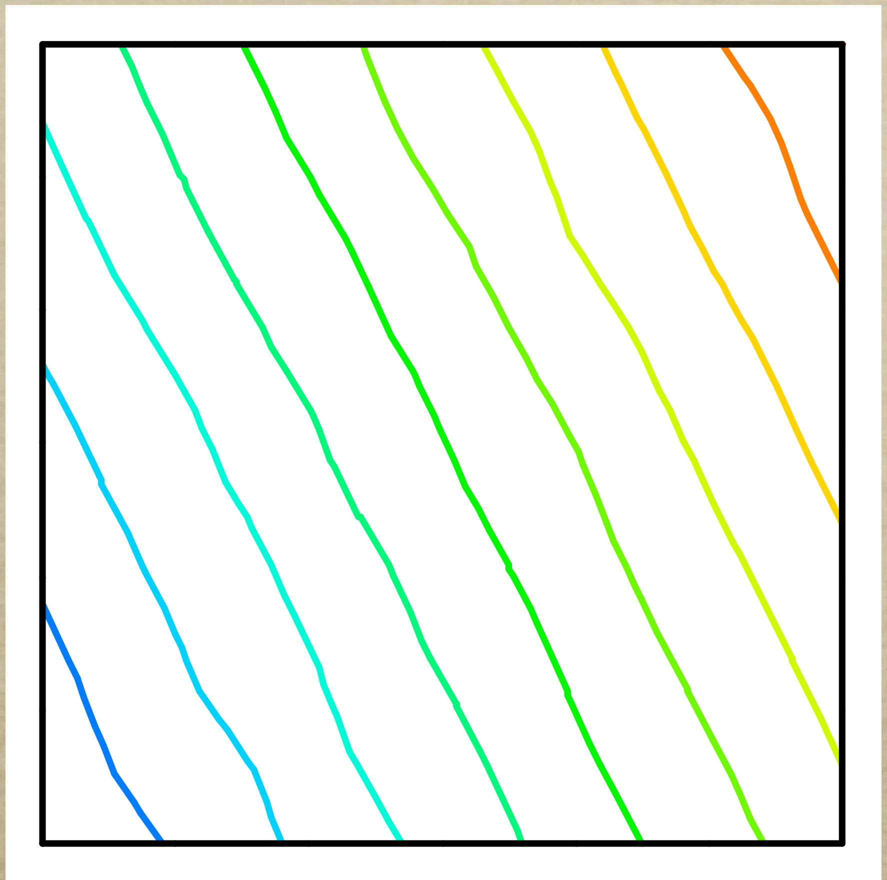
- *averaged nodal values?!*

$$\phi_i^0 := \frac{1}{|\Omega_i|} \int\limits_{\Omega_i} \phi^0(x)\, dx$$

Flux-based level set method

Consistency of finite volume scheme on a unstructured grid:



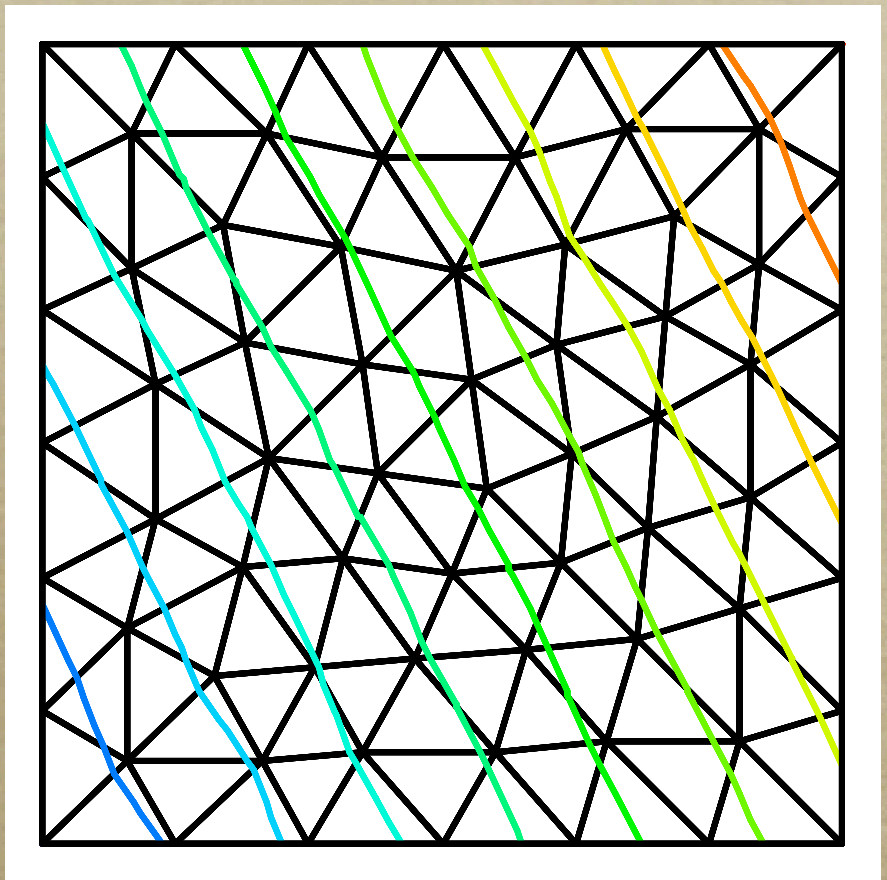


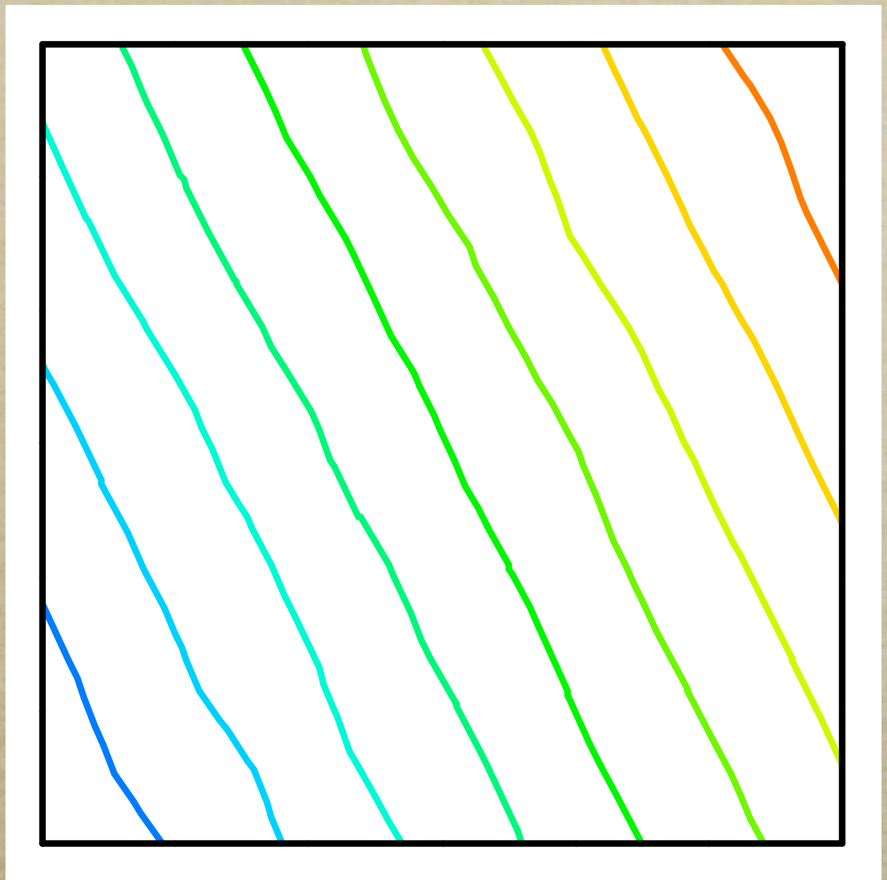
- *averaged nodal values?*

$$\phi_i^0 := \frac{1}{|\Omega_i|} \int\limits_{\Omega_i} \phi^0(x)\, dx$$

Flux-based level set method

Finite volume scheme on a nonuniform grid might be inconsistent



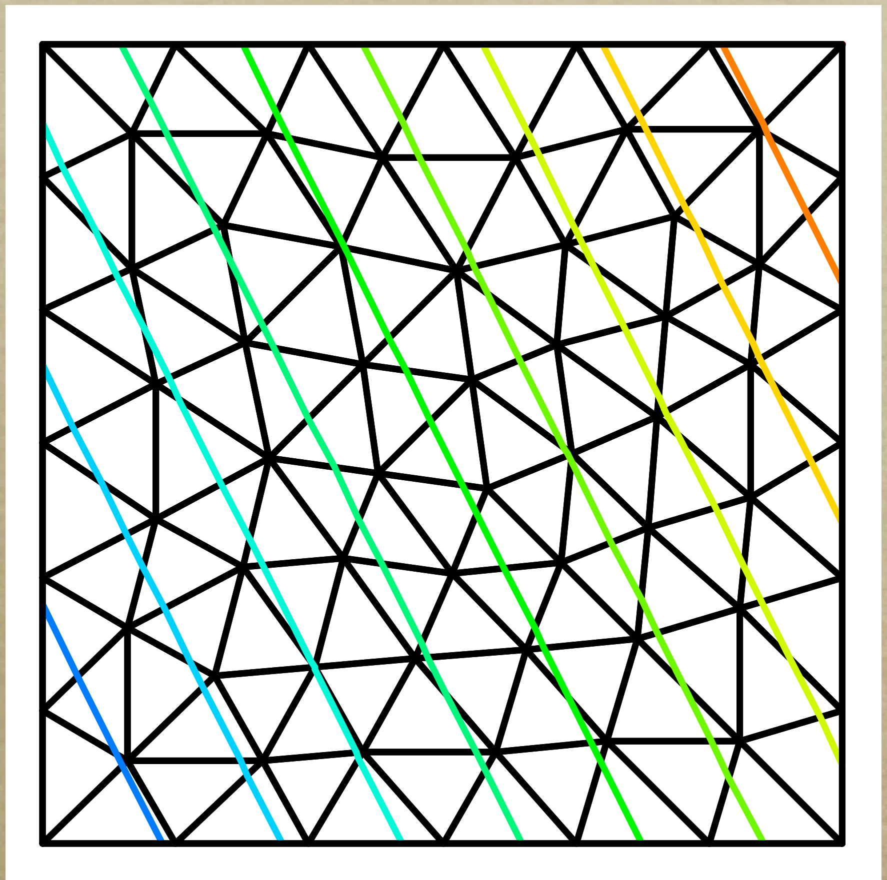


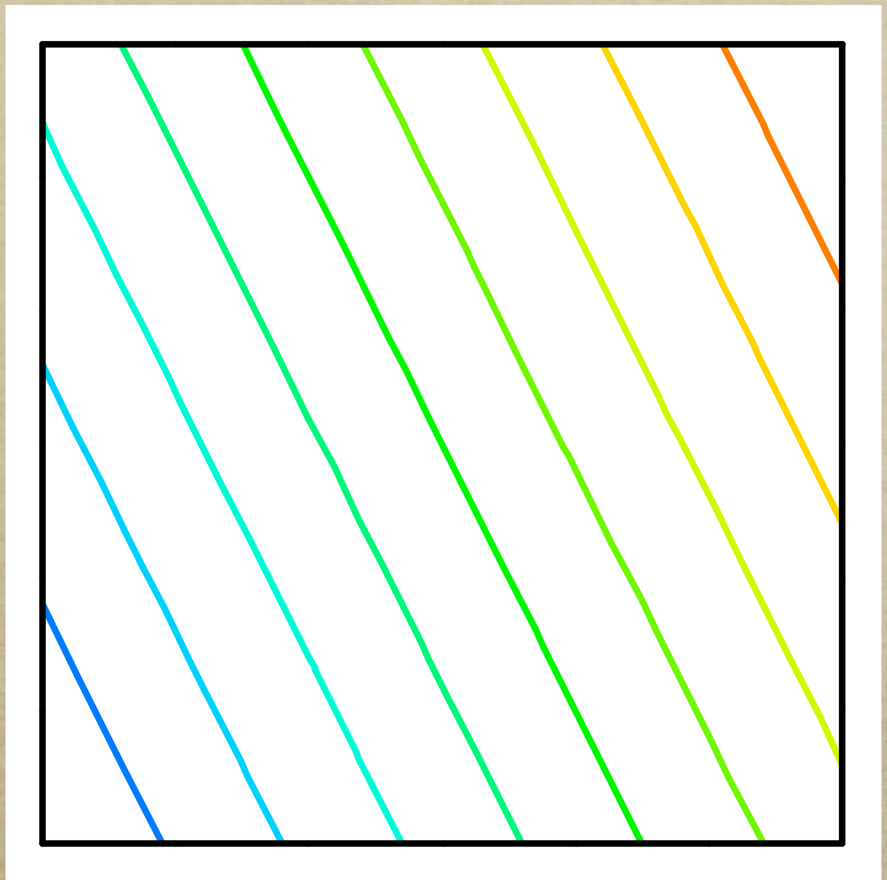
Cockburn, Gremaud, Yang: *A Priori Error Estimates for Numerical Methods for Scalar Conservation Laws*, 1996 (... non-consistent schemes can converge ...)

Barth, Ohlberger: *Finite volume methods: foundation and analysis*, 2004

Flux-based level set method

Finite volume scheme on a nonuniform grid can be consistent





- *nodal values!*

$$\phi_i^0 := \phi^0(x_i)$$

Flux-based level set method

Finite volume method - analytical formulation

$$\int_{\Omega_i} \phi(t^{n+1}) = \int_{\Omega_i} \phi(t^n) - \int_{t^n}^{t^{n+1}} \int_{\partial\Omega_i} \vec{n}_i \cdot \vec{V}\, \phi$$

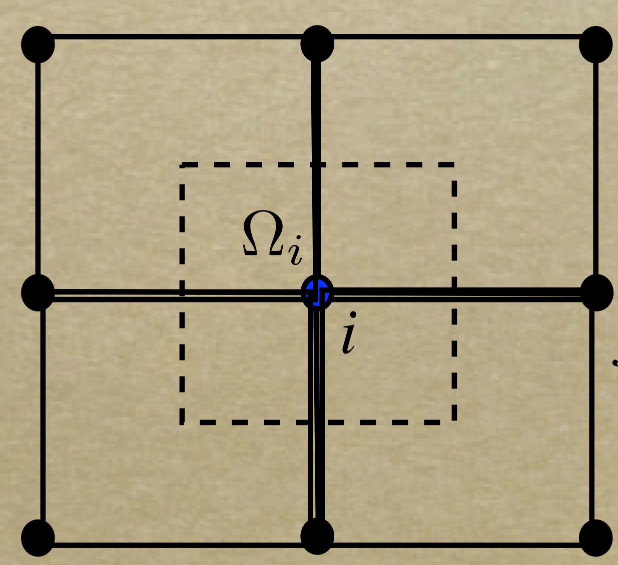
Finite volume method - discrete formulation

$$\phi_i^{n+1}|\Omega_i| = \phi_i^n|\Omega_i| - \Delta t^n \sum_j V_{ij}\phi_{ij}^{n+1/2}$$

Definition of gradients in nodes

$$\nabla \phi_i^n = ?$$

- *many choices*



Flux-based level set method

Finite volume method

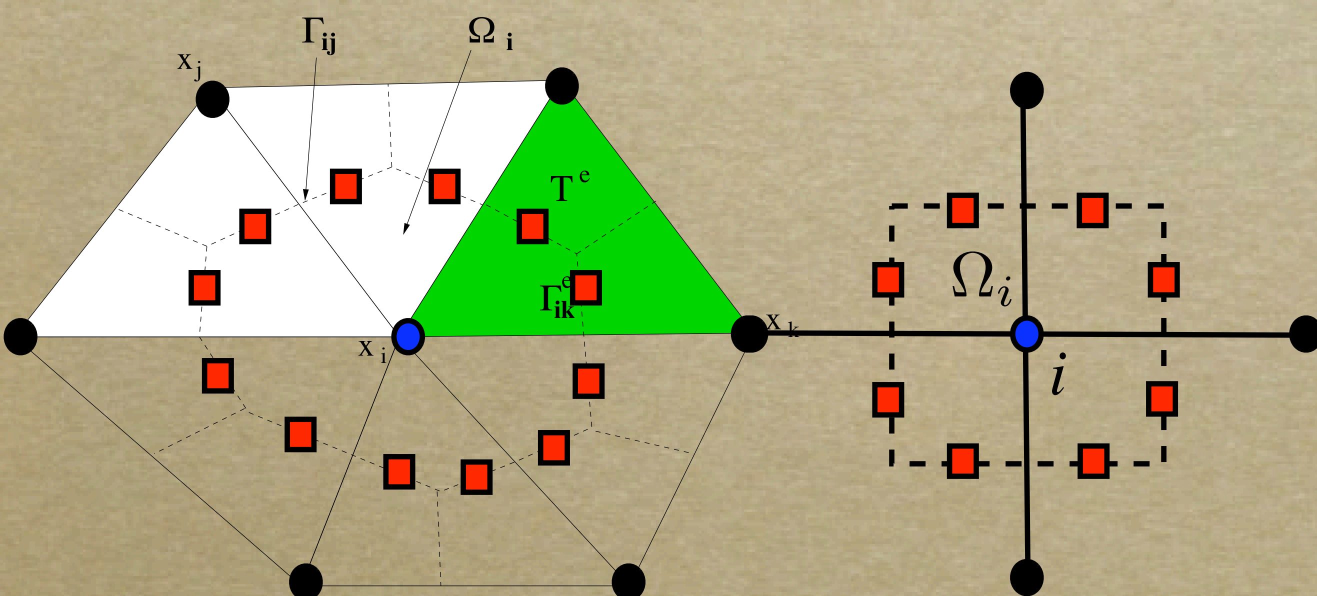
$$\int_{\Omega_i} \phi(t^{n+1}) = \int_{\Omega_i} \phi(t^n) - \int_{t^n}^{t^{n+1}} \int_{\partial\Omega_i} \vec{n}_i \cdot \vec{V}\, \phi$$

Flux-based form

$$\phi_i^{n+1}|\Omega_i| = \phi_i^n|\Omega_i| - \Delta t^n \sum_j V_{ij}\phi_{ij}^{n+1/2}$$

Definition of fluxes?

- *2nd order accurate!*



Flux-based level set method

Finite volume method

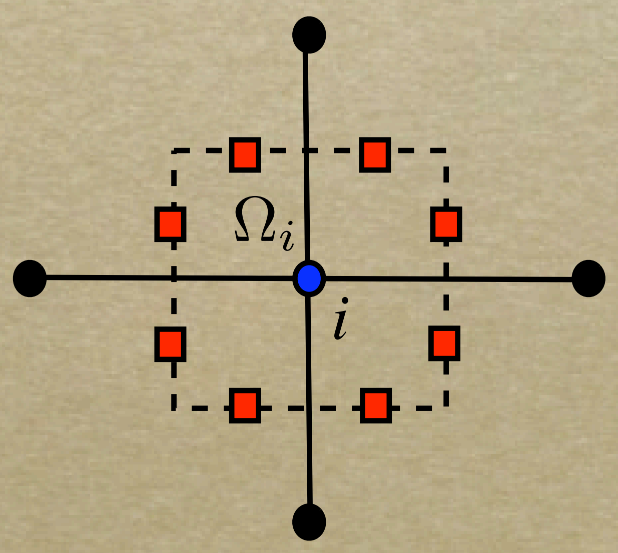
$$\int_{\Omega_i} \phi(t^{n+1}) = \int_{\Omega_i} \phi(t^n) - \int_{t^n}^{t^{n+1}} \int_{\partial\Omega_i} \vec{n}_i \cdot \vec{V} \phi$$

Flux-based form

$$\phi_i^{n+1} |\Omega_i| = \phi_i^n |\Omega_i| - \Delta t^n \sum_j V_{ij} \phi_{ij}^{n+1/2}$$

Numerical approximation of $\nabla \cdot \vec{V} = 0$

$$\sum_j V_{ij} = 0, \quad V_{ij} \approx \int_{\Gamma_j} \vec{n}_i \cdot \vec{V}$$



Flux-based level set method

Finite volume method

$$\int_{\Omega_i} \phi(t^{n+1}) = \int_{\Omega_i} \phi(t^n) - \int_{t^n}^{t^{n+1}} \int_{\partial\Omega_i} \vec{n}_i \cdot \vec{V} \, \phi$$

Flux-based form

$$\phi_i^{n+1} |\Omega_i| = \phi_i^n |\Omega_i| - \Delta t^n \sum_j V_{ij} \phi_{ij}^{n+1/2}$$

Approximation of fluxes - 2nd order consistent scheme?

$$\phi_{ij}^{n+1/2} = \, ?$$

$$V_{ij} > 0$$

Flux-based level set method

Finite volume method

$$\int_{\Omega_i} \phi(t^{n+1}) = \int_{\Omega_i} \phi(t^n) - \int_{t^n}^{t^{n+1}} \int_{\partial\Omega_i} \vec{n}_i \cdot \vec{V}\, \phi$$

Flux-based form

$$\phi_i^{n+1}|\Omega_i| = \phi_i^n|\Omega_i| - \Delta t^n \sum_j V_{ij} \phi_{ij}^{n+1/2}$$

Approximation of fluxes - 2nd order consistent scheme

$$\phi_{ij}^{n+1/2} = \phi_{ij}^n - \frac{\Delta t}{2} \vec{V}_i \cdot \nabla \phi_i^n$$

$$V_{ij} > 0$$

Flux-based level set method

Finite volume method

$$\int_{\Omega_i} \phi(t^{n+1}) = \int_{\Omega_i} \phi(t^n) - \int_{t^n}^{t^{n+1}} \int_{\partial\Omega_i} \vec{n}_i \cdot \vec{V}\,\phi$$

Flux-based form

$$\phi_i^{n+1}|\Omega_i| = \phi_i^n|\Omega_i| - \Delta t^n \sum_j V_{ij}\phi_{ij}^{n+1/2}$$

Approximation of fluxes - 2nd order consistent scheme

$$\phi_{ij}^{n+1/2} = \phi_i^n + (x_{ij} - x_i)\cdot\nabla\phi_i^n - \frac{\Delta t}{2}\vec{V}_i\cdot\nabla\phi_i^n$$

$$V_{ij} > 0$$

Flux-based level set method

Finite volume method

$$\int_{\Omega_i} \phi(t^{n+1}) = \int_{\Omega_i} \phi(t^n) - \int_{t^n}^{t^{n+1}} \int_{\partial\Omega_i} \vec{n}_i \cdot \vec{V} \, \phi$$

Flux-based form

$$\phi_i^{n+1} |\Omega_i| = \phi_i^n |\Omega_i| - \Delta t^n \sum_j V_{ij} \phi_{ij}^{n+1/2}$$

Approximation of fluxes - Taylor expansion

$$\phi_{ij}^{n+1/2} = \phi_i^n + (x_{ij} - x_i) \cdot \nabla \phi_i^n + \frac{\Delta t}{2} \, \partial_t \phi_i^n$$

$$V_{ij} > 0$$

Flux-based level set method

Conservation laws with source term

$$\partial_t \phi + \nabla \cdot \left(\phi \vec{V}\right) = \phi \nabla \cdot \vec{V}$$

Flux-based level set method

Conservation laws with source term

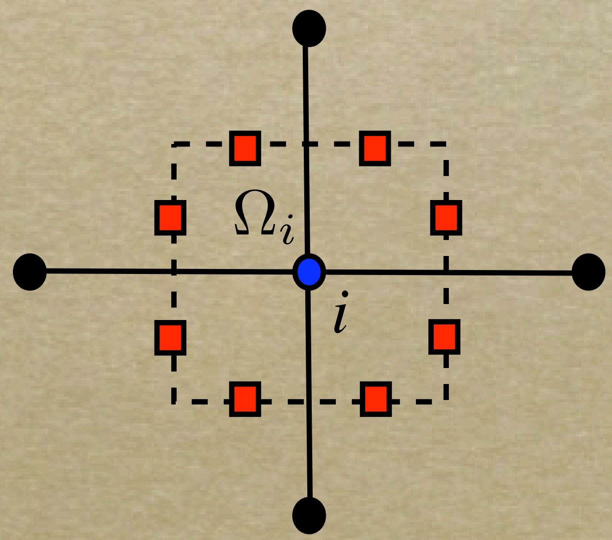
$$\partial_t \phi + \nabla \cdot \left(\phi \vec{V} \right) = \phi \nabla \cdot \vec{V}$$

Flux-based level set method

$$\phi_i^{n+1} |\Omega_i| = \phi_i^n |\Omega_i| - \Delta t^n \sum_j V_{ij} \left(\phi_{ij}^{n+1/2} - \phi_i^{n+1/2} \right)$$

Numerical approximation of $\nabla \cdot \vec{V}$

$$\nabla \cdot \vec{V} \approx \sum_j V_{ij}$$



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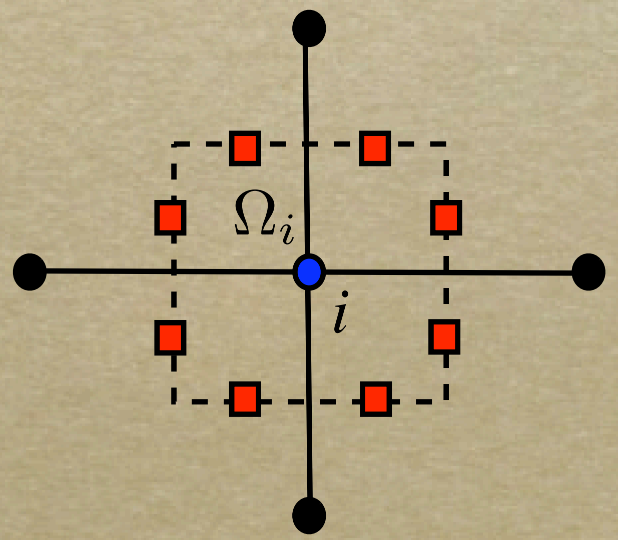
Approximation of source

$$\phi_i^{n+1/2} = \,?$$



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Approximation of source - first order scheme

$$\phi_i^{n+1/2} = \phi_i^n$$

Frolkovic, Mikula: *Flux-based level set method: a finite volume method ...*, ANM 2007

Flux-based level set method

Conservation laws with source term

$$\partial_t \phi + \nabla \cdot \left(\phi \vec{V} \right) = \phi \nabla \cdot \vec{V}$$

Flux-based level set method

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Approximation of source - second order scheme

$$\phi_i^{n+1/2} = \sum \alpha_{ij} \phi_{ij}^{n+1/2}$$

Frolkovic, Mikula: *"High-resolution flux-based level set method"*, SIAM SJSC 2007

Flux-based level set method

Conservation laws with source term

$$\partial_t \phi + \nabla \cdot \left(\phi \vec{V} \right) = \phi \nabla \cdot \vec{V}$$

Flux-based level set method

$$\phi_i^{n+1} |\Omega_i| = \phi_i^n |\Omega_i| - \Delta t^n \sum_j V_{ij} \left(\phi_{ij}^{n+1/2} - \phi_i^{n+1/2} \right)$$

Approximation of source - Taylor expansion

$$\phi_i^{n+1/2} = \phi_i^n + \frac{\Delta t}{2} \partial_t \phi_i^n$$

Flux-based level set method

Conservation laws with source term

$$\partial_t \phi + \nabla \cdot \left(\phi \vec{V} \right) = \phi \nabla \cdot \vec{V}$$

Flux-based level set method

$$\phi_i^{n+1} |\Omega_i| = \phi_i^n |\Omega_i| - \Delta t^n \sum_j V_{ij} \left(\phi_{ij}^{n+1/2} - \phi_i^{n+1/2} \right)$$

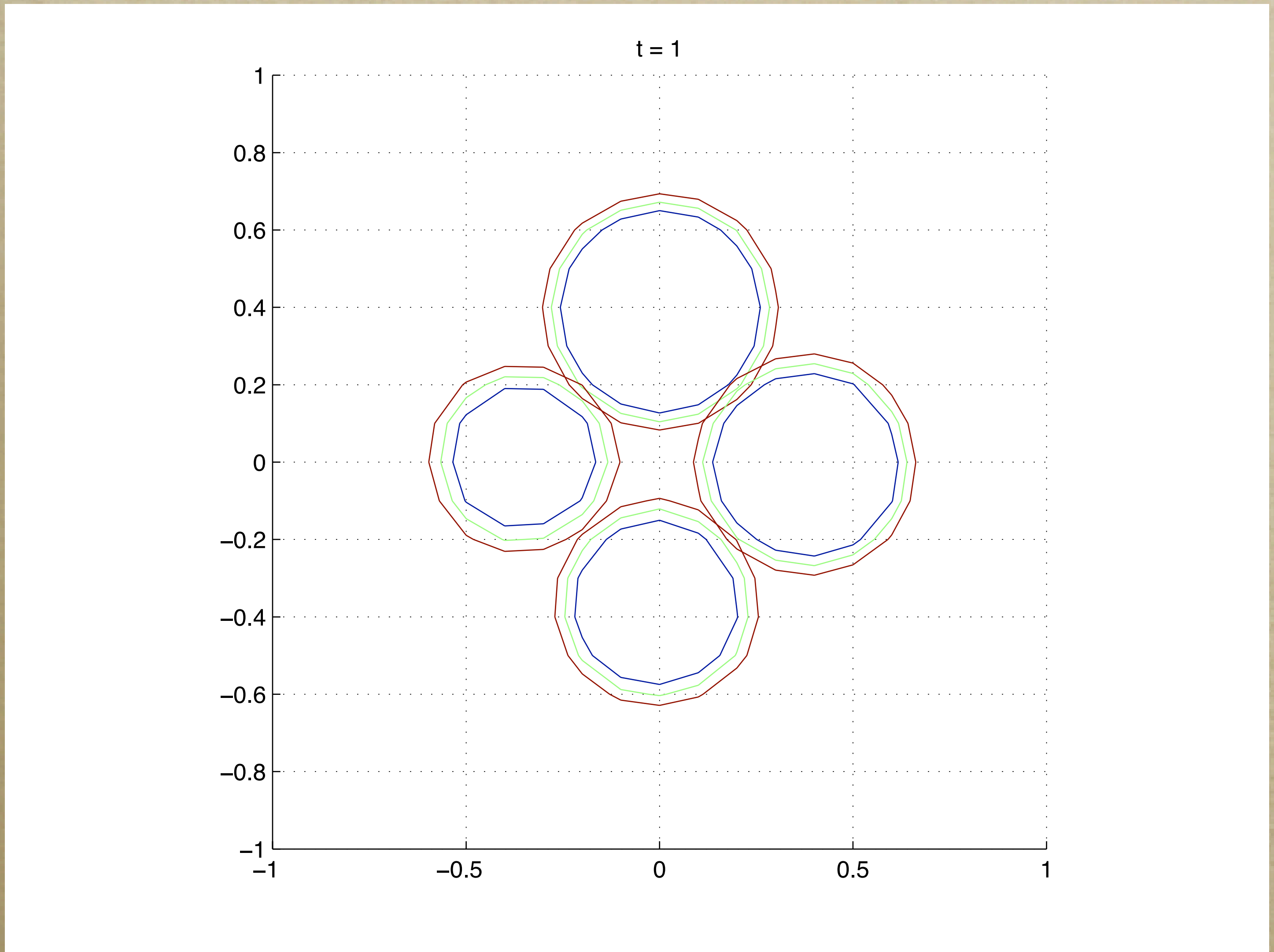
Approximation of source

$$\phi_i^{n+1/2} = \phi_i^n - \frac{\Delta t}{2} \vec{V}_i \cdot \nabla \phi_i^n$$

Frolkovic, Wehner: *Flux-based level set method on rectangular grids ..., CVS* 2008

Benchmarks

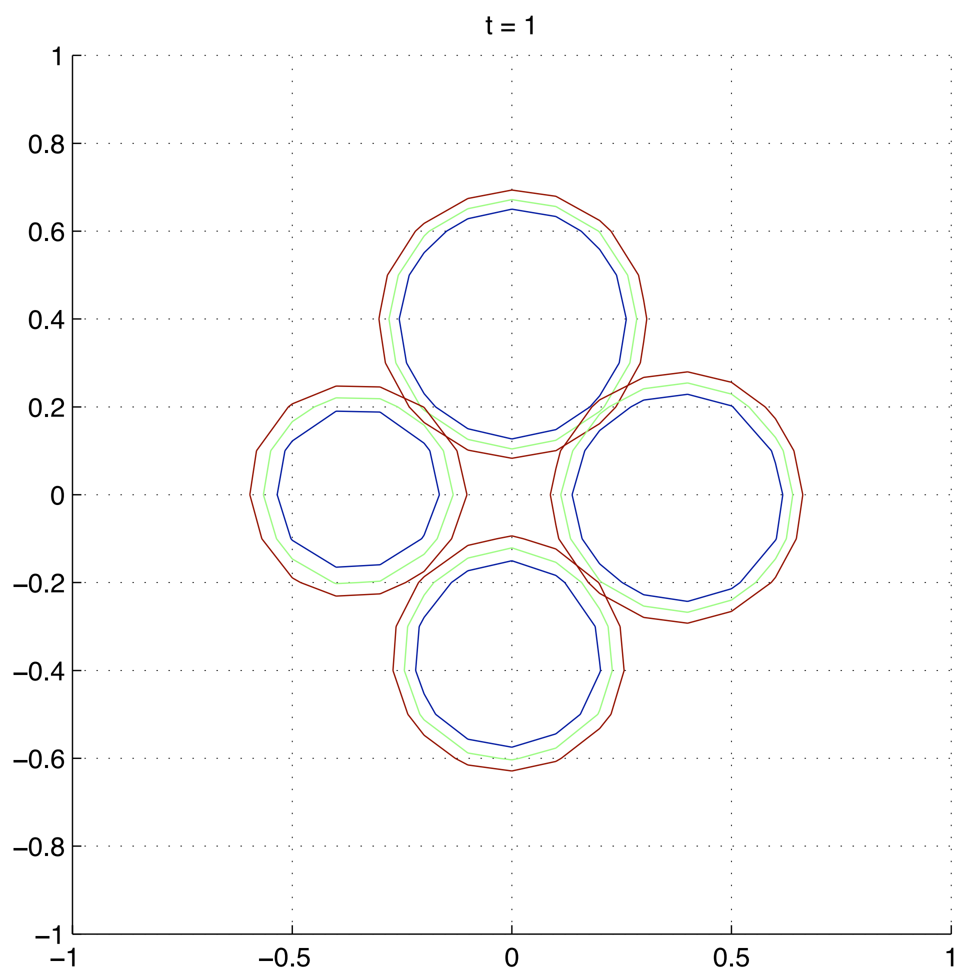
Rotation of a circle

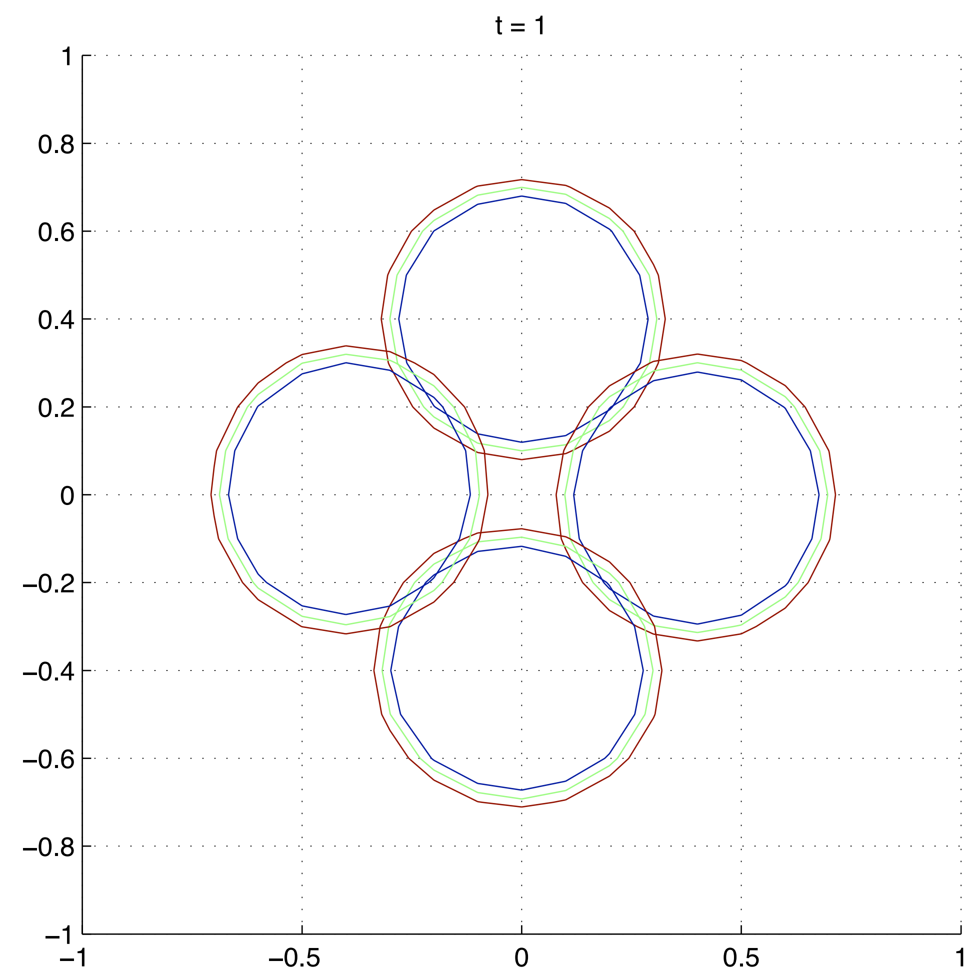


2nd order ENO

Benchmarks

Rotation of a circle



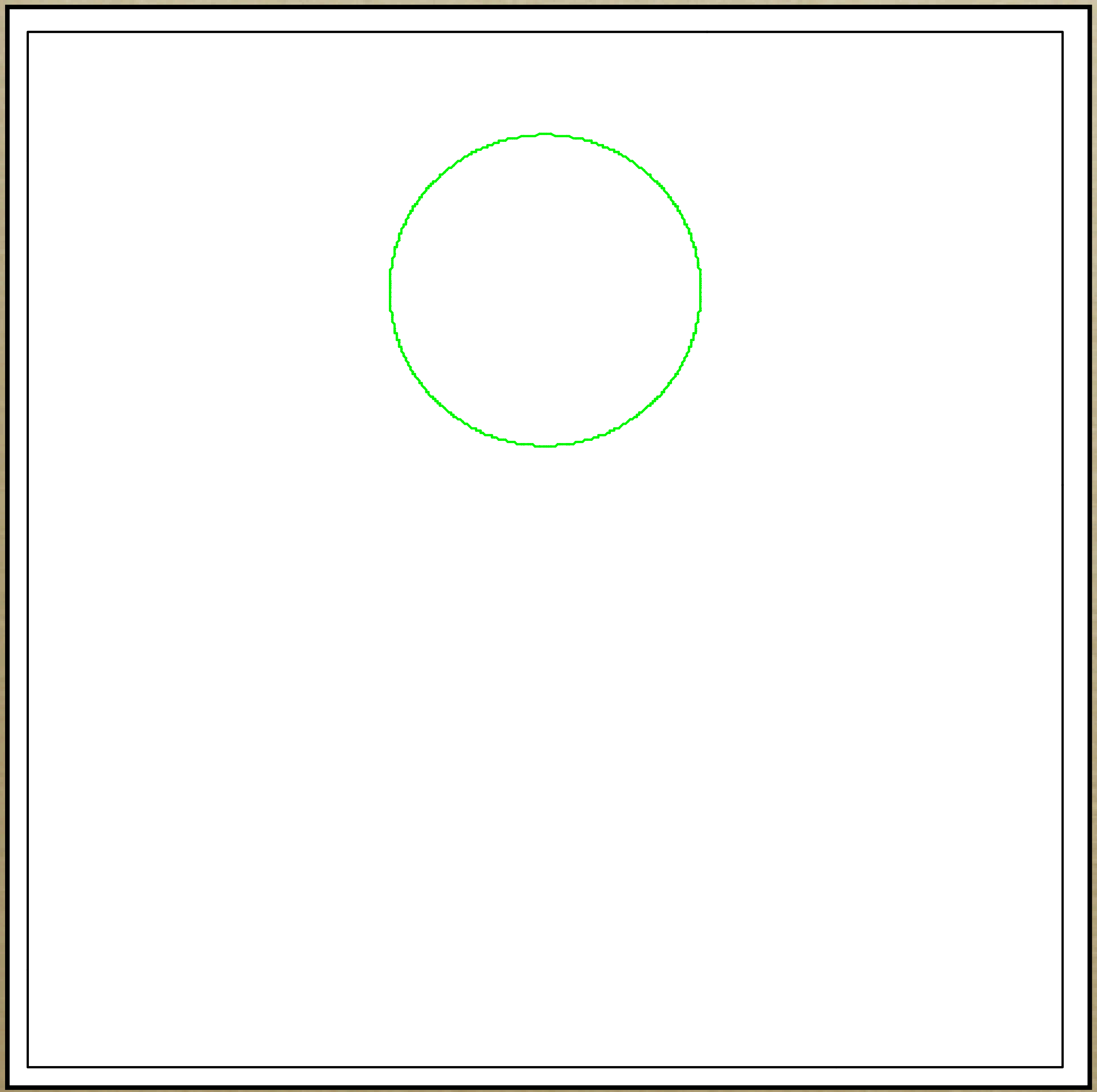


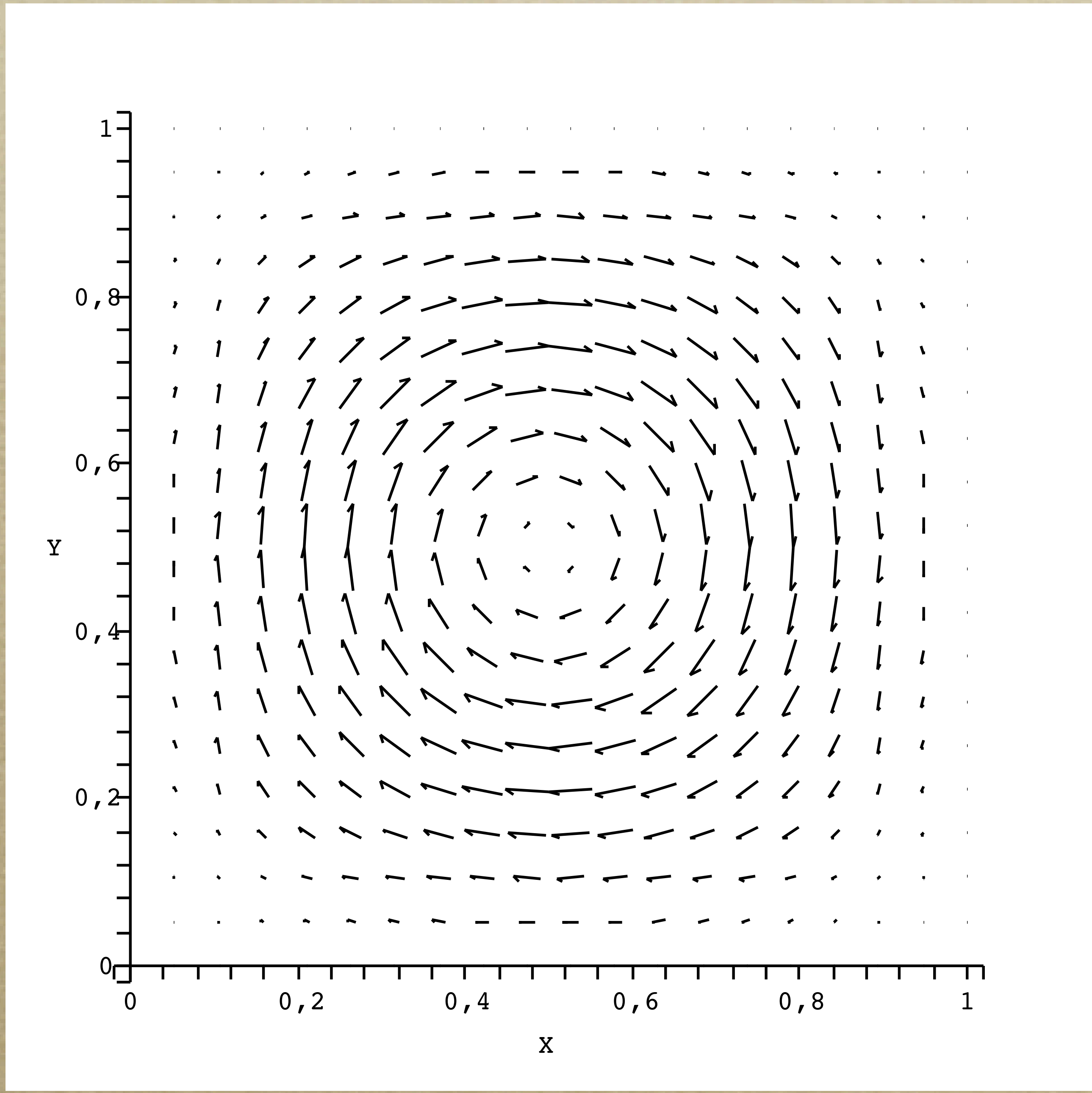
2nd order ENO　　　　*Flux-based LSM*

Benchmarks

Single-vortex example

$$\vec{V} = (-\partial_y \Psi, \partial_x \Psi)\,, \qquad \Psi = \frac{1}{\pi}\cos(\pi t/8)\sin^2(\pi x)\sin^2(\pi y)$$





Benchmarks

Single-vortex example

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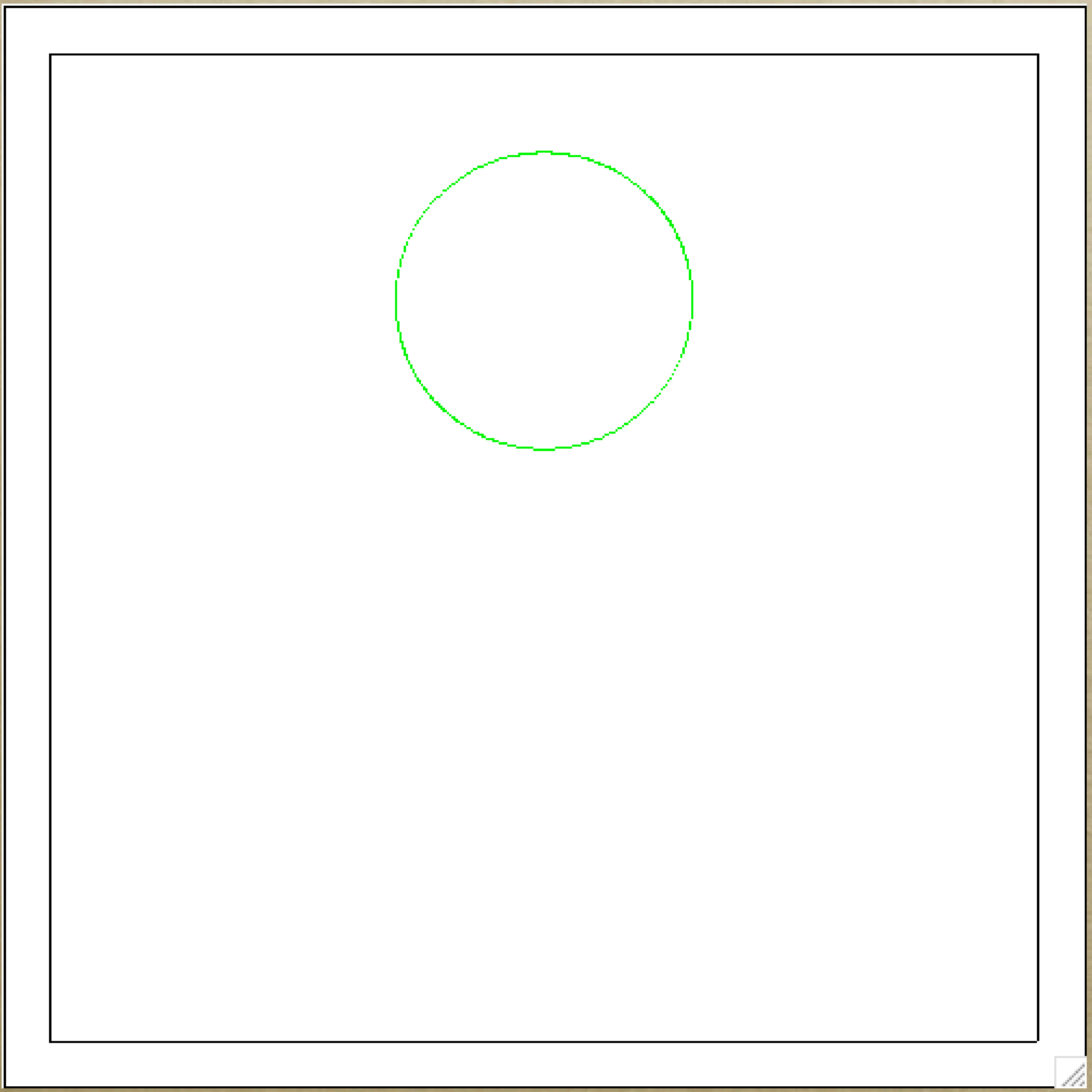
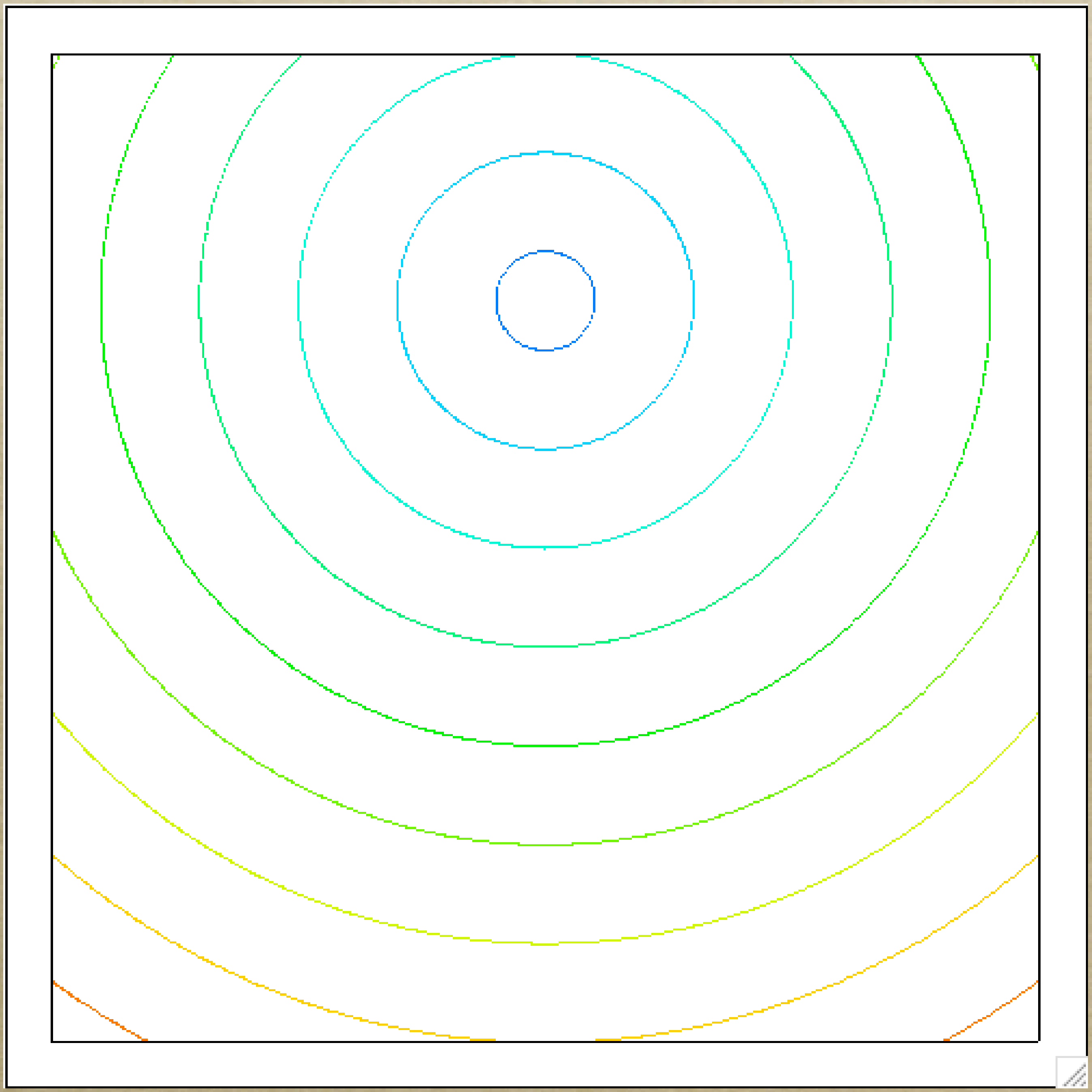
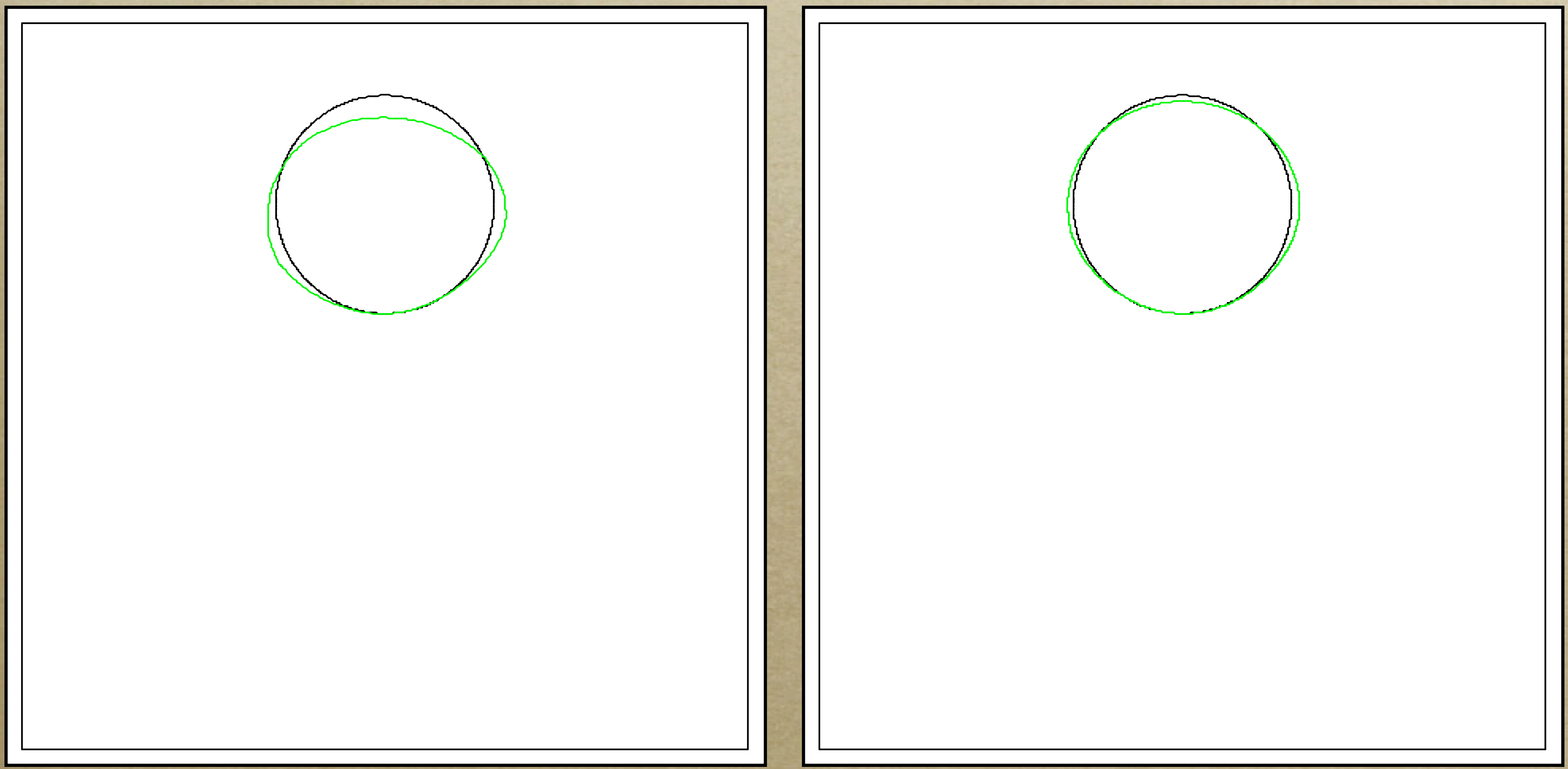




Benchmarks

Single-vortex example

- *approximation of the area: left 100^2 nodes, right 200^2 nodes*



Benchmarks

Single-vortex example

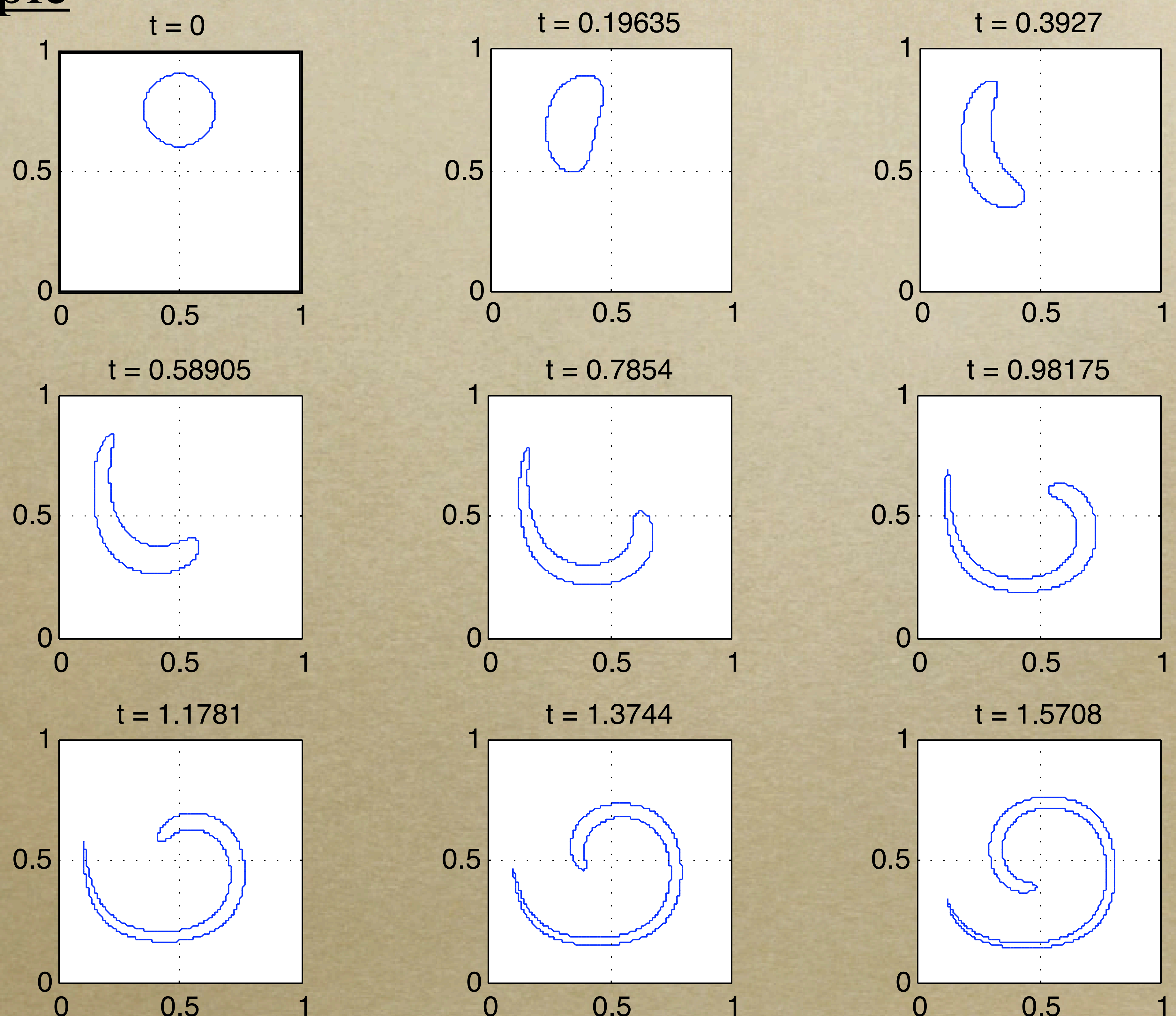
- *the reconstructed gradient similar to Beam-Warming scheme*

I	N	Solution	EOC	Area	EOC
50	500	1.30E-2		-4.02E-2	
100	1000	4.43E-3	1.56	-3.29E-3	3.61
200	2000	1.43E-3	1.63	2.73E-3	0.27
300	3000	6.25E-4	2.05	1.68E-3	1.20
400	4000	3.35E-4	2.17	8.07E-4	2.55
500	5000	2.06E-4	2.18	3.44E-4	3.83
600	6000	1.39E-4	2.17	1.37E-4	5.05
700	7000	9.93E-5	2.17	5.86E-5	5.50
800	8000	7.44E-5	2.17	3.37E-5	4.14

Benchmarks

Single-vortex example

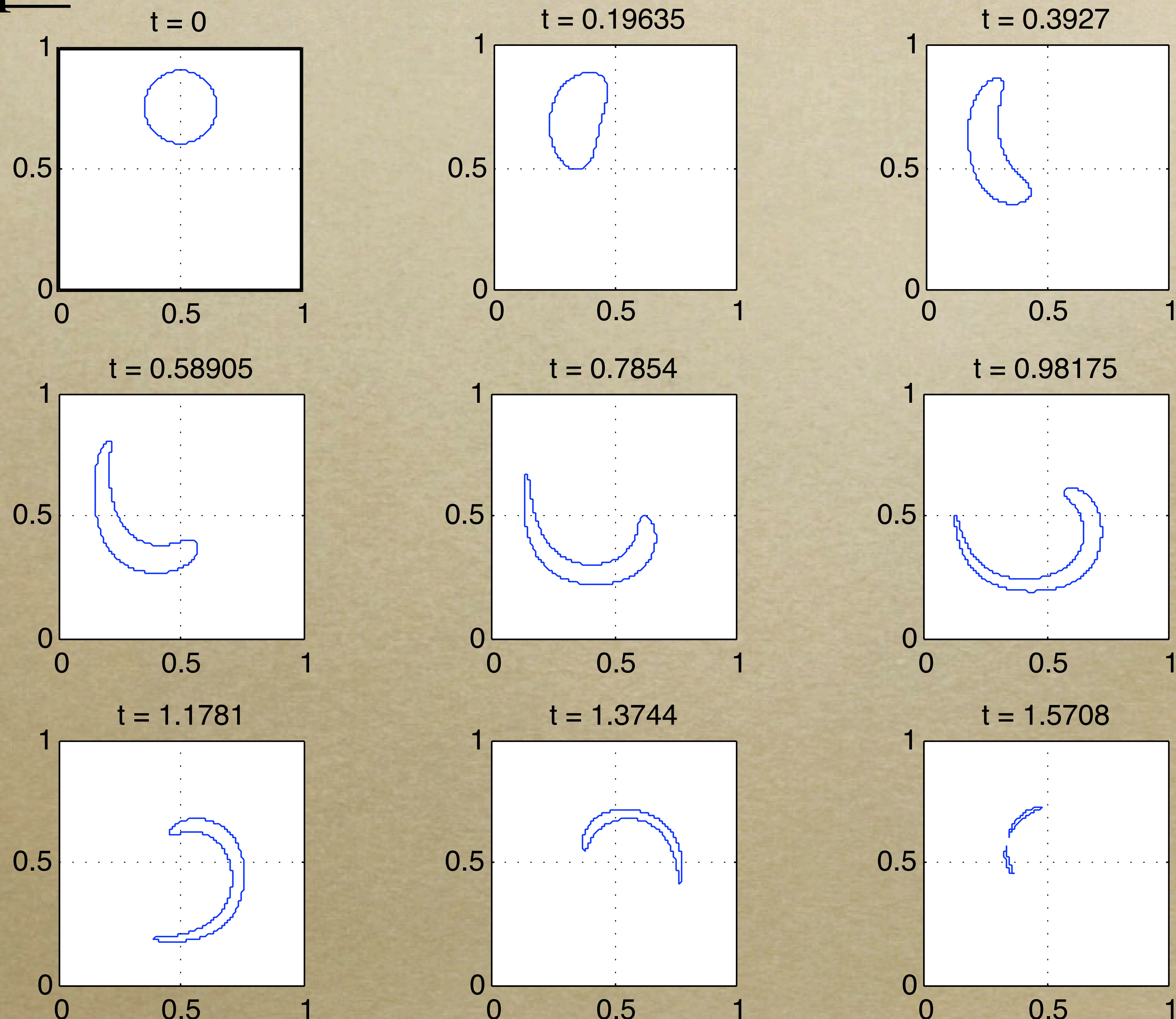
- *2nd order*
- *200x200 grid*



Benchmarks

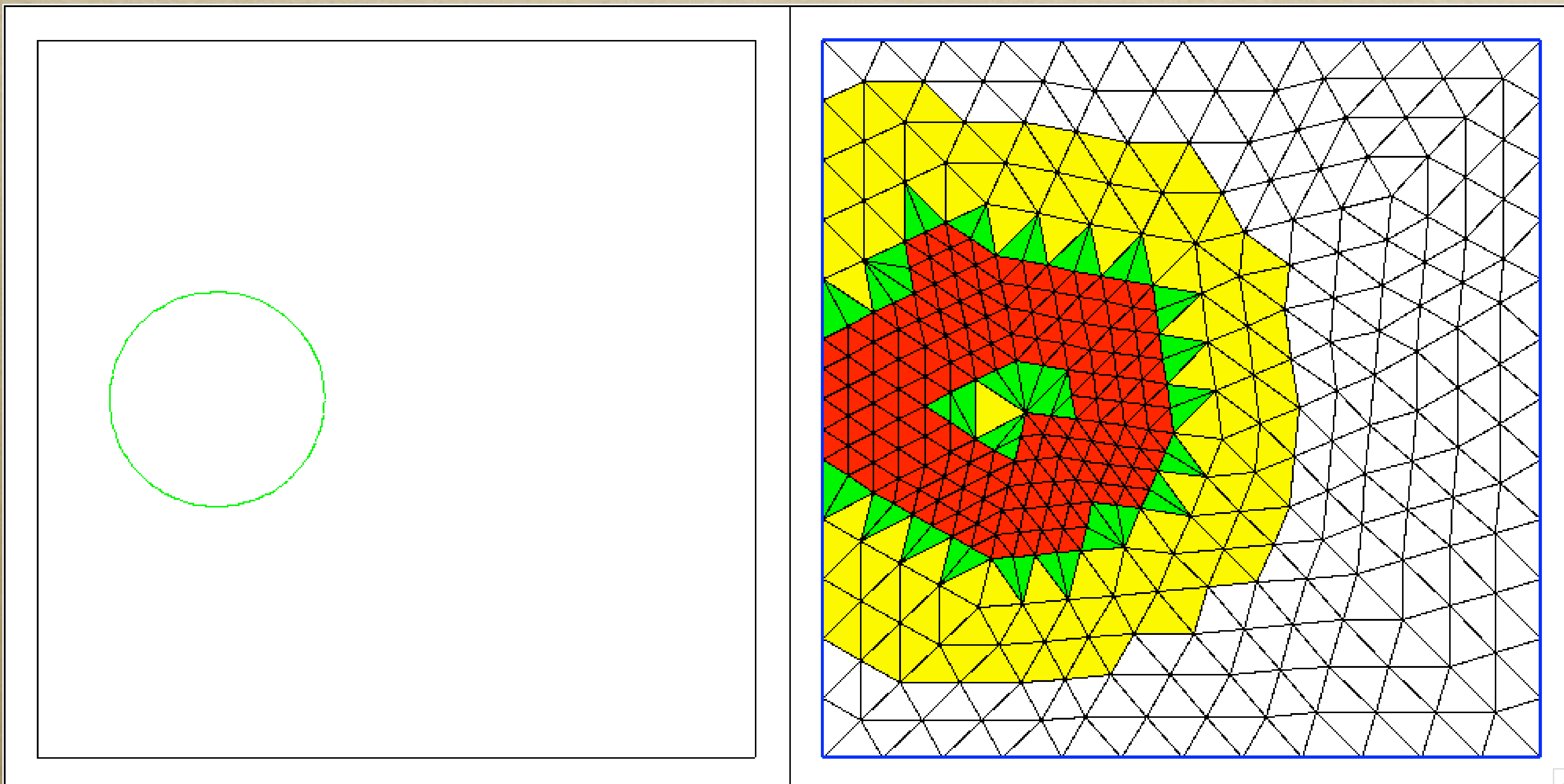
Single-vortex example

- *1st order*
- *200x200 grid*



Benchmarks

Local grid refinement and coarsening



Conclusions

-
- *flux-based level set method*
  - *“simple”*
- *MATLAB version available (Ch. Wehner)*
- *Unstructured Grids (UG) shall be available*
- *see math.sk/frolkovi for Quicktime version of this lecture*