Interface coupling of conservation laws

Edwige Godlewski
Laboratoire Jacques-Louis Lions
Université Pierre et Marie Curie - Paris 6

June 12, 2008
FVCA5, Aussois

Collaborations

Work initiated with

- Pierre-Arnaud Raviart
  plasma problem - CEA Bruyères

Collaborations

Work initiated with

- Pierre-Arnaud Raviart
  plasma problem - CEA Bruyères
- going on with a joint working group *LJLL-CEA Saclay* with Frédéric Coquel
  and Annalisa Ambroso, Christophe Chalons,
  Frédéric Lagoutière, Pierre-Arnaud Raviart, Nicolas Seguin
  Jean-Marc Hérard (Edf), Samuel Kokh
  + younger people: Benjamin Boutin, Filipa Caetano, Olivier Hurisse (Edf), Thomas Galie

Outline

- Interface coupling: main features, examples
- Mathematical model
- Coupling condition: state / flux coupling
- Regularization for state coupling
    - Dafermos regularization
    - finite volume scheme
- Relaxation state coupling solver for flux coupling
- Conclusion and future directions

Interface coupling: main features

Given two codes

- two (compressible) fluid codes simulating fluid flow of the same 'nature', taking into account different specificities not coupled phenomena *(monophysics)*
- fixed interface *(multidomain)*
- 'thin' interface, the codes interact
  exchange of information at the interface *(strong coupling)*
- need of a robust procedure
  understand the physics at the interface *('intelligent' coupling)*
- use existing codes
  few modifications in each domain

$\rightarrow$ give a numerical coupling procedure to 'couple' the codes.
First: examples; then: what is the mathematical model?

Examples

- 'Real' examples of coupling codes in thermohydraulics
  - homogeneous models: HRM-HEM (taking into account porosity or assuming thermodynamic equilibrium)
  - 1D - 2D, 1D - 3D models (taking into account symmetry or keeping multidimensional effects)
  - bifluid - drift flux models
  - other example: two plasma models with different current densities (neglected in some part)
- Some mathematical models of coupling
  - (scalar) conservation laws
  - linear systems (of the same dimension)
  - relaxation (2x2) system / relaxed (scalar) conservation law
  - Euler systems in Lagrangian coordinates
  - " " systems: barotropic (2x2)/ with energy (3x3)
  - linearly degenerate systems (relaxing to Euler)

Mathematical model

- Two hyperbolic systems of conservation laws (possibly nonconservative)

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial}{\partial x}\mathbf{f}_\alpha(\mathbf{u}) = \mathbf{0},\ \alpha = L, x < 0, R, x > 0,\ t > 0 \qquad (1)$$

-
-

Mathematical model

- Two hyperbolic systems of conservation laws (possibly nonconservative)

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial}{\partial x}\mathbf{f}_\alpha(\mathbf{u}) = \mathbf{0}, \;\; \alpha = L, x < 0, R, x > 0, \;\; t > 0 \qquad (1)$$

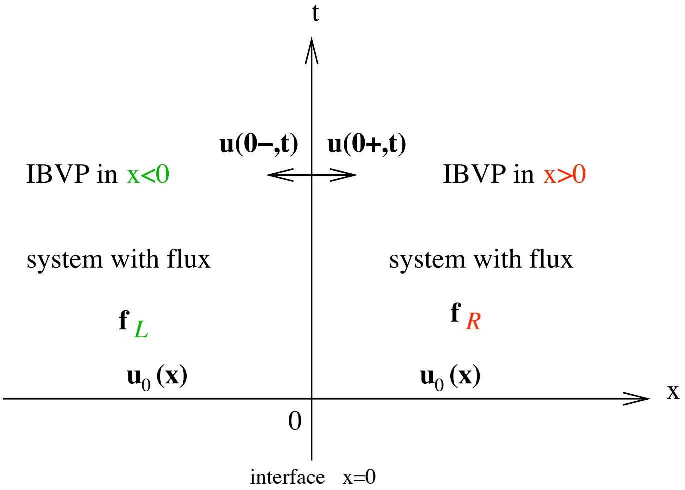
- 'compatibility' between systems (or not)
  1. plasma models: same equations, only one flux component is discontinuous
  2. models 1D-2D: 2D system reduces to the 1D system
  3. $p-$system coupled with Euler (in Lagrangian coord.) are compatible
  4. multiphase models: 7 equations (2 velocities) and drift - flux?
-

Mathematical model

- Two hyperbolic systems of conservation laws (possibly nonconservative)

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial}{\partial x}\mathbf{f}_\alpha(\mathbf{u}) = \mathbf{0},\ \ \alpha = L, x < 0, R, x > 0,\ \ t > 0 \qquad (1)$$

- 'compatibility' between systems (or not)
  1. plasma models: same equations, only one flux component is discontinuous
  2. models 1D-2D: 2D system reduces to the 1D system
  3. $p-$system coupled with Euler (in Lagrangian coord.) are compatible
  4. multiphase models: 7 equations (2 velocities) and drift - flux?
- two boundary value problems, one on each side of the interface $x = 0$ (*thin* interface, no 'interface model')
  coupling model through the 'choice' of boundary conditions (which physical variables can/should be transmitted?)


t
u(0−,t)
u(0+,t)
IBVP in x<0
IBVP in x>0
system with flux
system with flux
$\mathbf{f}_L$
$\mathbf{f}_R$
$\mathbf{u}_0(\mathbf{x})$
$\mathbf{u}_0(\mathbf{x})$
x
0
interface x=0

$$\mathbf{u} = (\rho, \rho v, \rho s, \rho s_e)^T$$

$$\mathbf{f}_L(\mathbf{u}) = (\rho v, \rho v^2 + P, \rho s v, \rho s_e v - \beta s_e)^T$$

$$\mathbf{f}_R(\mathbf{u}) = (\rho v, \rho v^2 + P, \rho s v, \rho s_e v)^T$$

$P = p + p_e = (\rho s)^\gamma + (\rho s_e)^\gamma$ $(\gamma = 5/3)$
only the 4th equation (entropy conservation of electrons) changes at interface

$$\frac{\partial}{\partial t}(\rho s_e) + \frac{\partial}{\partial x}(\rho s_e v_e) = 0$$

with

$$v_e = v - \frac{\beta}{\rho}, \ x < 0, \quad v_e = v, \ x > 0$$

eigenvalues $\lambda = v, v_e, v \pm c$, may change sign: many possible cases to order the $\lambda_{k,L}, \lambda_{k,R}$ (and nonlinear effects)

Comments

Why a thin interface ? why this mathematical model ?
several levels of answer

- codes should not be modified: only the (boundary) data
- need to understand what a 'natural' scheme computes
- in case of non uniqueness, instability linked to resonance is avoided (ex. plasma)
- if one 'regularizes', for large time, behaves like a *coupled problem* (CRP)
- thickening requires more physics

Link with equations with discontinuous coefficients

1. Riemann problems: Cauchy problem for $\mathbf{f}_\alpha$ with initial data

$$\mathbf{u}(x,0) = \begin{cases} \mathbf{u}_\ell, & x < 0 \\ \mathbf{u}_r, & x > 0 \end{cases} \tag{2}$$

   self-similar solution $\mathbf{W}_R(\frac{x}{t};\mathbf{u}_\ell,\mathbf{u}_r)$ for $\mathbf{f}_R$, $\mathbf{W}_L(\frac{x}{t};\mathbf{u}_\ell,\mathbf{u}_r)$ for $\mathbf{f}_L$.
   Riemann problems (RP) are used to
   - express the Coupling Condition (CC): *state coupling*
   - exhibit some explicit solutions: $\mathbf{W}_c(\frac{x}{t};\mathbf{u}_\ell,\mathbf{u}_r)$ solution of a coupled Riemann problem (CRP) (1) (2)
   - test numerical schemes
   - building block in a numerical scheme (Godunov)
2.
3.

1. Riemann problems: Cauchy problem for $\mathbf{f}_\alpha$ with initial data

$$\mathbf{u}(x,0) = \begin{cases} \mathbf{u}_\ell, & x < 0 \\ \mathbf{u}_r, & x > 0 \end{cases} \tag{2}$$

   self-similar solution $\mathbf{W}_R(\frac{x}{t}; \mathbf{u}_\ell, \mathbf{u}_r)$ for $\mathbf{f}_R$, $\mathbf{W}_L(\frac{x}{t}; \mathbf{u}_\ell, \mathbf{u}_r)$ for $\mathbf{f}_L$.
   Riemann problems (RP) are used to
   - express the Coupling Condition (CC): *state coupling*
   - exhibit some explicit solutions: $\mathbf{W}_c(\frac{x}{t}; \mathbf{u}_\ell, \mathbf{u}_r)$ solution of a coupled Riemann problem (CRP) (1) (2)
   - test numerical schemes
   - building block in a numerical scheme (Godunov)
2. Relaxation: $\partial_t \mathbf{U} + \partial_x \mathbf{F}(\mathbf{U}) = \mathbf{S}(\mathbf{U})/\varepsilon$
   - as example of compatible models: coupling a relaxation system and the relaxed system ($\varepsilon \to 0$)
   - in a numerical procedure: coupling of larger LD systems ('relaxing' to the L, R systems) leads to *flux coupling*
3.

1. Riemann problems: Cauchy problem for $\mathbf{f}_\alpha$ with initial data

$$\mathbf{u}(x,0) = \begin{cases} \mathbf{u}_\ell, & x < 0 \\ \mathbf{u}_r, & x > 0 \end{cases} \tag{2}$$

self-similar solution $\mathbf{W}_R(\frac{x}{t};\mathbf{u}_\ell,\mathbf{u}_r)$ for $\mathbf{f}_R$, $\mathbf{W}_L(\frac{x}{t};\mathbf{u}_\ell,\mathbf{u}_r)$ for $\mathbf{f}_L$.
Riemann problems (RP) are used to

- express the Coupling Condition (CC): *state coupling*
- exhibit some explicit solutions: $\mathbf{W}_c(\frac{x}{t};\mathbf{u}_\ell,\mathbf{u}_r)$ solution of a coupled Riemann problem (CRP) (1) (2)
- test numerical schemes
- building block in a numerical scheme (Godunov)

2. Relaxation: $\partial_t \mathbf{U} + \partial_x \mathbf{F}(\mathbf{U}) = \mathbf{S}(\mathbf{U})/\varepsilon$

- as example of compatible models: coupling a relaxation system and the relaxed system ($\varepsilon \to 0$)
- in a numerical procedure: coupling of larger LD systems ('relaxing' to the L, R systems) leads to *flux coupling*

3. Regularization

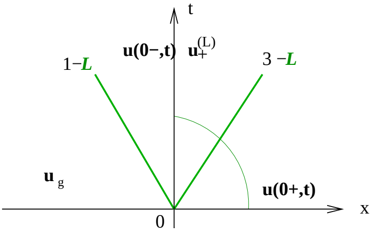
- numerical coupling (FV method): provides numerical viscosity
- Dafermos regularization of (1)(2): particular vanishing viscosity

Coupling Condition

- Given $\mathbf{b}$, IBVP in $x > 0$, one cannot impose $\mathbf{u}(0+, t) = \mathbf{b}$
  $\longrightarrow$ *weak formulation* of the boundary condition:
  $\mathbf{u}(0+, t) \in \mathcal{O}_R(\mathbf{b})$ means $\mathbf{u}(0+, t) = \mathbf{W}_R(0+; \mathbf{b}, \mathbf{u})$ for some $\mathbf{u} \in \mathbb{R}^p$
  $\mathbf{W}_R(0+; \mathbf{u}_l, \mathbf{u}_r)$ solution of the Riemann problem with $\mathbf{f}_R$
  $\mathcal{O}_R(\mathbf{b})$ = traces at $x = 0$ of all possible RP between $\mathbf{b}$ and a right state
- define the sets $\mathcal{O}_{L,R}(\mathbf{u}(0\pm, t))$
- coupling condition:

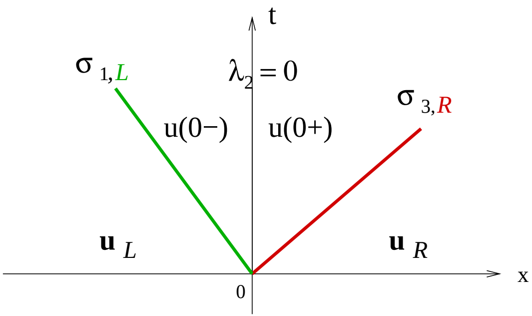
$$(\mathrm{CC}) \quad \boxed{\mathbf{u}(0-, t) \in \mathcal{O}_L(\mathbf{u}(0+, t)), \;\; \mathbf{u}(0+, t) \in \mathcal{O}_R(\mathbf{u}(0-, t))}$$

Two Euler systems in Lagrangian coordinates
$\mathbf{u} = (\tau, v, e), \mathbf{f}_\alpha(\mathbf{u}) = (-v, p, pv),\ p = p_\alpha(\tau, \varepsilon),\ \lambda_2 = 0$ eigenvalue

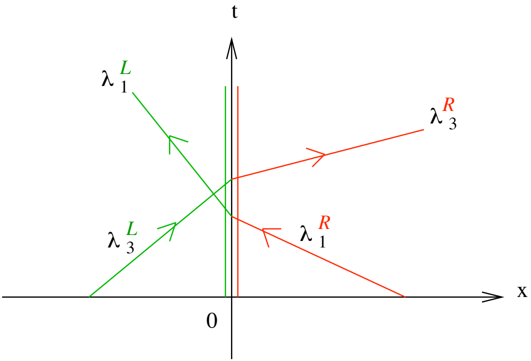


Coupling condition CC
$\mathbf{u}(0-, t) \in \mathcal{O}_L(\mathbf{u}(0+, t))$ (left), $\mathbf{u}(0+, t) \in \mathcal{O}_R(\mathbf{u}(0-, t))$(right)
$\mathbf{u}(0-) = \mathbf{W}_L(0-; \mathbf{u}_-, \mathbf{u}(0+)),\quad \mathbf{u}(0+) = \mathbf{W}_R(0+; \mathbf{u}(0-), \mathbf{u}_+)$

same Euler systems: coupled Riemann problem (CRP) = Cauchy problem for (1) with Riemann data $\mathbf{u}_L, \mathbf{u}_R$



Example: solution of a CRP with two shocks $1 - L$ and $3 - R$, one stationary wave $\mathbf{u}(0-)$, $\mathbf{u}(0+)$
(easy because $\lambda_{1,L}(\mathbf{u}) < 0 < \lambda_{3,R}(\mathbf{u})$, no change of sign)



$\lambda_{L,1} < 0 < \lambda_{L,3}$, $\lambda_{R,1} < 0 < \lambda_{R,3}$ $\lambda_{L,2} = \lambda_{R,2} = 0$ *characteristic* case

heuristic: transmission of 2 quantities (justified by a linearized analysis)

coupled RP (CRP): only $1L-$ waves, $0-$wave and $3R-$waves

State coupling / Flux coupling

$$\partial_t \mathbf{u} + \partial_x \mathbf{f}_\alpha(\mathbf{u}) = \mathbf{0},\ \alpha = L, x < 0, R, x > 0,\ t > 0 \quad (1)$$

- State coupling
  • when $x = 0$ is *non characteristic* the coupling condition CC 'often' yields continuity $\boxed{\mathbf{u}(0+, t) = \mathbf{u}(0-, t)}$
  conservative variables are *transmitted*, NOT the flux
  • when $x = 0$ is *characteristic* not all, only part of the conservative variables can be *transmitted*
- $\neq$ Flux coupling = conservative approach, (1) written with H Heaviside

$$\partial_t \mathbf{u} + \partial_x((1 - H(x))\mathbf{f}_L(u) + H(x)\mathbf{f}_R(\mathbf{u})) = 0, x \in \mathbb{R}$$

yields $\boxed{\mathbf{f}_L(\mathbf{u}(0-, t)) = \mathbf{f}_R(\mathbf{u}(0+, t))}$ the flux is *transmitted*

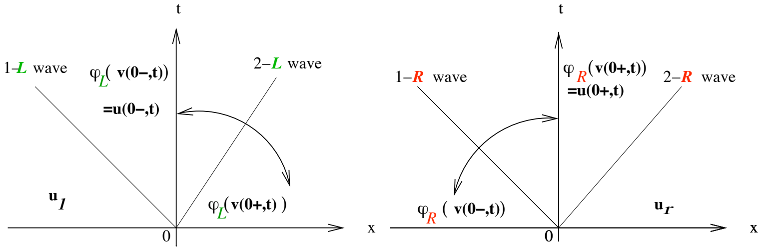
1. A natural link exists between flux coupling and equations with discontinuous coefficients
   A conservative form (given by physics) involves some *natural* entropy condition
2. Even for 'identical' systems ($\mathbf{f}_L = \mathbf{f}_R$), the conservative formulation is a choice for transmission: one *decides* to 'transmit' the flux. In some cases, it is not physical (ex. nozzles with discontinuous but constant section, the rate of flow is not conserved)
   $\longrightarrow$ we choose to study all possibilities: state and flux coupling
3. One can model the *transmission* of other variables

- change of variables : $\mathbf{u} \in \Omega \to \mathbf{v} \in \Omega_{\mathbf{v}}$ (conservative/primitive)
- $\mathbf{v} \to \mathbf{u} = \varphi_\alpha(\mathbf{v}); \alpha = L, R$ admissible i.e. $\varphi'_\alpha(\mathbf{v})$ isomorphism of $\mathbb{R}^p$
- $\mathbf{c}$ given by *physics* (pressure), $\mathbf{b}_L = \varphi_L(\mathbf{c})$, $\mathbf{b}_R = \varphi_R(\mathbf{c})$, set
  $\mathcal{O}_L(\mathbf{b}_L) = \{\mathbf{w} = \mathbf{W}_L(0-; \mathbf{u}_-, \mathbf{b}_L); \mathbf{u}_- \in \Omega\}$
  $\mathcal{O}_R(\mathbf{b}_R) = \{\mathbf{w} = \mathbf{W}_R(0+; \mathbf{b}_R, \mathbf{u}_+); \mathbf{u}_+ \in \Omega\}$
  sets of admissible boundary values for L, R
- *transmission of variables* $\mathbf{v}$ obtained by

$$\mathbf{u}(0-,t) \in \mathcal{O}_L(\varphi_L(\mathbf{v}(0+,t)))$$
$$\mathbf{u}(0+,t) \in \mathcal{O}_R(\varphi_R(\mathbf{v}(0-,t)))$$

(note that $\varphi_L(\mathbf{v}(0+,t)) \neq \mathbf{u}(0+,t) = \varphi_R(\mathbf{v}(0+,t))$*)*
It yields 'continuity' of (or part of) $\mathbf{v}$: $\boxed{\mathbf{v}(0-,t) = \mathbf{v}(0+,t)}$

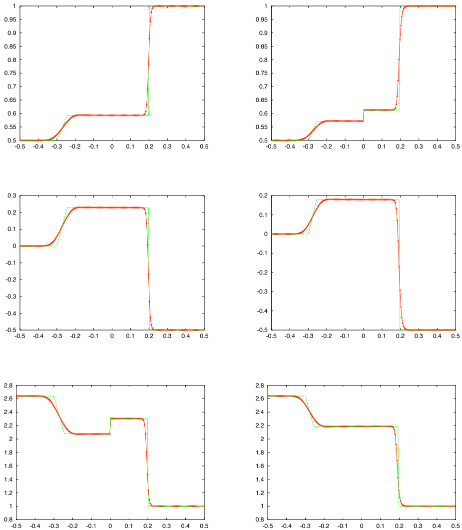
Barotropic Euler system in Lagrangian coordinates $\mathbf{u} = (\tau, v)^T$,
$\mathbf{f}(\mathbf{u}) = (-v, p)^T, \lambda_1 = -C < 0 < \lambda_2 = +C \quad (C = \sqrt{-p'(\tau)})$
two systems with $p = p_\alpha(\tau), \ \alpha = L, R$
interface $x = 0$, *non characteristic* separates the $1-$ and $2-$waves
CC by transmission of $\mathbf{v} = (v, p)$ yields continuity of $\mathbf{v} = (v, p)$



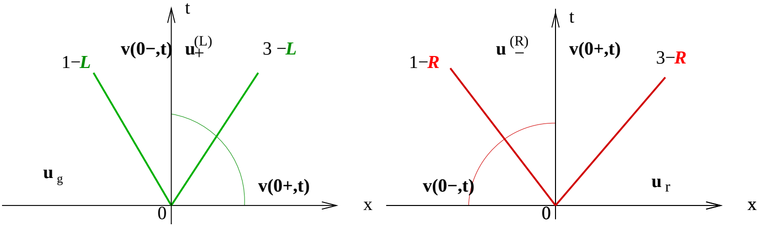
left RP : $\mathbf{v}(0-) \rightarrow \mathbf{v}(0+)$
$2L-$wave

right RP : $\mathbf{v}(0-) \rightarrow \mathbf{v}(0+))$
by a $1R-$ wave

τ, v, p in transmission of $\mathbf{u} = (\tau, v)$ left vs $\mathbf{v} = (v, p)$ right



Again two Euler systems in Lagrangian coordinates with gamma law: $\mathbf{u} = (\tau, v, e), \mathbf{f}_\alpha(\mathbf{u}) = (-v, p, pv)$, $p = (\gamma_\alpha - 1)\varepsilon/\tau$



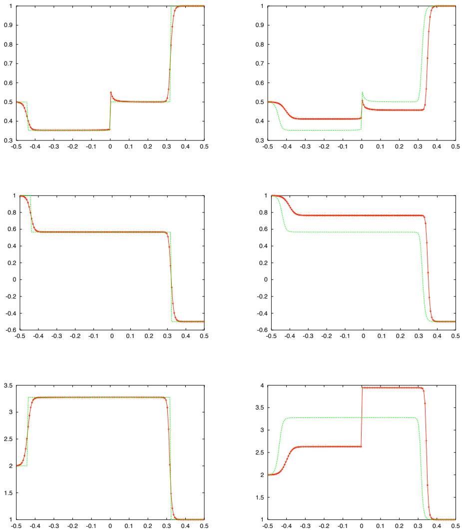
CC in primitive variable $\mathbf{v} = (\tau, v, p)$ yields

$$\boxed{p(0-, t) = p(0+, t), v(0-, t) = v(0+, t)}$$

intersection of two wave curves in $(v, p)-$plane:
$\mathbf{v}(0+) \in \tilde{\mathcal{C}}_L^3(\mathbf{v}(0-)) \cap \tilde{\mathcal{C}}_R^1(\mathbf{v}(0-)) = \{\mathbf{v}(0-)\}$
CC in conservative variable $\mathbf{u} = (\tau, v, e)$ yields

$$\boxed{\varepsilon/\tau(0-, t) = \varepsilon/\tau(0+, t), v(0-, t) = v(0+, t)}$$

τ, v, p for Euler with CC $\mathbf{v} = (\tau, v, p)$ Left, vs $\mathbf{u} = (\tau, v, e)$ Right



Comments

Nonuniqueness of CC; different CC give different solutions
$\longrightarrow$ need of a *physical* criteria for choosing the transmitted variables (besides conservation of mass), conservation of some stationary solutions (material wave)?
Some difficulties linked to the present approach:

- non conservative systems
- singular source terms
- possible resonance: the eigenvalues may change sign (ex. Euler system in Eulerian coordinates) at $x = 0$
- non uniqueness of the solution

A natural answer: add viscosity

- Dafermos regularization (does not bring uniqueness)
- numerical (uniqueness, but which solution is computed?)

Regularization

State coupling = non-conservative approach

$$\partial_t u + \partial_x f(u, a) = \mathcal{M}, \quad x \in \mathbb{R},\ t > 0,\ \partial_t a = 0,$$

$f(u, a) = af_L(u) + (1 - a)f_R(u)$, $\mathcal{M}$ measure (Dirac), weight jump

$$[f(u, a)] = f_R(u(0+, t)) - f_L(u(0-, t))$$

Riemann data for a: $a_L = 1, a_R = 0$ and $\partial_t a = 0$
$\Rightarrow a(x)$ is a Heaviside function, $\partial_x a = -\delta_0$,

$$\begin{cases} \partial_t u + \partial_x \Big(af_L(u) + (1 - a)f_R(u) \Big) + \Big(f_R(u) - f_L(u) \Big) \partial_x a = 0 \\ \partial_t a = 0 \end{cases}$$

If u continuous, $(f_R(u) - f_L(u))\partial_x a$ (non conservative product) is well defined. Write the 1st equation

$$\partial_t u + (af_L{}'(u) + (1 - a)f_R{}'(u))\partial_x u = 0$$

System with eigenvalues 0 and $\lambda(u, a) = af_L{}'(u) + (1 - a)f_R{}'(u)$.
Extends to v-coupling

Non conservative system

$$\begin{cases} \partial_t u + \lambda(u,a)\partial_x u = 0 \\ \partial_t a = 0 \end{cases}$$

add a regularization term

$$\begin{cases} \partial_t u_\varepsilon + \lambda(u_\varepsilon, a_\varepsilon))\partial_x u_\varepsilon = t\varepsilon\partial_{xx} u_\varepsilon \\ \partial_t a_\varepsilon = t\varepsilon^2\partial_{xx} a_\varepsilon \end{cases}$$

initial data $u_\varepsilon(x,0) = u_0(x), a_\varepsilon(x,0) = a_0(x)$

$$u_0(x) = \begin{cases} u_L, \ x < 0 \\ u_R, \ x > 0 \end{cases} \quad a_0(x) = \begin{cases} 1, \ x < 0 \\ 0, \ x > 0. \end{cases}$$

Regularization with t in the RHS was proposed by Dafermos. It corresponds to a classical viscous regularization in variable $\xi = x/t$, $T = \ln t$ and allows to study the approximation of self-similar solutions.

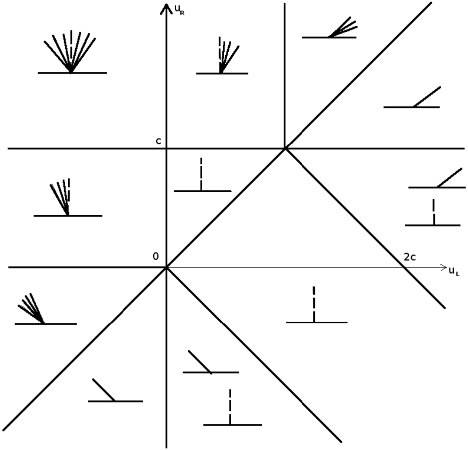
Look for self similar solutions: $\xi = x/t$, $u_\varepsilon(\xi), a_\varepsilon(\xi)$
$u_\varepsilon, a_\varepsilon$ exist, $\exists u$, '$u_{\varepsilon_k} \to u$' as $\varepsilon \to 0$,
- u solution of the CRP, entropy solution in $x < 0$, $x > 0$
- at interface possible boundary layer $\longrightarrow$ *zoom*: *fast* variable

$y = \xi/\varepsilon$ $\mathcal{U}_\varepsilon(y) = u_\varepsilon(\varepsilon y)$, $\mathcal{A}_\varepsilon(y) = a_\varepsilon(\varepsilon y)$.

- $\mathcal{A}_\varepsilon(y)$ converges to $\mathcal{A}(y) = (1 - \mathrm{erf}(y/\sqrt{2}))/2$, $\mathcal{A}(-\infty) = 1, \mathcal{A}(+\infty) = 0$, non trivial profile connecting 1 to 0 thanks to ε^2 (if ε, $\mathcal{A}(y) = 1/2$)
- possible non trivial profiles for $\mathcal{U}$. If f_α strictly convex:

-left: $\mathcal{U}(-\infty) = u(0-)$ or $\mathcal{U}(-\infty) < u(0-)$
$$f_L'(\mathcal{U}(-\infty)) < 0 < f_L'(u(0-))$$
-right: $\mathcal{U}(+\infty) = u(0+)$ or $\mathcal{U}(+\infty) > u(0+)$
$$f_R'(\mathcal{U}(+\infty)) > 0 > f_R'(u(0+))$$
Structure of the discontinuity $u(0-), u(0+)$: $u(0-), \mathcal{U}(-\infty)$ L-stationary shock, $\mathcal{U}(-\infty), \mathcal{U}(+\infty)$, $\mathcal{U}(+\infty), u(0+)$ R- stationary shock. Rules out some *unstable* solutions, possible nonuniqueness

solution of the CRP, in the plane (u_L, u_R)



$f_L(u) = u^2/2, f_R(u) = (u-c)^2/2, c > 0$

Numerical coupling by a two-flux FV method

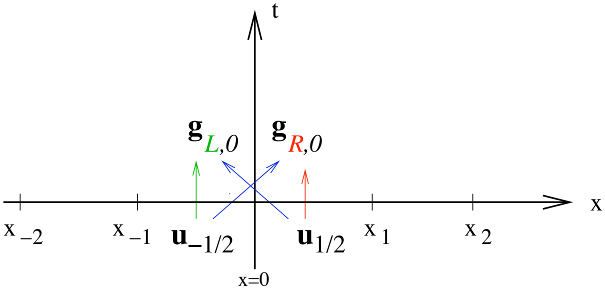
Finite volume method: Δx, Δt, $\mu = \frac{\Delta t}{\Delta x}$, $t_n = n\,\Delta t,\ n \in \mathbb{N}$
cell (x_j, x_{j+1}), center $x_{j+1/2} = \left(j + \frac{1}{2}\right)\Delta x,\ j \in \mathbb{Z}$,
$\mathbf{u}^0_{j+1/2} = \frac{1}{\Delta x}\int_{x_j}^{x_{j+1}} \mathbf{u}_0(x)dx,\ j \in \mathbb{Z}$
two numerical fluxes $\mathbf{g}_L$, $\mathbf{g}_R$, $\mathbf{g}_\alpha$ consistent with $\mathbf{f}_\alpha$
3-point monotone scheme (under CFL condition):

$$\mathbf{g}^n_{\alpha,j} = \mathbf{g}_\alpha\left(\mathbf{u}^n_{j-1/2}, \mathbf{u}^n_{j+1/2}\right)$$

- $\mathbf{u}^{n+1}_{j-1/2} = \mathbf{u}^n_{j-1/2} - \mu\left(\mathbf{g}^n_{L,j} - \mathbf{g}^n_{L,j-1}\right), \quad j \leq 0$
- $\mathbf{u}^{n+1}_{j+1/2} = \mathbf{u}^n_{j+1/2} - \mu\left(\mathbf{g}^n_{R,j+1} - \mathbf{g}^n_{R,j}\right), \quad j \geq 0$
- $x_0 = 0$ is a boundary between two cells: two fluxes for $j = 0$
  $\mathbf{g}^n_{\alpha,0} = \mathbf{g}_\alpha\left(\mathbf{u}^n_{-1/2}, \mathbf{u}^n_{+1/2}\right), \alpha = L, R$

two numerical fluxes at $x = 0$,

- $\mathbf{g}^n_{\alpha,0} = \mathbf{g}_\alpha\left(\mathbf{u}^n_{-1/2}, \mathbf{u}^n_{+1/2}\right)$, $\alpha = L, R$ ensures **u**-state coupling

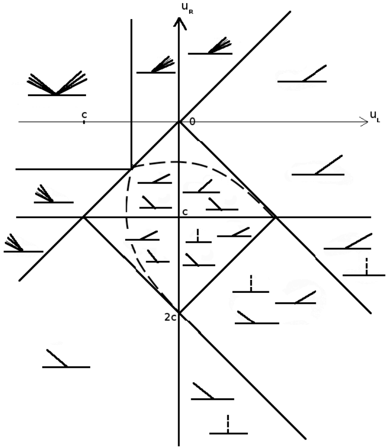


- $\mathbf{g}^n_{L,0} = \mathbf{g}_L(\mathbf{u}^n_{-1/2}, \varphi_L(\mathbf{v}^n_{+1/2}))$, $\mathbf{g}^n_{0,R} = \mathbf{g}_R(\varphi_R(\mathbf{v}^n_{-1/2}), \mathbf{u}^n_{+1/2})$
ensures **v**-state coupling

If* the two-flux FV scheme converges ($u_\Delta \to u$) in some 'sensible way', (*proven in the scalar case, with rather general assumptions)
then u is solution of the coupled problem with our CC.
In case of

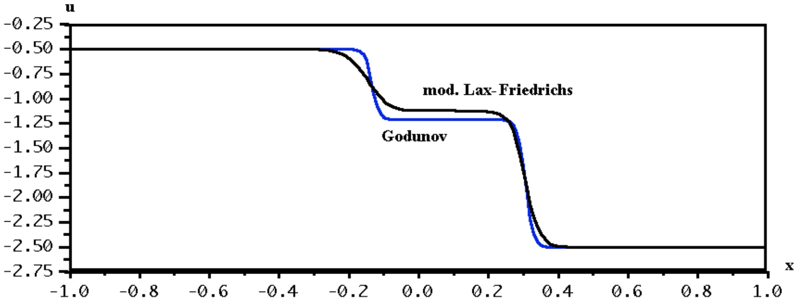
- uniqueness, u is the unique solution
- non-uniqueness, u is a solution, which solution?

scalar quadratic case: $f_L(u) = u^2/2, f_R(u) = (u - c)^2/2, c < 0$:
possible solutions obtained by Dafermos regularization



double shock missing in central area

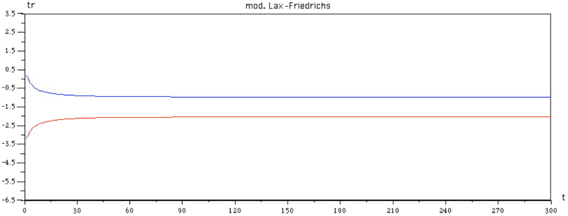
scalar quadratic case: $f_L(u) = u^2/2, f_R(u) = (u+4)^2/2$
CRP with $u_L = -0.5, u_R = -2.5,\ f_L'(u_L) < 0, f_R'(u_R) > 0$



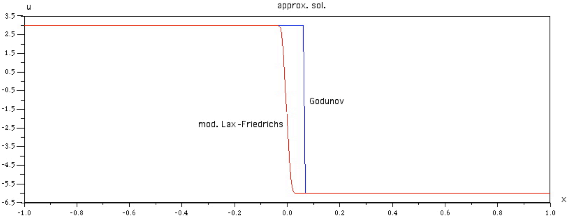
exact solution: 2 shocks computed with Godunov's scheme and Lax-Friedrichs modified: $u_G^m = -1,21, u_{LF}^m = -1,12$

which solution is computed?

$f_L(u) = u^2/2, f_R(u) = (u+3)^2/2$, CRP with $u_L = 3, u_R = -6$

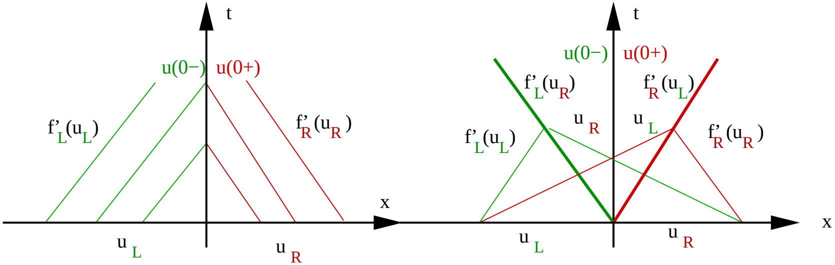


traces for mod. Lax-Friedrichs



computed solution: a $R-$shock with Godunov's scheme,
$L-$shock + stationary discontinuity + $R-$shock with mod. L.-F.

$f_L(u) = u^2/2, f_R(u) = (u+3)^2/2$, same CRP with $u_L = 3, u_R = -6$
data such that $f_L'(u_L) > 0, f_L'(u_R) < 0, f_R'(u_L) > 0, f_R'(u_R) < 0,$

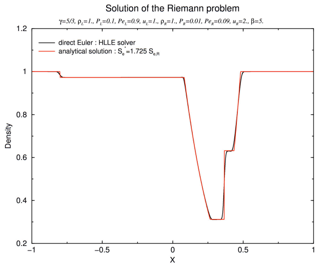


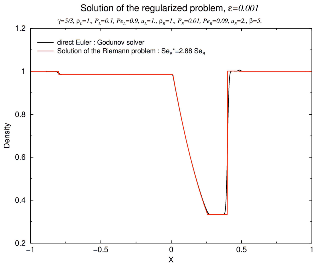
mod. LF computes a compound discontinuity with boundary layer: L−shock $u_L \to u(0-)$, discontinuity $u(0-) \to u(0+)$, R−shock $u(0+) \to u_R$

$\mathbf{u} = (\rho, \rho v, \rho s, \rho s_e)^T$, $\mathbf{f}_L(\mathbf{u}) = (\rho v, \rho v^2 + P, \rho s v, \rho s_e v - \beta s_e)^T$
$\mathbf{f}_R(\mathbf{u}) = (\rho v, \rho v^2 + P, \rho s v, \rho s_e v)^T$, $P = p + p_e = (\rho s)^\gamma + (\rho s_e)^\gamma$
$(\gamma = 5/3)$ differ only by the la 4th equation with
$v_e = v - \frac{\beta}{\rho}$, $x < 0$, $\quad v_e = v$, $x > 0$
We study the case
$v_e(0-) < (v-c)(0-) < 0 < v(0-) < (v+c)(0-)$ and
$(v-c)(0+) < 0 < v_e(0+) = v(0+) < (v+c)(0+)$

- continuity of $\mathbf{u}$ at $x = 0$
- nonuniqueness of the solution of the CRP
- parametrize the solutions by s_e electron entropy of state $\mathbf{u}(0)$
- numerical results: for a coupled CRP, the computed solution depends little on the scheme
- differs from the solution computed after regularization of the initial data

CRP, initial data: $\rho_L = \rho_R$, u, p discontinuous, $\beta = 5$





left: computed CRP, $s_e(0) = 1,725 s_{e,R}$ (fitted value) and exact solution computed with this value

right: regularization of u, p by spline, one determines $s_e(0) = 2,88 s_{e,R}$ (fitted), and exact solution computed with this value

Numerical flux coupling for Euler

For Euler, $\mathbf{u} = (\varrho, \varrho u, \varrho e)$, flux $(\varrho u, \varrho u^2 + p, (\varrho e + p)u)$, the eigenvalues may change sign, the flux is not an admissible change of variables.

Numerical flux coupling via a *global relaxation coupling solver*

- a larger relaxation system relaxing towards Euler as $\epsilon \to 0$
- a numerical coupling of the convective part of the relaxation systems with judicious choice of CC
- a splitting method: convection + instantaneous relaxation $\epsilon = 0$

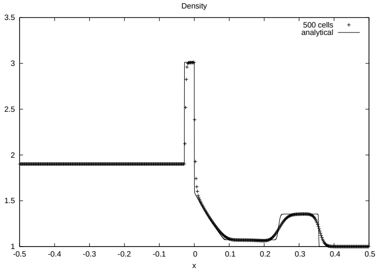
Results in a standard finite volume method: if it 'converges' to $\mathbf{u}$, $\mathbf{u}$ is solution of a *coupled problem* with *continuous flux*, and entropy solution in $x < 0$ and $x > 0$

Euler (barotropic case): $\mathbf{u} = (\varrho, \varrho u)$, flux $\mathbf{f}(\mathbf{u}) = (\varrho u, \varrho u^2 + p)$
Relaxation system (Suliciu):

$$\begin{cases} \partial_t \varrho + \partial_x(\varrho u) = 0 \\ \partial_t(\varrho u) + \partial_x(\varrho u^2 + \Pi) = 0 \\ \partial_t(\varrho \mathcal{T}) + \partial_x(\varrho \mathcal{T} u) = \lambda \varrho(\tau - \mathcal{T}) \end{cases}$$

with $\Pi_\alpha = \tilde{\Pi}(\tau, \mathcal{T}) \equiv \tilde{p}_\alpha(\mathcal{T}) + a^2(\mathcal{T} - \tau)$, $\tau = 1/\varrho, \tilde{p}(\tau) = p(\varrho)$.
Formally $\mathcal{T} \to \tau$, $\Pi \to p$ as $\lambda \to \infty$.
3 LD fields, RP are easily computed $\to$ Godunov's scheme:
$j \neq 0$ (left and right) $\mathbf{g}^n_{\alpha,j} = \mathbf{f}_\alpha(\mathbf{W}_\alpha(0; \mathbf{u}^n_{j-1/2}, \mathbf{u}^n_{j+1/2}))$
$j = 0$, solve a CRP with transmission of $\mathbf{v} = (\tau, u, \Pi)$ then
$\mathbf{g}^n_{\alpha,0} = \mathbf{f}_\alpha(\mathbf{W}_c(0; \mathbf{v}^n_{-1/2}, \mathbf{v}^n_{+1/2}))$, $\mathbf{g}^n_{L,0} = \mathbf{g}^n_{R,0}$
$\to$ results in a *conservative consistent scheme* for Euler, entropy scheme in $x < 0, x > 0$

Example of a (numerical) flux coupling: a CRP for Euler



data $\varrho_L = 1.902, u_L = 1.6361, p_L = 2.4598; \varrho_R = 1, u_R = 2, p_R = 1$
(computation by Thomas Galié)
exact solution: $1L-$shock, stationary (coupling) wave, $1R-$ sonic rarefaction, $2R-$CD and $3R-$shock

Developments: theoretical results

1. Scalar case
   - Existence theorem in some generic situations (and uniqueness in some cases)
   - convergence of the two-flux scheme (monotone, E-scheme)
   - Coupled Riemann problem
   - coupling of the 2x2 relaxation system with the relaxed equation with F. Caetano
   - Dafermos regularization with Benjamin Boutin
2. The case of systems
   - coupling of linear systems
   - multiple choice of transmitted variables
   - coupling of Lagrange-type systems (characteristic interface)
   - coupling Euler system (3x3) and p-system (2x2)
   - coupling two Euler systems (Eulerian coordinates)
     - coupled Riemann problem (state coupling, not easy)
     - relaxation model: explicit solution of CRP for a relaxation system with LD fields; flux coupling

1. Plasma model: *same model (same pde)*, one neglects the current density. Case of non uniqueness
2. Coupling two Euler systems: *same model*, different closure laws
   - choice of transmitted variables
   - example u, p for a material wave
   - choice of scheme (relaxation, Lagrange+projection)
   - examples of coupled Riemann problem
3. Coupling multi-phase models: HRM-HEM, two different but *consistent* models, 4 equations / 3 equations
   HEM is obtained from HRM through relaxation (thermodynamical equilibrium)
4. Work in progress: 4 equations (mixture model with drift) / 7 equations (bifluid model) *compatibility is not obvious*

- coupling bifluid and drift flux models:
  - asymptotic expansion of a bifluid model ($\rightarrow$ drift flux model)
  - asymptotic preserving schemes *(cf. N. Seguin)*
  - relaxation approximation of a bifluid ($\rightarrow$ a drift flux) model *(cf. A. Ambroso)*
- control of the transmission procedure, optimization
- relaxation for fluid models
- stability of solutions, linearized stability in 2d
- more convergence results (relaxation, approximation)
- thickened interface

Conclusion

- This work was necessary: it gives in many cases
  - a theoretical model for interface coupling
  - a better understanding of what can be transmitted
  - a robust coupling scheme

  and useful tools (even for other approaches)
- Some questions left
- It is not the ultimate approach
  - thickened interface
- Related topics of interest are
  - interface coupling with small scale phenomena
  - coupling more complex fluid systems (multiphysics)