Discretization schemes for linear diffusion operators on general nonconforming meshes

R. Eymard[1], **T. Gallouët**[2] **and R. Herbin**[2]

[1]Université Paris-Est, France, [2]Université de Provence, France

robert.eymard@univ-mlv.fr, thierry.gallouet, raphaele.herbin@cmi.univ-mrs.fr

1. Objectives

Approximation of the solution u to

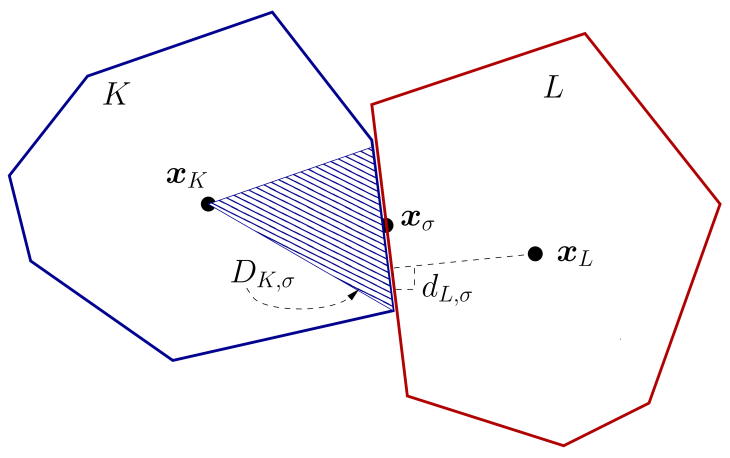
$$\begin{cases} u \in H_0^1(\Omega), \\ \displaystyle\int_\Omega \Lambda(\boldsymbol{x})\nabla u(\boldsymbol{x}) \cdot \nabla v(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = \int_\Omega f(\boldsymbol{x})v(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \quad \forall v \in H_0^1(\Omega) \end{cases}$$

with constraints:

- Any type of grid (conforming or not), 2D or 3D
- sparse, symmetric, positive and definite matrix
- approximation of u and ∇u (with convergence and error estimate)

2. Discrete unknowns

Discretization: $\mathcal{D} = (\mathcal{M}, \mathcal{E}, \mathcal{P})$ (control volumes, interfaces and points)



Discrete unknowns: $(v_K)_{K\in\mathcal{M}}$ and $(v_\sigma)_{\sigma\in\mathcal{E}}$

$X_\mathcal{D} = \{v = ((v_K)_{K\in\mathcal{M}}, (v_\sigma)_{\sigma\in\mathcal{E}}), v_K \in \mathbb{R}, v_\sigma \in \mathbb{R}\}$

$X_{\mathcal{D},0} = \{v \in X_\mathcal{D}$ such that $v_\sigma = 0,\ \forall\sigma \in \mathcal{E}_{\text{ext}}\}$

$H_\mathcal{M}(\Omega) \subset L^2(\Omega)$: set of piecewise constant functions on the control volumes

$\Pi_\mathcal{M} v \in H_\mathcal{M}(\Omega)$: piecewise constant function $\Pi_\mathcal{M} v(\boldsymbol{x}) = v_K$ for a.e. $\boldsymbol{x} \in K$

3. Discrete gradient

For $u \in X_\mathcal{D}$, construction of a discrete gradient:

- **Geometric formula.** For $K \in \mathcal{M}$ (using x_σ, center of σ):

$$\sum_{\sigma\in\mathcal{E}_K} \mathrm{m}(\sigma)\mathbf{n}_{K,\sigma}(\boldsymbol{x}_\sigma - \boldsymbol{x}_K)^T = \mathrm{m}(K)\mathrm{Id}$$

- **Rough gradient.** For $K \in \mathcal{M}$:

$$\nabla_K u = \frac{1}{\mathrm{m}(K)} \sum_{\sigma\in\mathcal{E}_K} \mathrm{m}(\sigma)(u_\sigma - u_K)\mathbf{n}_{K,\sigma}$$

- **Stabilized gradient.** For $K \in \mathcal{M}$ and $\sigma \in \mathcal{E}_K$:

$$\nabla_{K,\sigma} u = \nabla_K u + R_{K,\sigma} u\ \mathbf{n}_{K,\sigma},$$

with

$$R_{K,\sigma} u = \frac{\sqrt{d}}{d_{K,\sigma}}(u_\sigma - u_K - \nabla_K u \cdot (\boldsymbol{x}_\sigma - \boldsymbol{x}_K))$$

$\nabla_\mathcal{D} u(\boldsymbol{x}) = \nabla_{K,\sigma} u$ for a.e. $\boldsymbol{x} \in D_{K,\sigma}$

4. Numerical scheme, SUSHI

- **Reduction of the number of unknowns.** $\mathcal{E}_{\text{int}} = \mathcal{B} \cup \mathcal{H}$, $\mathcal{H} = \mathcal{E}_{\text{int}} \setminus \mathcal{B}$.

$$X_{\mathcal{D},\mathcal{B}} = \{v \in X_{\mathcal{D},0}, v_\sigma = \sum_{K\in\mathcal{M}} \beta_\sigma^K v_K,\ \sum_{K\in\mathcal{M}} \beta_\sigma^K = 1 \ \&\ \boldsymbol{x}_\sigma = \sum_{K\in\mathcal{M}} \beta_\sigma^K \boldsymbol{x}_K, \forall\sigma \in \mathcal{B}\}$$

- **Discrete equations.**

card($\mathcal{M}$) + card($\mathcal{H}$) equations and unknowns u_K for $K \in \mathcal{M}$ and u_σ for $\sigma \in \mathcal{H}$

$$\begin{cases} \text{Find } u \in X_{\mathcal{D},\mathcal{B}} \text{ such that:} \\ \displaystyle\int_\Omega \Lambda\nabla_\mathcal{D} u \cdot \nabla_\mathcal{D} v \mathrm{d}\boldsymbol{x} = \int_\Omega f\ \ \Pi_\mathcal{M} v \mathrm{d}\boldsymbol{x},\ \forall v \in X_{\mathcal{D},\mathcal{B}} \end{cases}$$

The so-called "local conservativity" is ensured for all $\sigma \in \mathcal{H}$ (taking in the scheme v associated to σ) but not for $\sigma \in \mathcal{B}$

In 2d, for $\Lambda = Id$ and with triangles or rectangles, the scheme is the classical finite volume scheme

5. Convergence of the approximate solution

Non degeneracy of the discretization. $h_\mathcal{D} = \sup\{\mathrm{diam}(K), K \in \mathcal{M}\}$

$$\theta_\mathcal{D} = \max\left(\max_{\sigma\in\mathcal{E}_{\text{int}}, K,L\in\mathcal{M}_\sigma} \frac{d_{K,\sigma}}{d_{L,\sigma}},\ \max_{K\in\mathcal{M},\sigma\in\mathcal{E}_K} \frac{h_K}{d_{K,\sigma}}\right)$$

$$\theta_{\mathcal{D},\mathcal{B}} = \max\left(\theta_\mathcal{D},\ \max_{K\in\mathcal{M},\sigma\in\mathcal{E}_K\cap\mathcal{B}} \frac{\sum_{L\in\mathcal{M}} |\beta_\sigma^L||\boldsymbol{x}_L - \boldsymbol{x}_\sigma|^2}{h_K^2}\right)$$

Let $\theta > 0$ and $u_\mathcal{D}$ the approximate solution with $\mathcal{D}$ and $\mathcal{B}$ such that $\theta_{\mathcal{D},\mathcal{B}} \le \theta$. Then $\Pi_\mathcal{M} u_\mathcal{D}$ converges in $L^2(\Omega)$ to the unique solution u and $\nabla_\mathcal{D} u_\mathcal{D}$ converges to ∇u in $L^2(\Omega)^d$ as $h_\mathcal{D} \to 0$.

6. Proof of the convergence result

- **Discrete norms**
  - Norm in $X_\mathcal{D}$:

$$\forall v \in X_\mathcal{D},\ |v|_X^2 = \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} \frac{\mathrm{m}(\sigma)}{d_{K,\sigma}}(v_\sigma - v_K)^2$$

  - Norm in $H_\mathcal{M}(\Omega)$ (Discrete H_0^1–norm)

For $v \in H_\mathcal{M}(\Omega)$ and for $\sigma \in \mathcal{E}_{\text{int}}$ with $\mathcal{M}_\sigma = \{K, L\}$, $D_\sigma v = |v_K - v_L|$ and $d_\sigma = d_{K,\sigma} + d_{L,\sigma}$

For $\sigma \in \mathcal{E}_{\text{ext}}$ with $\mathcal{M}_\sigma = \{K\}$, $D_\sigma v = |v_K|$ and $d_\sigma = d_{K,\sigma}$

$$\forall v \in H_\mathcal{M}(\Omega),\ \|v\|_{1,2,\mathcal{M}} = \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} m_\sigma d_{K,\sigma}(\frac{D_\sigma v}{d_\sigma})^2$$

  - Comparison of the norms in $X_{\mathcal{D},0}$

Thanks to the Cauchy-Schwarz inequality:

$$\|\Pi_\mathcal{M} v\|_{1,2,\mathcal{M}}^2 \le |v|_X^2,\ \forall v \in X_{\mathcal{D},0}.$$

- **Equivalence of norms**

Thanks to

$$(R_{K,\sigma} u)^2 \ge \frac{\lambda d}{1+\lambda}\left(\frac{u_\sigma - u_K}{d_{K,\sigma}}\right)^2 - \lambda d|\nabla_K u|^2 \left(\frac{|\boldsymbol{x}_\sigma - \boldsymbol{x}_K|}{d_{K,\sigma}}\right)^2.$$

Equivalence of norms:

$$\alpha|u|_X \le \|\nabla_\mathcal{D} u\|_{L^2(\Omega)} \le \beta|u|_X,\ \forall u \in X_\mathcal{D}$$

- **Estimate on $u_\mathcal{D}$, solution of the scheme**

Taking $v = u_\mathcal{D}$ in the scheme and using the preceding equivalence of norms and a discrete Sobolev inequality lead to an estimate on $u_\mathcal{D}$ in L^2 and for the norm of $X_\mathcal{D}$

- **Regularity of the possible limit and convergence of the gradient**

Let $\mathcal{F}$ be a family of discretization and, for $\mathcal{D} \in \mathcal{F}$, $u_\mathcal{D}$ (not necessarily solution of the scheme) such that:

- $u_\mathcal{D} \in X_{\mathcal{D},0}$
- $|u_\mathcal{D}|_X \le C$
- $\exists u \in L^2(\Omega)$ with $\lim_{h_\mathcal{D}\to 0} \|\Pi_\mathcal{M} u_\mathcal{D} - u\|_{L^2(\Omega)} = 0$

Then $u \in H_0^1(\Omega)$ and $\nabla_\mathcal{D} u_\mathcal{D}$ weakly converges in $L^2(\Omega)^d$ to ∇u as $h_\mathcal{D} \to 0$

- **Passing to the limit in the discrete equation**

For $\varphi \in C2_c(\Omega, \mathbb{R})$, let:

1. $P_\mathcal{D}\varphi = ((\varphi(\boldsymbol{x}_K))_{K\in\mathcal{M}}, (\varphi(\boldsymbol{x}_\sigma))_{\sigma\in\mathcal{E}})$ (so that $P_\mathcal{D}\varphi \in X_\mathcal{D}$)
2. $P_{\mathcal{D},\mathcal{B}}\varphi$ the element $v \in X_{\mathcal{D},\mathcal{B}}$ such that $v_K = \varphi(\boldsymbol{x}_K)$ for all $K \in \mathcal{M}$, $v_\sigma = 0$ for all $\sigma \in \mathcal{E}_{\text{ext}}$, $v_\sigma = \sum_{K\in\mathcal{M}} \beta_\sigma^K \varphi(\boldsymbol{x}_K)$ for all $\sigma \in \mathcal{B}$ and $v_\sigma = \varphi(\boldsymbol{x}_\sigma)$ for all $\sigma \in \mathcal{E}_{\text{int}} \setminus \mathcal{B}$
3. $P_\mathcal{M}\varphi(\boldsymbol{x}) = \varphi(\boldsymbol{x}_K)$ for a.e. $\boldsymbol{x} \in K$, for all $K \in \mathcal{M}$

Taking $v = P_{\mathcal{D},\mathcal{B}}\varphi$ as test function in the scheme and using $\Pi_\mathcal{M}(P_{\mathcal{D},\mathcal{B}}) = P_\mathcal{M}\varphi$, the conclusion follows with the two following consistency results:

$$\|\nabla_\mathcal{D} P_\mathcal{D}\varphi - \nabla\varphi\|_{(L^\infty(\Omega))^d} \le Ch_\mathcal{D}$$

$$\lim_{h_\mathcal{D}\to 0} \|P_{\mathcal{D},\mathcal{B}}\varphi - P_\mathcal{D}\varphi\|_X = 0$$

Leading to:

$$\lim_{h_\mathcal{D}\to 0} \|\nabla_\mathcal{D} P_{\mathcal{D},\mathcal{B}}\varphi - \nabla\varphi\|_{(L^2(\Omega))^d} = 0$$