Convergence Rate of Finite Volume Schemes for Hyperbolic Conservation Laws on Riemannian Manifolds

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Plan of the talk

1. Motivation
2. A class of conservation laws on manifolds
3. Finite volume schemes on Riemannian manifolds
4. The convergence result
5. Remark on higher oder schemes
6. Prospects

Shallow water equations

in $\mathbb{R}^2$:

$$\begin{pmatrix} h \\ hv \end{pmatrix}_t + \nabla \cdot \begin{pmatrix} hv \\ hv \otimes v + \frac{h^2}{2g} \end{pmatrix} = 0 \quad \text{in } \mathbb{R}^2 \times (0, T),$$

here g is a gravitational constant, h the height and v the velocity.

on the sphere S^2 :

$$\begin{pmatrix} h \\ hv \end{pmatrix}_t + \nabla_g \cdot \begin{pmatrix} hv \\ hv \otimes v + \frac{h^2}{2g} \end{pmatrix} = 0 \quad \text{in } S^2 \times (0, T),$$

where $\nabla_g \cdot$ denotes the divergence operator on the sphere. In polar coordinates $\nabla_g \cdot$ is given by

$$\nabla_g \cdot f := \frac{1}{\sin\theta} \left((f^\phi \sin\theta)_\phi + (f^\theta \sin\theta)_\theta \right).$$

2. A class of conservation laws on manifolds

We will consider the scalar hyperbolic conservation law

$$\begin{aligned} u_t + \nabla_g \cdot \Big(v(\cdot)\tilde{f}(u(\cdot)) \Big) &= 0, \quad \text{in } M \times \mathbb{R}_+, \\ u(\cdot, 0) &= u_0, \quad \text{on } M \end{aligned} \qquad (P_M)$$

for $u = u(x,t) \in \mathbb{R}$, where v is a smooth vector-field with $\nabla_g \cdot v = 0$ and $\tilde{f}$ depends locally Lipschitz continuous on u. Here M denotes a closed manifold with Riemannian metric g.

In the special case $M = \mathbb{R}^d$ the divergence coincides with the classical divergence. For brevity we define

$$f(x,u) := v(x)\tilde{f}(u).$$

Entropy solution

Definition:

$u \in L^\infty(M \times \mathbb{R}_+)$ is called an **entropy solution** of (P_M) if

$$\int_{M\times\mathbb{R}_+} \left[|u-\kappa|\varphi_t + g_x\big(f(x, u\top\kappa) - f(x, u\bot\kappa), \nabla_g \varphi\big)\right] dv_g\, dt$$
$$+ \int_M |u_0 - \kappa|\varphi(\cdot, 0)\, dv_g \geq 0$$

for all $\kappa \in \mathbb{R}$ and all $\varphi \in C_0^\infty(M \times \mathbb{R}_+, \mathbb{R}_+)$.

Note: Every entropy solution is a weak solution.

Existence and Uniqueness

Theorem(Ben-Artzi, LeFloch '06):

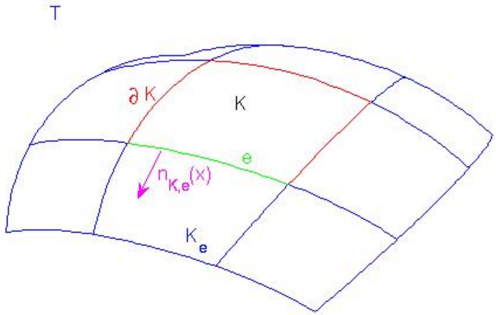
For $u_0 \in L^\infty(M)$ and

$$\nabla_g \cdot f(x,s) = 0 \ \forall s \in \mathbb{R}$$

there exists a unique entropy solution u of (P_M).
If u, v are entropy solutions to some initial data u_0, v_0 respectively, it holds

$$\|v(\cdot,t) - u(\cdot,t)\|_{L^1(M)} \leq \|v_0 - u_0\|_{L^1(M)}.$$

3. Finite volume schemes on Riemannian manifolds



Numerical fluxes

Definition: For every $K \in \mathcal{T}^h$ and $e \in \partial K$ we consider a locally Lipschitz continuous **numerical flux function**

$$f_{K,e} : \mathbb{R}^2 \longrightarrow \mathbb{R}$$

satisfying

- $f_{K,e}(u,u) = \frac{1}{|e|} \int_e g_x\left(f(x,u), n_{K,e}(x)\right) dv_e \quad \forall\, u \in \mathbb{R}$ (Consistency),
- $f_{K,e}(u,v) = -f_{K_e,e}(v,u) \quad \forall\, u,v \in \mathbb{R}$ (Conservation),
- $\frac{\partial}{\partial u} f_{K,e}(u,v) \geq 0$ and $\frac{\partial}{\partial v} f_{K,e}(u,v) \leq 0 \quad \forall\, u,v \in \mathbb{R}$ (Monotonicity).

These conditions are e.g. fulfilled by the Lax-Friedrichs flux, for an appropriate viscosity coefficient.

Semi-discrete scheme

We consider the

Semi-discrete scheme

$$
\begin{aligned}
(u_K)_t(t) &= -\sum_{e\in\partial K} \frac{|e|}{|K|} f_{K,e}(u_K(t), u_{K_e}(t)), \\
u_K(0) &:= \frac{1}{|K|}\int_K u_0(x)dv_g, \qquad\qquad (P_{M,h}) \\
u^h(x,t) &:= u_K(t), \text{ for } x \in K.
\end{aligned}
$$

Note: This scheme is always conservative and in the $d = 1$-case it is TVD. Furthermore

$$
\operatorname{essinf}_{p\in M} u_0(p) \leq u^h(x,t) \leq \operatorname{esssup}_{p\in M} u_0(p) \qquad \text{for all } (x,t) \in M \times \mathbb{R}_+.
$$

Convergence on manifolds

Theorem (Amorim, Ben-Artzi, LeFloch '05):

Let u be the entropy solution of (P_M). Under some technical assumptions on f and the grid the approximate solution $u^h : M \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ given by the finite volume method $(P_{M,h})$ satisfies

$$u_h \to u \quad \text{a.e.}$$

when h tends to zero.

Note:

- The proof is actually done for a fully discrete scheme.
- Result for $M = \mathbb{R}^n$: Coquel & LeFloch '95, Kröner & Rokyta '94, Kröner, Noelle & Rokyta '95.

Open question: Convergence rate

Convergence rates in $\mathbb{R}^d$

Theorem (Vila '94, Eymard, Gallouët, Ghilani, Herbin '98):

Let $u_0 \in L^\infty(\mathbb{R}^d) \cap \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^d)$ and provided some technical assumptions on f and the grid there is the following error estimate for the finite volume scheme $(P_{M,h})$:
For every compact $E \subset \mathbb{R}^d \times \mathbb{R}_+$ there exists $C(E, f, u_0) \geq 0$ such that

$$\int_E \left| u^h(x,t) - u(x,t) \right| \, dx \, dt \leq C h^{\frac{1}{4}}.$$

Note: For $d = 1$ the exponent can be improved to $\frac{1}{2}$.

Idea: Modify the methods of Eymard et. al. to prove a $h^{\frac{1}{4}}$-error estimate on manifolds of dimension $d \geq 2$ and a $h^{\frac{1}{2}}$-error estimate for 1-dimensional manifolds.

Convergence rates on manifolds

Theorem (G. '07):

Consider the Cauchy Problem (P_M) on a d-dimensional Riemannian manifold (M, g) with $d = 1, 2$ and

- initial data $u_0 \in \mathrm{BV}(M) \cap L^\infty(M)$.
- u be the entropy solution of (P_M)
- u^h the approximate solution given by the finite volume scheme $(P_{M,h})$

then we have the following error estimate:
For every $T > 0$ exists a constant $C(M, g, f, u_0, T)$, such that

$$\int_{M\times[0,T]} \left| u^h(x,t) - u(x,t) \right| \, dv_g \, dt \leq \begin{cases} Ch^{\frac{1}{2}} & : \quad d = 1 \\ Ch^{\frac{1}{4}} & : \quad d = 2. \end{cases}$$

Sketch of the proof

For $K \in \mathcal{T}^h, e \in \partial K$

$$C_{K,e} := \{(c,d) \in [u_K \perp u_{K_e}, u_K \top u_{K_e}]^2 : (u_{K_e} - u_K)(d-c) \geq 0\}.$$

Weak BV-estimate

Consider the Cauchy Problem (P_M). Let u^h the approximate solution given by $(P_{M,h})$. There exists a constant $C(f, u_0, T, \alpha, M, g) > 0$, such that for h sufficiently small

$$\int_0^T \sum_{K \in \mathcal{T}^h} \sum_{e \in \partial K} \max_{(c,d) \in C_{K,e}} |e| |f_{K,e}(c,d) - f_{K,e}(c,c)| \, dt \leq \begin{cases} C & : \quad d = 1 \\ Ch^{-\frac{1}{2}} & : \quad d = 2. \end{cases}$$

The better estimate for $d = 1$ follows directly from the TVD property of the scheme.

Sketch of the proof

Entropy inequality for the approximate solution

For h sufficiently small there exists a constant $C > 0$, such that we have

$$\int_{M\times\mathbb{R}_+} (|u^h - \kappa)|\varphi_t + g_x(f(x, u^h \top \kappa) - f(x, u^h \bot \kappa), \nabla_g \varphi)\, dv_g\, dt$$
$$+ \int_M |u_0 - \kappa|\varphi(\cdot, 0)\, dv_g$$
$$\geq \begin{cases} -Ch\|\nabla_g \varphi\|_{L^1} & : \quad d = 1 \\ -Ch^{\frac{1}{2}}\|\nabla_g \varphi\|_{L^1} & : \quad d = 2 \end{cases}$$

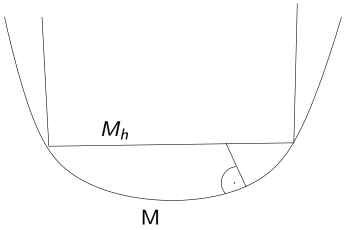
for every $\kappa \in [\text{essinf } u_0, \text{esssup } u_0]$ and $\varphi \in C_0^\infty(M \times \mathbb{R}_+, \mathbb{R}_+)$.

Then the proof is based on a doubling of variables argument.

Ideas for higher order schemes

We need a notion of polynomial reconstruction. One idea:

- analysed by Dziuk and Elliot '07 for parabolic equations.
- Approximate M by a polyhedral surface M_h whose nodes lie on M and such that the normal projection from M to M_h is bijective.
- Do the reconstruction on M_h and project the polynomials to M.
- If we define the whole scheme on M_h, this introduces a geometry error of order h^2.



Prospects

- Generalisation to Riemannian manifolds in higher dimensions.
- Construction of higher order schemes.
- Generalisation to manifolds with Lorentzian metric, to be able to treat problems from general relativity. (The well-posedness of such problems was treated by Ben Artzi & LeFloch).

Properties of the grid

To prove convergence rates for $d = 2$ we need the following properties of the grid

- Every K is a geodesicly convex, curved polygon.
- For $i \neq j$ the section $K_i \cap K_j$ is empty, a point or a common face.
- Each face is a segment of a geodesic line.
- There are constants $\alpha, h > 0$, such that

$$\alpha h^2 \leq |K|, \quad (1)$$

$$\delta(K) \leq h, \quad (2)$$

where $\delta(K)$ denotes the diameter of K.