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Renormalization of the Numerical Diffusion for an Upwind Finite Volume Method. Application to the Simulation of Kelvin-Helmholtz Instability

Jean-Michel Ghidaglia

Centre de Mathématiques et de Leurs Applications

ENS Cachan ⇔, CNRS UMR 8536

and LRC MESO ENS CACHAN - CEA DAM DIF

Work in collaboration with

F. Dauvergne, F. Pascal and J.-M. Rovarch

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1 Introduction

One of the most challenging problem in CFD is the building of a numerical scheme which behaves well for both compressible and (almost) incompressible flows.

Compressible flows : FINITE VOLUMES ($M < \infty$)
Incompressible flows : FINITE ELEMENTS ($M = 0$)

$$M \to 0$$

2 Kelvin-Helmholtz Instability

$2D$ Euler's equations :

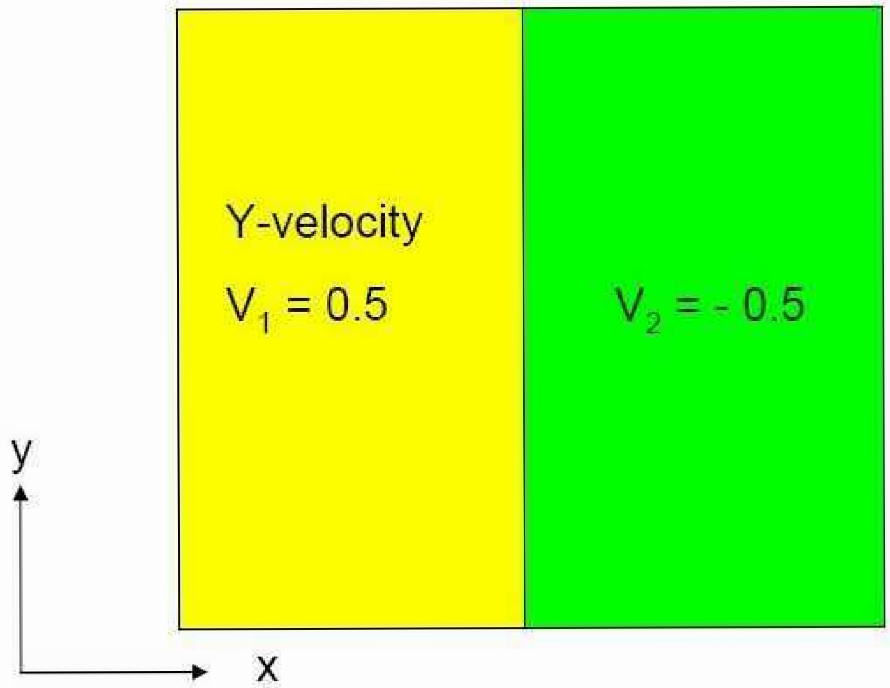
$$\frac{\partial \rho}{\partial t} + div(\rho u) = 0, \tag{1}$$

$$\frac{\partial (\rho u)}{\partial t} + div(\rho u \otimes u + p1) = 0, \tag{2}$$

$$\frac{\partial (\rho E)}{\partial t} + div(\rho E u + p u) = 0, \tag{3}$$

where ρ denotes the density, e the internal energy, $p = p(\rho, e)$ the pressure, $u = (u_x, u_y)$ the velocity, $E = e + \frac{1}{2}|u|^2$ the total specific energy.

KELVIN-HELMHOLTZ TEST PROBLEM – used for dissipation studies



Size of box 1 x 1

Computational cells

64 times 64

$\Delta V = | V_1 - V_2 | = 1$

Density = 1, Perfect gas equation of state ($\gamma = 5/3$)

Pressure = 15, Sound speed = 5 (default case)

Mach no. $\Delta V / c = 0.2$ (default case)

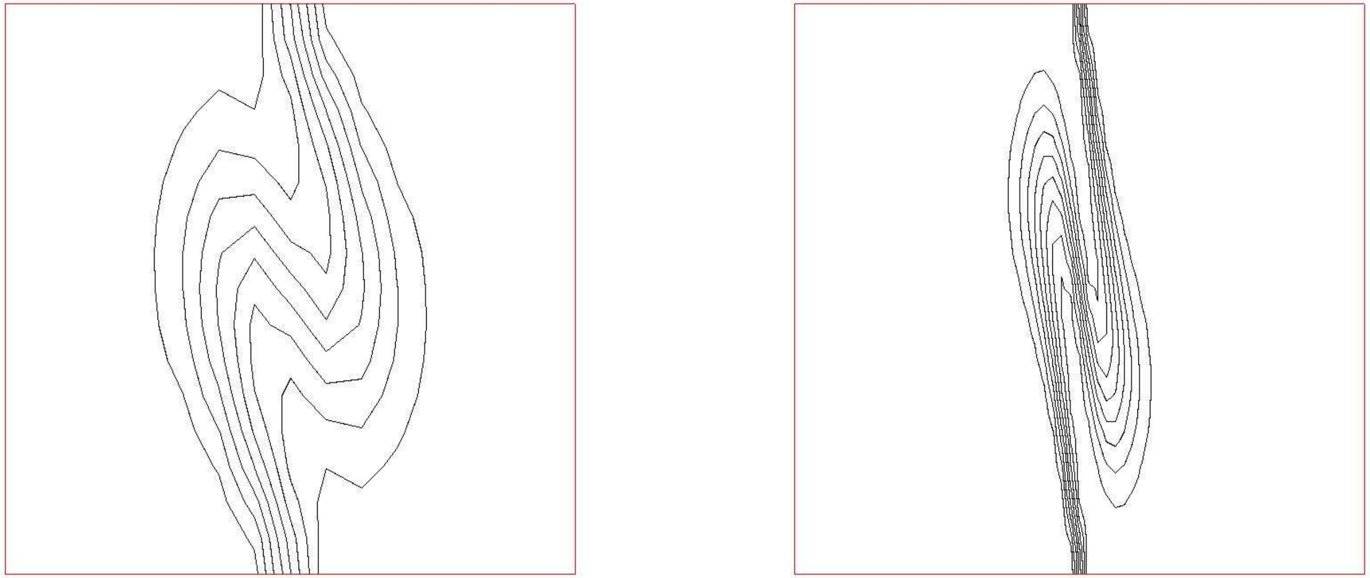
3 Finite Volume Approximation

$$|K|\frac{dv_K}{dt} + \sum_{L\in\mathcal{N}(K)} |K\cap L|\,\Phi(v_K, v_L; K, L) = 0\,,$$

$$\Phi(v, w; K, L) = \frac{F(v) + F(w)}{2}\cdot\nu_{K,L} - \mathcal{U}(v, w; K, L)\frac{F(w) - F(v)}{2}\cdot\nu_{K,L}\,.$$

$$\mathcal{U}(v, w; K, L) = \mathrm{sgn}(A_{\nu_{K,L}}(\mu_{K,L}))$$

$$p_{j+\frac{1}{2}} = \frac{p_j + p_{j+1}}{2} - \frac{\mathrm{sgn}(u)}{2}(M\Delta p + \frac{\rho u}{M}\Delta u) + o(\Delta^2 + M^2)$$



(a) $M = 0.2$.　　(b) $M = 0.02$.

Figure 1: Interface at $t = 1$.

4 Renormalization

$$\Phi(v,w;K,L) = \frac{F(v)+F(w)}{2}\cdot\nu_{K,L} - \mathcal{U}(v,w;K,L)\frac{F(w)-F(v)}{2}\cdot\nu_{K,L}\,.$$

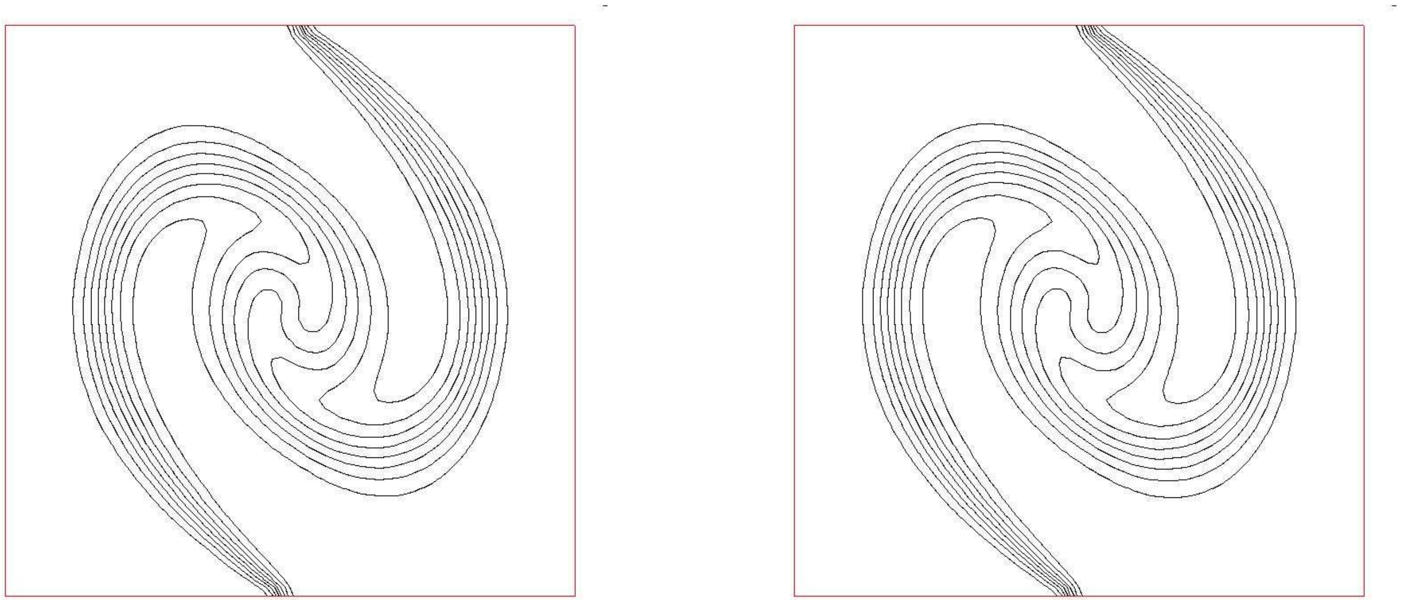
$$\mathcal{U}(v,w;K,L;\beta) = \operatorname{sgn}\left(A_{\nu_{K,L}}(\mu_{K,L})\,P(\mu_{K,L},\beta)\right)\,,$$

where $P(\mu_{K,L},\beta)$ is the preconditioning matrix proposed E. Turkel *J. Comp. Phys.*in 1987 with β a parameter of the order of the Mach number. Here: time consistent scheme.

Proposition 1 *For the renormalized FVCF scheme ($\beta = M$), we have*

$$p_{j+\frac{1}{2}} = \frac{p_j + p_{j+1}}{2} - \frac{\operatorname{sgn}(u)}{2\sqrt{5}}\left(\Delta p + 3\rho u \Delta u\right) + o(\Delta^2 + M^2)\,.$$

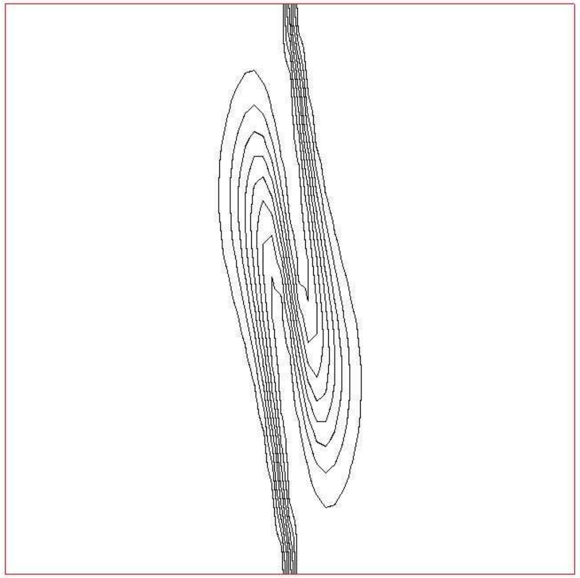
$$\left(p_{j+\frac{1}{2}} = \frac{p_j + p_{j+1}}{2} - \frac{\operatorname{sgn}(u)}{2}(M\Delta p + \frac{\rho u}{M}\Delta u) + o(\Delta^2 + M^2)\right)$$



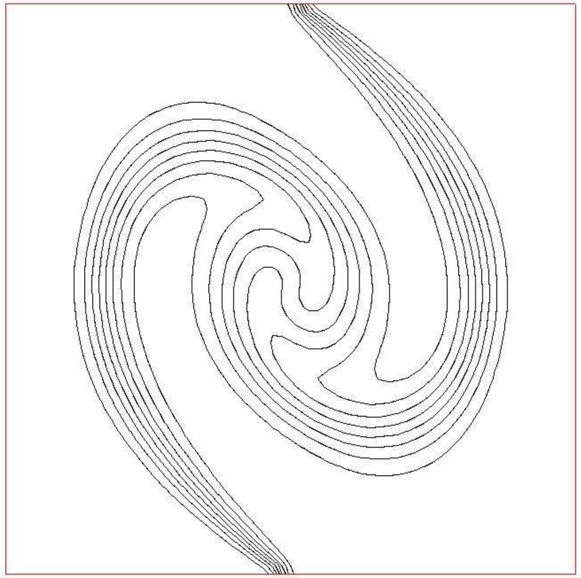
(a) $M = 0.2, \beta = M$.

(b) $M = 0.02, \beta = M$.

Figure 2: Interface at $t = 1$ with renormalization.



(a) $M = 0.02, \beta = 1.$



(b) $M = 0.02, \beta = M.$

Figure 3: Interface at $t = 1$ without and with renormalization.

5 Explicit versus Implicit schemes

Euler explicit scheme reads:

$$\frac{|K|}{\Delta t^n}(v_K^{n+1} - v_K^n) + \sum_{L \in \mathcal{N}(K)} |K \cap L| \, \Phi(v_K^n, v_L^n; K, L; \beta) = 0 \,. \qquad (4)$$

Bypass the C.F.L. condition: implicit Euler discretization:

$$\frac{|K|}{\Delta t^n}(v_K^{n+1} - v_K^n) + \sum_{L \in \mathcal{N}(K)} |K \cap L| \, \Phi(v_K^{n+1}, v_L^{n+1}; K, L; \beta) = 0 \,. \qquad (5)$$

Stability

Tool for the study of stability: Von Neuman analysis (amplification matrix).

	without renormalization	*with renormalization*
Explicit	$u\frac{\Delta t}{\Delta x} = \mathcal{O}(M)$	$u\frac{\Delta t}{\Delta x} = \mathcal{O}(M^2)$
Implicit	unconditionally	unconditionally

Table 1: Stability (time discretization)

6 Bibliography

[BIR 05] BIRKEN P., MEISTER A., “Stability of preconditioned finite volume schemes at low Mach numbers”, *BIT*, vol. 45(3), 2005, p. 463-480.

[GHI 96] GHIDAGLIA J., KUMBARO A., COQ G. L., “Une méthode volumes finis à flux caractéristiques pour la résolution numérique de lois de conservation”, *C. R. Acad. Sci. Paris Série I*, vol. 332, 1996, p. 981-988.

[GHI 97] GHIDAGLIA J.-M., KUMBARO A., LE COQ G., TAJCHMAN M., “A finite volume implicit method based on characteristic flux for solving hyperbolic systems of conservation laws”, *Proceedings of the Conference on Nonlinear Evolution Equations and Infinite-dimensional Dynamical Systems (Shanghai, 1995)*, World Sci. Publ., River Edge, NJ, 1997, p. 50–65.

[GHI 02] GHIDAGLIA J.-M., “Flux schemes for solving nonlinear systems of conservation laws”, CHATTOT J.-J., HAFEZ M., Eds., *Innovative Methods for Numerical Solutions of PDEs. Proceedings of the symposium on progress in numerical solutions of PDE's, Arcachon, France, July 11–13, 1998*, Singapore: World Scientific, 2002, p. 232-242.

[GHI 05] GHIDAGLIA J.-M., PASCAL F., “The normal flux method at the boundary for multidimensional finite volume approximations in CFD”, *Eur. J. Mech., B, Fluids*, vol. 24(1), 2005, p. 1-17.

[GUI 99] GUILLARD H., VIOZAT C., “On the behaviour of upwind schemes in the low Mach number limit”, *Comput. Fluids*, vol. 28(1), 1999, p. 63-86.

[GUI 04] GUILLARD H., MURRONE A., “On the behavior of upwind schemes in the low Mach number limit. II: Godunov type schemes”, *Comput. Fluids*, vol. 33(4), 2004, p. 655-675.

[MAT 04] “Mathematical and Numerical aspects of Low Mach Number Flows, International Conference, Porquerolles”, http://www-sop.inria.fr/smash/LOMA/, 2004.

[PAS 02] PASCAL F., “Sur des méthodes d'approximation effectives et d'analyse numérique pour les équations de la mécanique des fluides”, HDR, Université Paris-Sud, 2002.

[STR 04] STRIKWERDA J. C., *Finite Differences And Partial Differential Equations*, Society for Industrial and Applied Mathematics, Philadelphia, PA, USA, 2004.

[THO 08] THORNBER B., MOSEDALE A., DRIKAKIS D., YOUNGS D., WILLIAMS R.J.R., “An improved reconstruction method for compressible flows with low Mach number features”, *J. Comp. Phys.*, to appear, 2008.

[TUR 87] TURKEL E., “Preconditioned methods for solving the incompressible and low speed compressible equations”, *J. Comp. Phys.*, vol. 72, 1987, p. 277-298.

[TUR 93] TURKEL E., “Review of preconditionning methods for fluid dynamics”, *App. Num. Math.*, vol. 12, 1993, p. 257-284.

[VIO 97] VIOZAT C., “Implicit upwind scheme for low Mach number compressible flows”, report 3084, 1997, INRIA.

[YOU 08] YOUNGS D., WILLIAMS R., “Turbulent mixing in spherical implosions”, *Int. J. Numer. Meth. Fluids*, to appear, 2008.