Two-phase flows with granular stress

T. Gallouët, P. Helluy, J.-M. Hérard, J. Nussbaum

LATP Marseille
IRMA Strasbourg
EDF Chatou
ISL Saint-Louis

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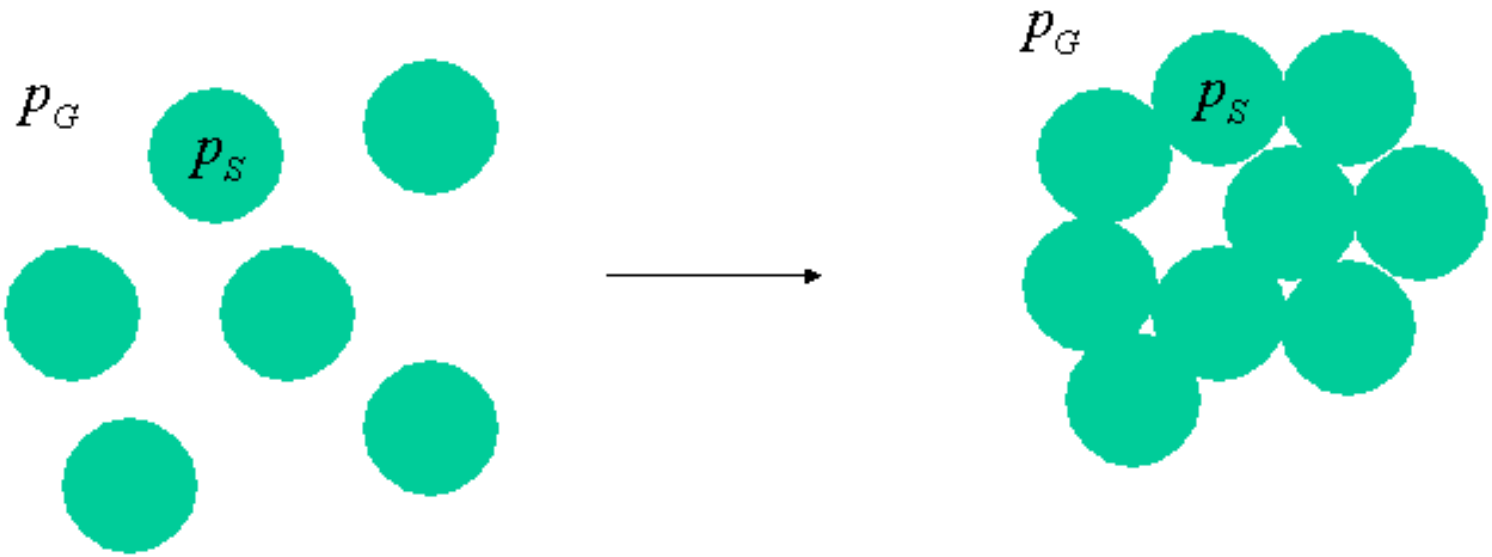
Context

- Flow of weakly compressible grains (powder, sand, *etc.*) inside a compressible gas;
- importance of the granular stress when the grains are packed;
- averaged model (the solid phase is represented by an equivalent continuous media);
- two-velocity, two-pressure mathematical model;
- relaxation of the pressure equilibrium in order to gain stability;
- stable and useful approximation for real applications.

Granular stress R

Gas phase = (G)
Solid phase =(S)



Dilute case: $p_S - p_G = 0$.

Packed case: $p_S - p_G = R > 0$.

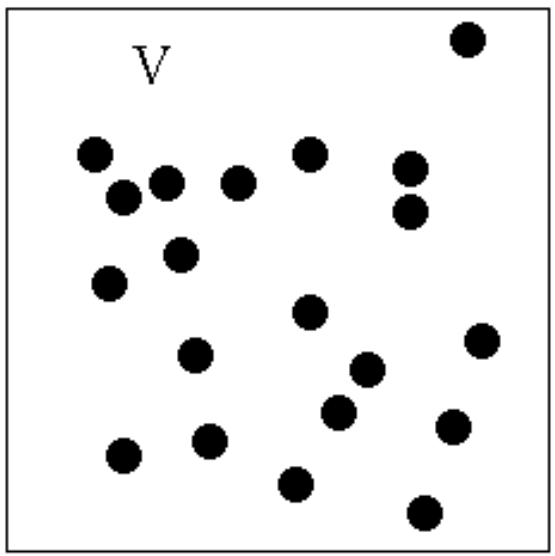
Averaging process

Averaging in a small volume V around point x

particle $i = 1 \cdots N$

mass M_i

volume V_i



Discrete model

$$\alpha_S(x) = \frac{\sum_i V_i}{V}, \quad \alpha_G = 1 - \alpha_S$$

$$\rho_S(x) = \frac{\sum_i M_i}{\sum_i V_i}.$$

Averaged model

Outlines

Two-phase granular flow models

Unstable one-pressure model

A gas phase (G) $k = 1$, a solid phase (S) $k = 2$.
Seven unknowns: partial densities $\rho_k(x,t)$, velocities $u_k(x,t)$, internal energies $e_k(x,t)$, gas volume fraction $\alpha_1(x,t)$.

Pressure law: $p_k = p_k(\rho_k, e_k)$.

Other definitions: $m_k = \alpha_k \rho_k, \quad \alpha_2 = 1 - \alpha_1, \quad E_k = e_k + \dfrac{u_k^2}{2}$.

The whole six-equation PDE system reads [3]

$$\begin{aligned} &m_{k,t} + (m_k u_k)_x = 0, \\ &(m_k u_k)_t + (m_k u_k^2 + \alpha_k p_k)_x - p_1 \alpha_{k,x} = 0, \\ &(m_k E_k)_t + ((m_k E_k + \alpha_k p_k) u_k)_x + p_1 \alpha_{k,t} = 0, \end{aligned} \tag{1}$$

with the algebraic relation

$$p_2 = p_1 + R.$$

More stable two-pressure model

The previous system is generally not hyperbolic. Therefore, we relax the algebraic pressure relation.
The seven-equation PDE system reads [1]

$$
\begin{aligned}
&m_{k,t} + (m_k u_k)_x = 0, \\
&(m_k u_k)_t + (m_k u_k^2 + \alpha_k p_k)_x - p_1 \alpha_{k,x} = 0, \\
&(m_k E_k)_t + ((m_k E_k + \alpha_k p_k) u_k)_x + p_1 \alpha_{k,t} = 0, \\
&\alpha_{1,t} + u_2 \alpha_{1,x} = P.
\end{aligned}
\tag{2}
$$

And the source term P has to contain the granular modeling.

Entropy and closure

The gas entropy s_1 satisfies the standard relation (T_k is the temperature of phase k)

$$T_1 ds_1 = de_1 - \frac{p_1}{\rho_1^2} d\rho_1.$$

For the entropy s_2 of the solid phase, we take into account the granular stress.

$$\underbrace{T_2 ds_2}_{\text{energy variation}} = \underbrace{de_2}_{\text{internal energy variation}} \underbrace{-\frac{p_2}{\rho_2^2} d\rho_2}_{\text{pressure work}} \underbrace{-\frac{R}{m_2} d\alpha_2}_{\text{granular work}}$$

After some computations, we find the following entropy dissipation equation

$$(\sum_k m_k s_k)_t + (\sum_k m_k u_k s_k)_x = \frac{P}{T_2} (p_1 + R - p_2).$$

A natural choice that ensures positive dissipation is

$$\boxed{P = \frac{1}{\tau_p}\alpha_1\alpha_2(p_1 + R - p_2), \quad \tau_p \geqslant 0}.$$

The choice of the pressure relaxation parameter τ_p is very important. We can also write

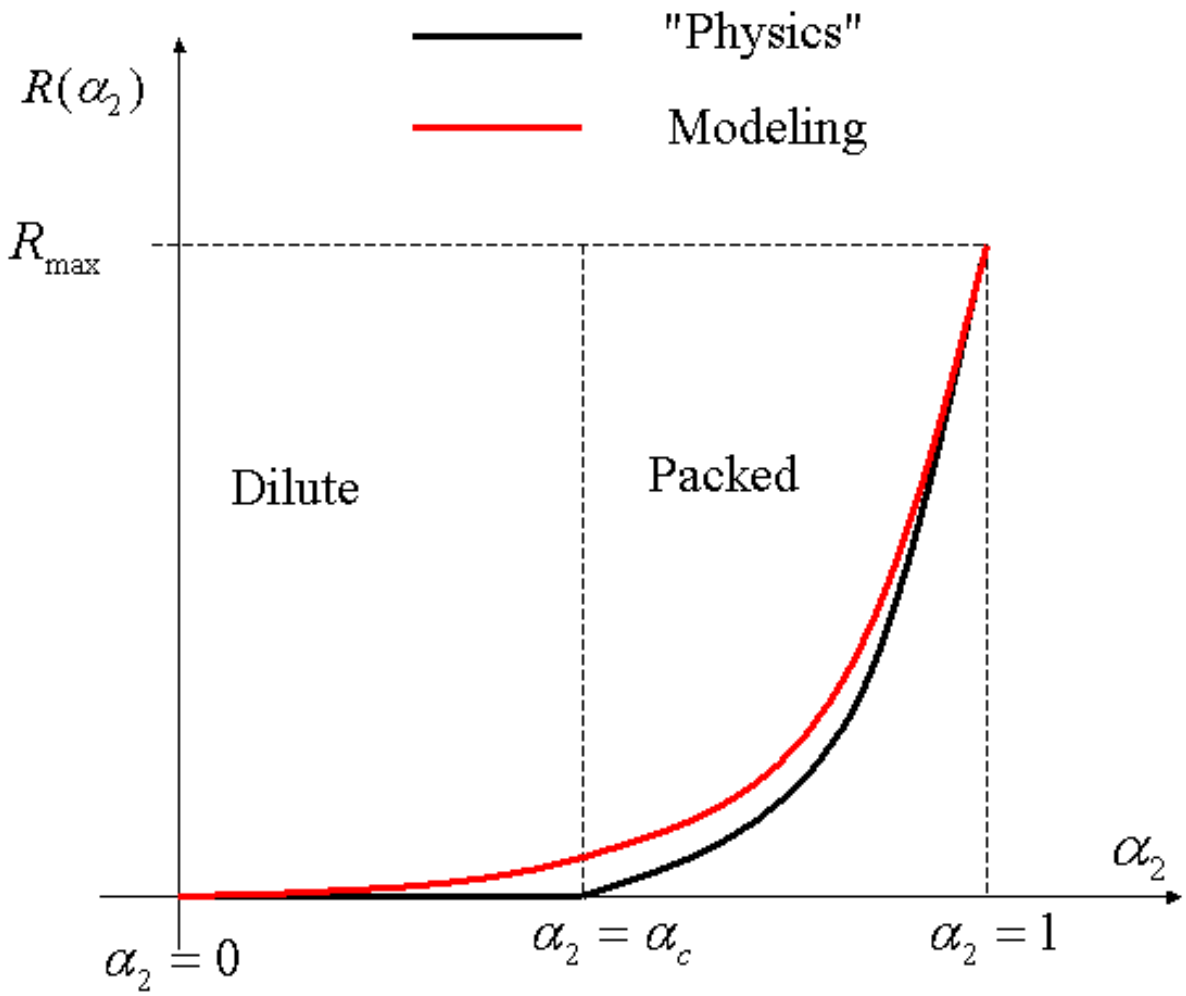
$$p_1 + R - p_2 = \frac{\tau_p}{\alpha_1\alpha_2}\left(\alpha_{1,t} + u_2\alpha_{1,x}\right).$$

- for $\tau_p = t_{eq} = 0$, we recover the instantaneous six-equation equilibrium model $p_2 = p_1 + R$;
- $\tau_p = t_{eq}p_{ref}$ where t_{eq} is the characteristic time of the pressure equilibrium and p_{ref} a reference pressure;
- for $\tau_p > 0$, the system should be more stable.

Thermodynamically coherent granular stress

Shape of the granular stress

Usually, $R = R(\alpha_2)$ [4]



Entropy and granular stress

How to choose the granular stress R for compressible solid phases?
Exemple: for a stiffened gas equation of state

$$p_2 = p_2(\rho_2, e_2) = (\gamma_2 - 1)\rho_2 e_2 - \gamma_2 \pi_2 \quad , \quad \gamma_2 > 1.$$

We suppose that $R = R(\rho_2, \alpha_2)$. We express that the entropy is a closed differential form ($1/T_2$ is an integrating factor of the energy form)

$$ds_2 = \frac{1}{T_2}\left(de_2 - \frac{p_2}{\rho_2^2} d\rho_2 - \frac{R}{m_2} d\alpha_2 \right).$$

We find

$$R(\rho_2, \alpha_2) = \rho_2^{\gamma_2} r(\alpha_2).$$

Particular choice

$$r(\alpha_2) = \lambda \alpha_2^{\gamma_2} \quad \Rightarrow \quad \boxed{R = \lambda m_2^{\gamma_2}}.$$

Hyperbolicity study

Stability of the relaxed system ($\tau_p > 0$)

Let

$$Y = (\alpha_1, \rho_1, u_1, s_1, \rho_2, u_2, s_2)^T.$$

In this set of variables the system becomes

$$Y_t + B(Y)Y_x = (P, 0, 0, 0, 0, 0, 0)^T$$

$$B(Y) = \begin{bmatrix} u_2 & & & & & & \\ \frac{\rho_1(u_1-u_2)}{\alpha_1} & u_1 & \rho_1 & & & & \\ & \frac{c_1^2}{\rho_1} & u_1 & \frac{p_{1,s_1}}{\rho_1} & & & \\ & & & u_1 & & & \\ & & & & u_2 & \rho_2 & \\ \frac{p_1-p_2}{m_2} & & & & \frac{c_2^2}{\rho_2} & u_2 & \frac{p_{2,s_2}}{\rho_2} \\ & & & & & & u_2 \end{bmatrix}, \quad c_k = \sqrt{\frac{\gamma_k(p_k+\pi_k)}{\rho_k}}.$$

$$\det(B(Y)-\lambda I) = (u_2-\lambda)^2(u_1-\lambda)(u_1-c_1-\lambda)(u_1+c_1-\lambda)(u_2-c_2-\lambda)(u_2+c_2-\lambda).$$

The system is hyperbolic [6, 2].

Instability of the one-pressure model ($\tau_p = 0$)

When $\tau_p \to 0$, formally, we end up with a standard single pressure model

$$p_2 = p_1 + R.$$

We can remove an equation (for example the volume fraction evolution) and we find a six-equation system

$$Z = (\rho_1, u_1, s_1, \rho_2, u_2, s_2)^T.$$
$$Z_t + C(Z)Z_x = 0.$$

In the simple case where $R = \lambda m_2^{\gamma_2}$, let

$$\Delta = \alpha_1\alpha_2 + \delta(\alpha_1\rho_2 a_2^2 + \alpha_2\rho_1 c_1^2),$$

and

$$\delta = \frac{\alpha_2^{1-1/\gamma_2}}{\lambda\gamma_2\rho_2^{\gamma_2}}.$$

Then we find

$$C(Z) = \begin{bmatrix} u_1 + \frac{\rho_1\rho_2 c_1^2 a_2^2 \delta^2 (u_1 - u_2)}{\Delta} & \frac{\alpha_1\rho_1(\alpha_2 + \rho_2 a_2^2 \delta)}{\Delta} & \frac{\rho_1\rho_2 a_2^2 \delta^2 (u_1 - u_2) p_{1,2,}}{\Delta} & \frac{\rho_1 a_2^2 \delta(\alpha_2 + \rho_2 a_2^2 \delta)(u_2 - u_1)}{\Delta} & \frac{\alpha_2\rho_1\rho_2 a_2^2 \delta}{\Delta} & \frac{\rho_1\delta(\alpha_2 + \rho_2 a_2^2 \delta)(u_2 - u_1) p_{2,2,}}{\Delta} \\ \frac{c_1^2}{\rho_1} & u_1 & \frac{p_{1,2,}}{\rho_1} & 0 & 0 & 0 \\ 0 & 0 & u_1 & 0 & 0 & 0 \\ \frac{\rho_2 c_1^2 \delta(\alpha_1 + \rho_1 c_1^2 \delta)(u_1 - u_2)}{\Delta} & \frac{\alpha_1\rho_1\rho_2 c_1^2 \delta}{\Delta} & \frac{\rho_2\delta(\alpha_1 + \rho_1 c_1^2 \delta)(u_1 - u_2) p_{1,2,}}{\Delta} & u_2 + \frac{\rho_1\rho_2 c_1^2 a_2^2 \delta^2 (u_2 - u_1)}{\Delta} & \frac{\alpha_2\rho_2(\alpha_1 + \rho_1 c_1^2 \delta)}{\Delta} & \frac{\rho_1\rho_2 c_1^2 \delta^2 (u_2 - u_1) p_{2,2,}}{\Delta} \\ \frac{(p_1 - p_2)\delta c_1^2}{\alpha_2\rho_2} & 0 & \frac{(p_1 - p_2)\delta p_{1,2,}}{\alpha_2\rho_2} & \frac{\delta(p_2 - p_1) + \alpha_2 c_2^2}{\alpha_2\rho_2} & u_2 & \frac{(\alpha_2 + \delta(p_2 - p_1)) p_{2,2,}}{\alpha_2\rho_2} \\ 0 & 0 & 0 & 0 & 0 & u_2 \end{bmatrix}$$

- The eigenvalues can be computed only numerically.
- We observe that when $R \longrightarrow 0$, the system is generally not hyperbolic.
- We observe also that when $R \longrightarrow \infty$, we recover hyperbolicity;
- When $R = 0$, the convection matrix has to be computed in another way.

Numerical approximation: splitting method

- The two-pressure model is hyperbolic and more stable.
- The one-pressure model is commonly used in applications on coarse meshes (high damping of the numerical viscosity).
- We can approximate the one-pressure model by the more general two-pressure model.
- A time step is made of a convection stage and a relaxation stage. In the relaxation stage, we apply the exact or relaxed pressure equilibrium.

Convection step

Let

$$w = (\alpha_1, m_1, m_1 u_1, m_1 E_1, m_2, m_2 u_2, m_2 E_2)^T,$$

The system can be written

$$w_t + f(w)_x + G(w) w_x = (P, 0, 0, 0, 0, 0, 0)^T$$

In the first half step the source term is omitted. We use a standard Rusanov scheme

$$\frac{w_i^* - w_i^n}{\Delta t} + \frac{f_{i+1/2}^n - f_{i-1/2}^n}{\Delta x} + G(w_i^n) \frac{w_{i+1}^n - w_{i-1}^n}{2\Delta x} = 0,$$

$f_{i+1/2}^n = f(w_i^n, w_{i+1}^n)$ numerical conservative flux,

$$f(a, b) = \frac{f(a) + f(b)}{2} - \frac{\lambda}{2}(b - a).$$

For λ large enough, the scheme is entropy dissipative. Typically, we take

$$\lambda = \max \left[\rho \left(f'(a) + G(a) \right), \rho \left(f'(b) + G(b) \right) \right].$$

Relaxation step

In the second half step, we have formally to solve

$$\begin{aligned}
&\alpha_{1,t} = P, \\
&m_{k,t} = u_{k,t} = 0, \\
&(m_k e_k)_t + p_1 \alpha_{k,t} = 0.
\end{aligned}$$

Because of mass and momentum conservation we have $m_k = m_k^*$ and $u_k = u_k^*$. In each cell we have to compute (α_1, p_1, p_2) from the previous state w^*

$$\begin{aligned}
&p_2 = p_1 + \lambda m_2^{\gamma_2} + \tau_p \frac{\alpha_2 - \alpha_2^*}{\alpha_1 \alpha_2 \Delta t}, \\
&m_1 e_1 + m_2 e_2 = m_1^* e_1^* + m_2^* e_2^*, \\
&(m_1 e_1 - m_1^* e_1^*) + p_1(\alpha_1 - \alpha_1^*) = 0.
\end{aligned}$$

After some manipulations, we have to solve

$$H(\alpha_2) = (\pi_2 - \pi_1)(\alpha_1 + (\gamma_1 - 1)(\alpha_1 - \alpha_1^*))(\alpha_2 + (\gamma_2 - 1)(\alpha_2 - \alpha_2^*))$$
$$+(\alpha_2^{2-\gamma_2} m_2^{\gamma_2} \theta(\alpha_2) + \frac{\tau_p}{(1-\alpha_2)} \frac{\alpha_2 - \alpha_2^*}{\Delta t} - A_2)(\alpha_1 + (\gamma_1 - 1)(\alpha_1 - \alpha_1^*))$$
$$+A_1(\alpha_2 + (\gamma_2 - 1)(\alpha_2 - \alpha_2^*)) = 0,$$

with

$$A_k = \alpha_k^*(p_k^* + \pi_k) > 0.$$

Theorem: for stiffened gas laws and the simple granular law, H is concave, $H(0) < 0$ and $H(\beta_2) > 0$ with

$$0 < \beta_2 = \frac{(1 - \alpha_2^*) + \alpha_2^* \gamma_1}{\gamma_1} < 1.$$

There is a unique α_2 in $]0, \beta_2[\subset]0, 1[$ such that $H(\alpha_2) = 0$.

Numerical application

Stability study

We have constructed an entropy dissipative approximation of a non-hyperbolic system !
What happens numerically ?
Consider a simple Riemann problem in the interval $[-1/2, 1/2]$. $\gamma_1 = 1.0924$ and $\gamma_2 = 1.0182$. We compute the solution at time $t = 0.00075$. The CFL number is 0.9.
Data:

	(L)	(R)
ρ_1	76.45430093	57.34072568
u_1	0	0
p_1	200×10^5	150×10^5
ρ_2	836.1239718	358.8982226
u_2	0	0
p_2	200×10^5	150×10^5
α_1	0.25	0.25

$\tau_p = 0,\ R = 0$

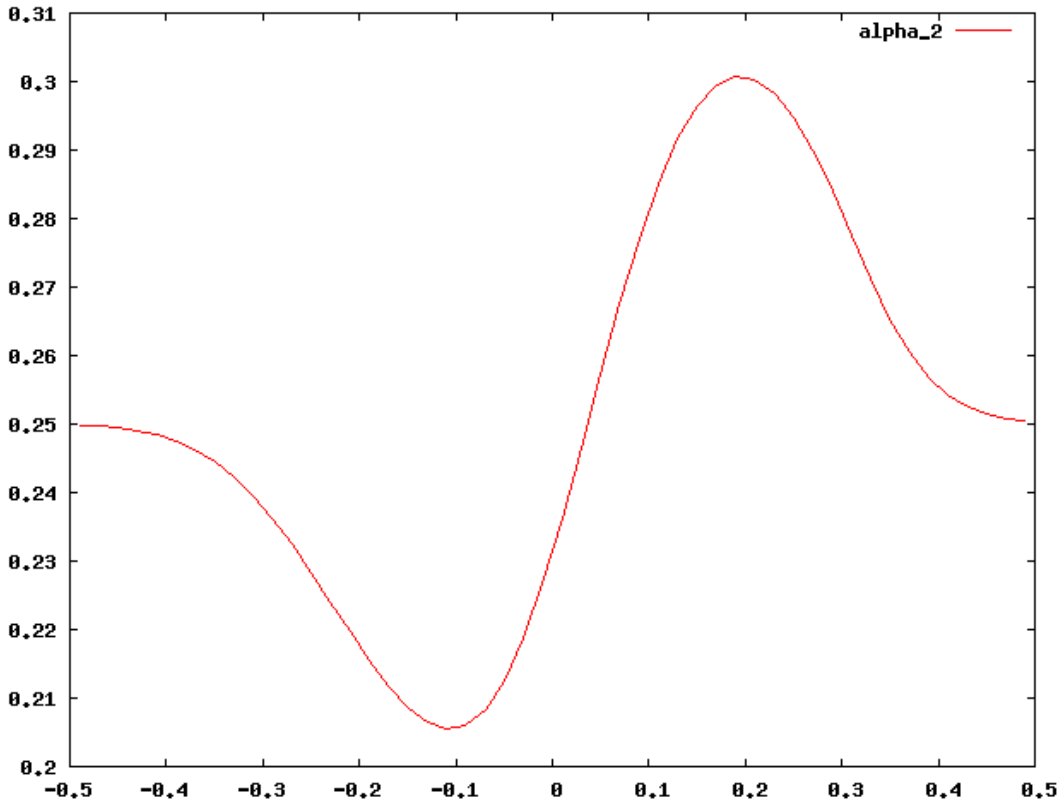


Figure: Void fraction, 50 cells, $\tau_p = 0,\ R = 0$.

$\tau_p = 0,\ R = 0$

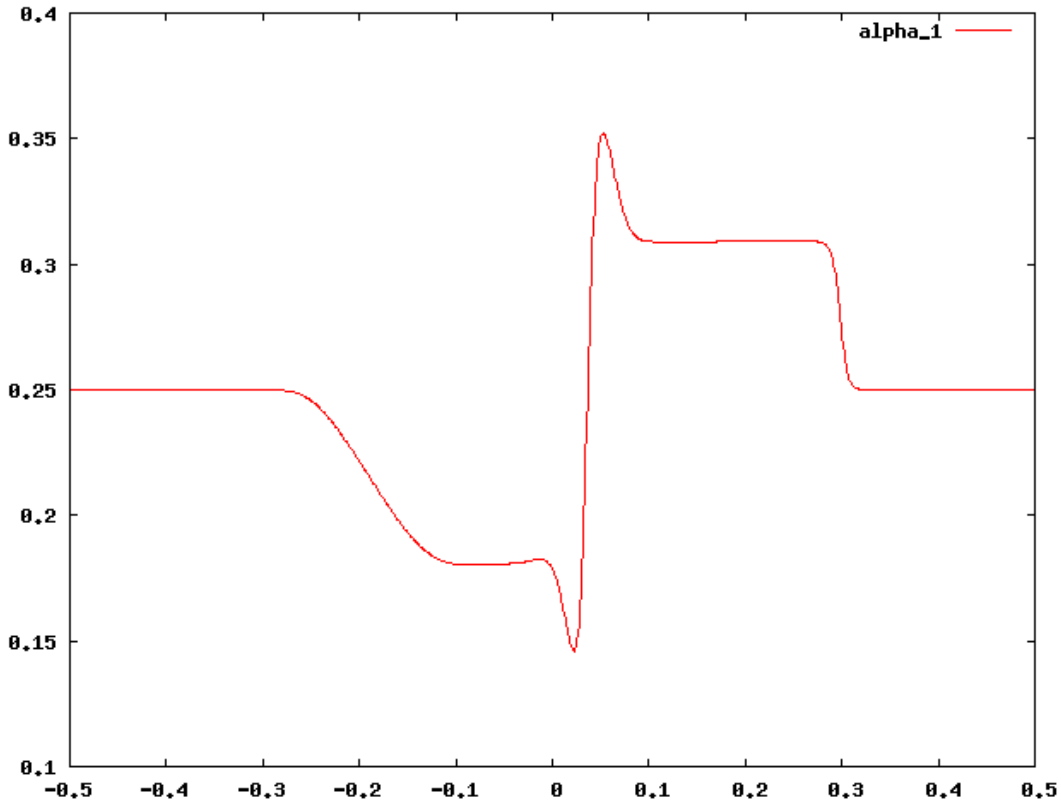


Figure: Void fraction, $1,000$ cells, $\tau_p = 0,\ R = 0$.

$\tau_p = 0,\ R = 0$

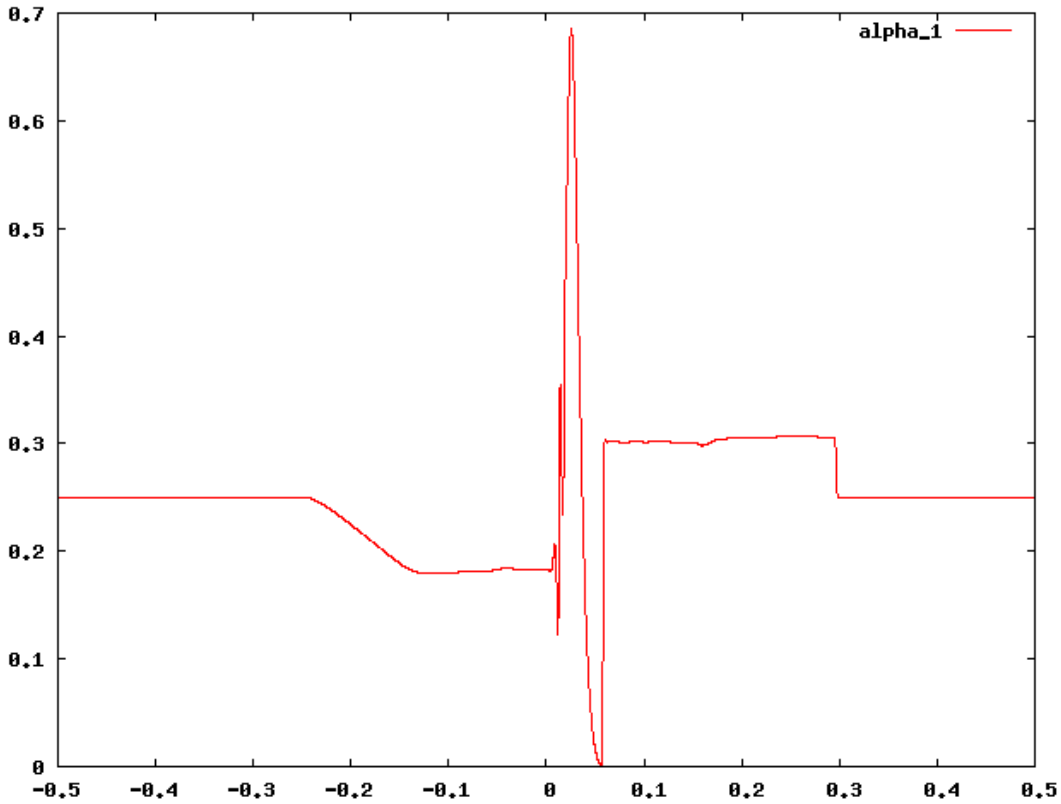


Figure: Void fraction, $10,000$ cells, $\tau_p = 0,\ R = 0$.

$\tau_p = 0,\ R = 0$

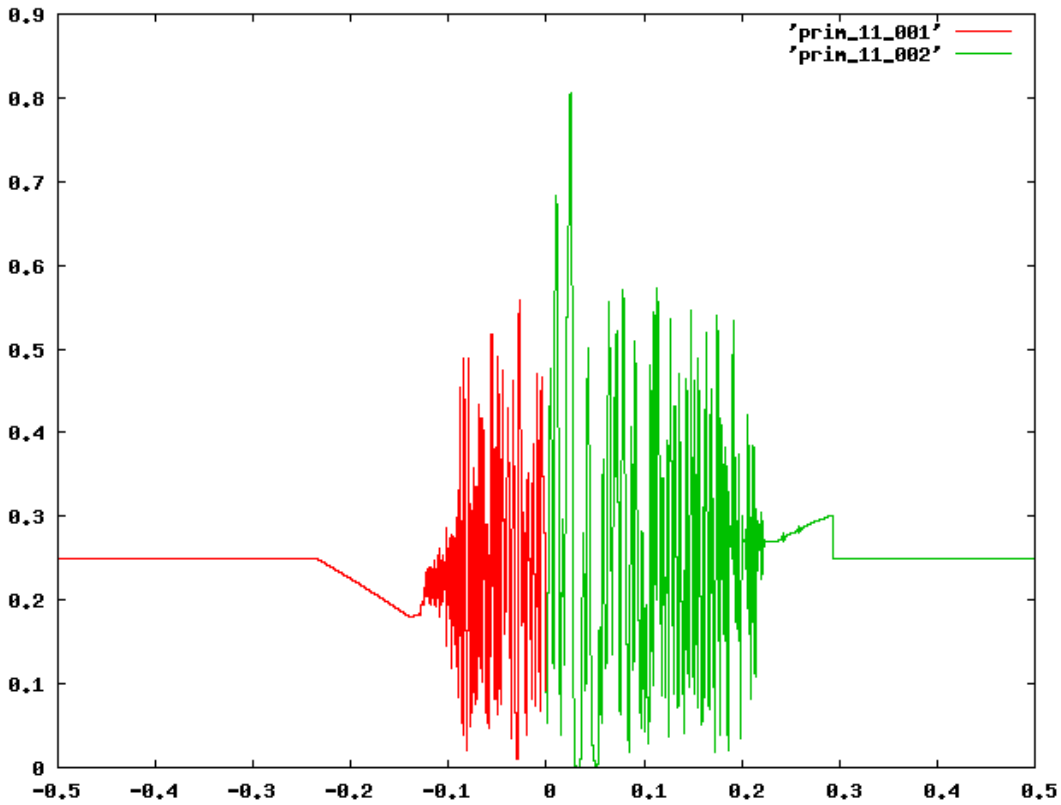


Figure: Void fraction, $100,000$ cells, $\tau_p = 0,\ R = 0$.

Linearly unstable but non-linearly stable...

$\tau_p > 0,\ R = 0$

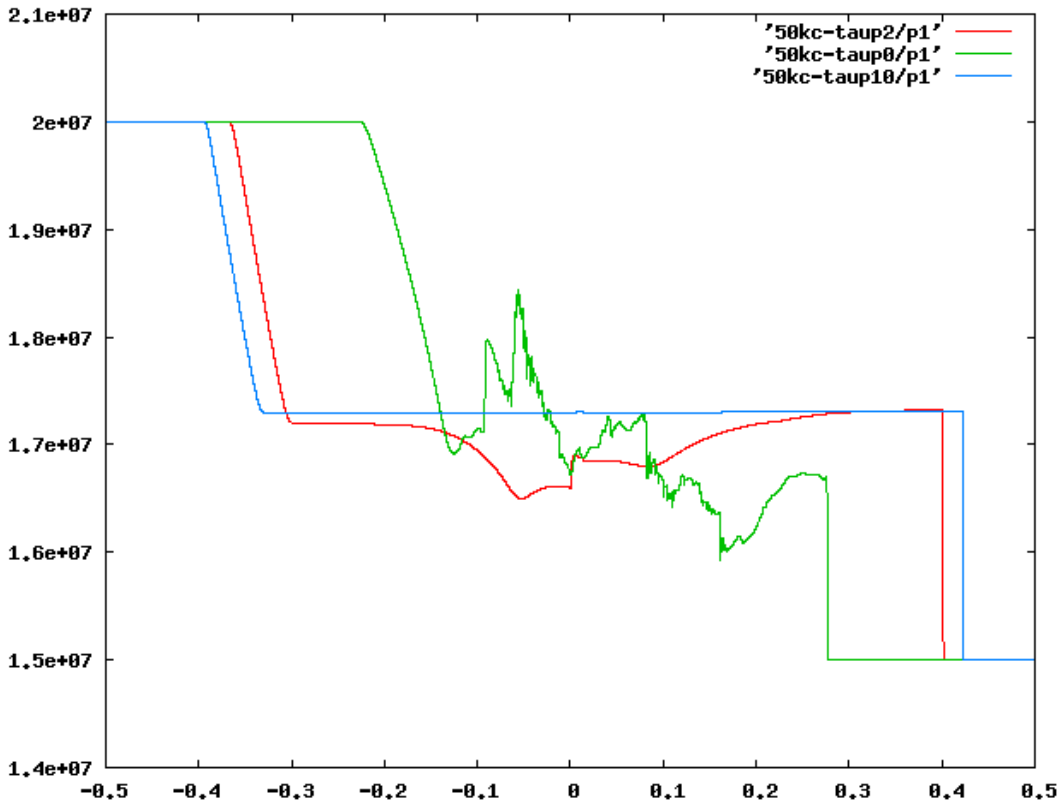


Figure: Pressure, $50,000$ cells, $\tau_p = 0, 2, 10,\ R = 0$.

The relaxation clearly stabilizes the results...

$\tau_p = 0,\ R > 0$

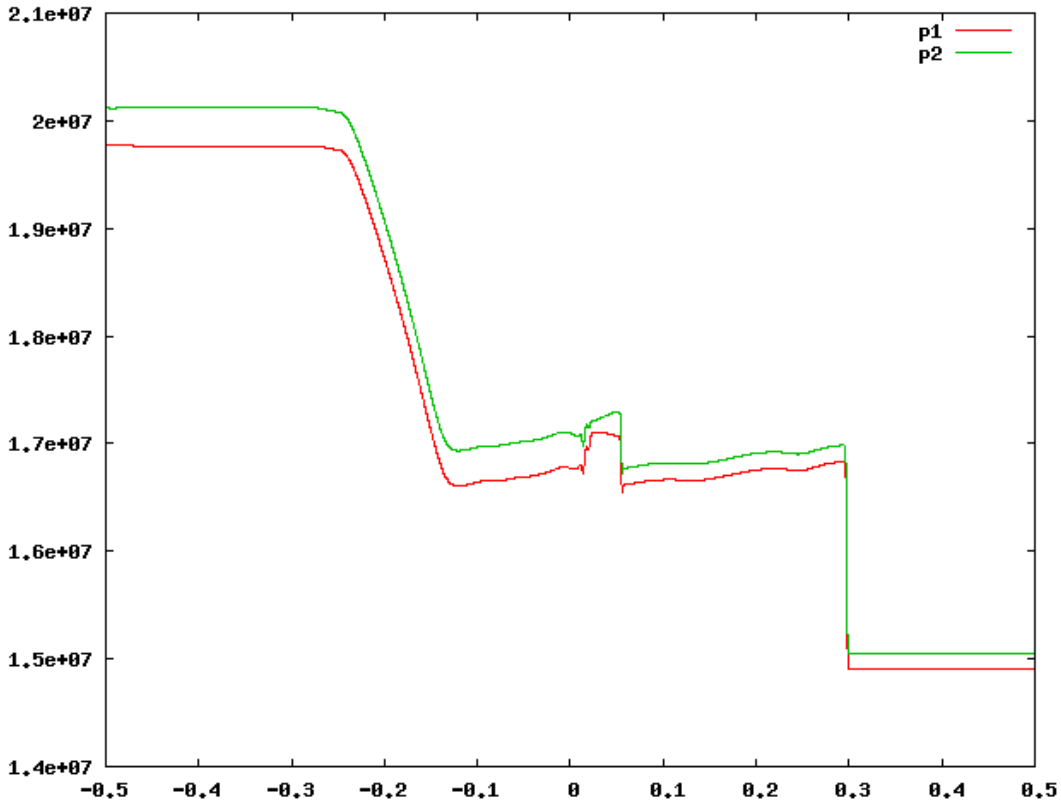


Figure: Pressures p_1 and p_2, $10,000$ cells, $\tau_p = 0,\ R = 500 m_2^{\gamma_2}$.

$\tau_p = 0,\ R > 0$

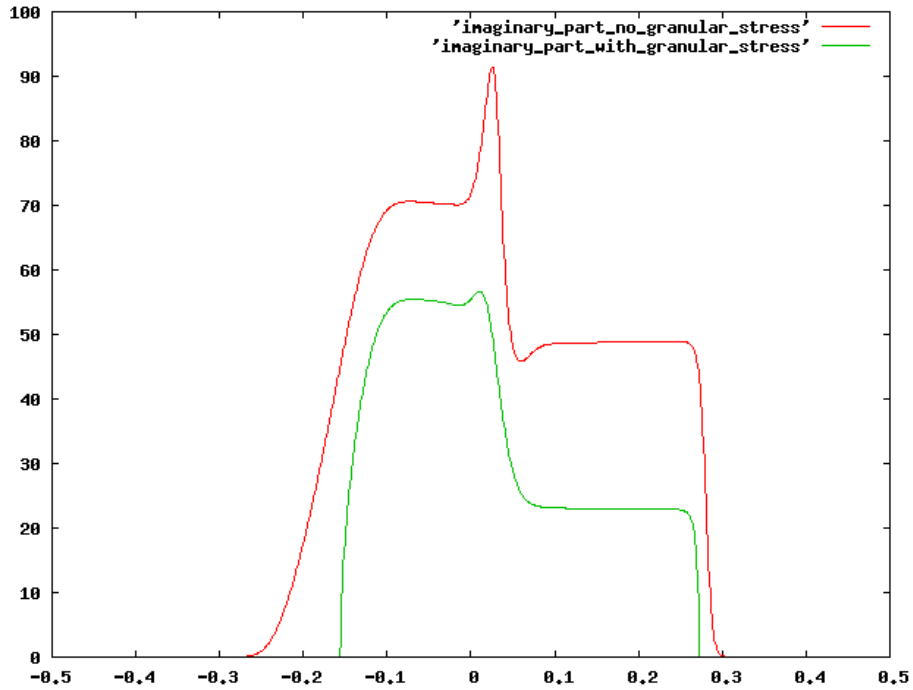


Figure: Eigenvalues: imaginary part $I = \sqrt{\sum_{i=1}^{6} \mathrm{Im}\,(\lambda_i)^2}$, $1,000$ cells, $R = 0$ or $R = 500 m_2^{\gamma_2}$.

The granular stress slightly improves the stability...

Combustion chamber

We consider now a simplified gun [3, 5]. The right boundary of the computational domain is moving. We activate the granular stress and other source terms (chemical reaction and drag), which are all entropy dissipative. The instabilities would occur on much finer grids...
Other source terms are added.

$$
\begin{aligned}
&m_{k,t} + (m_k u_k)_x = \pm M,\\
&(m_k u_k)_t + (m_k u_k^2 + \alpha_k p_k)_x - p_1 \alpha_{k,x} = \pm Q,\\
&(m_k E_k)_t + ((m_k E_k + \alpha_k p_k) u_k)_x + p_1 \alpha_{k,t} = S_k,\\
&\alpha_{k,t} + v_2 \alpha_{k,x} = \pm P.
\end{aligned}
$$

The mass transfer term is defined by the simplified relations

$$\begin{aligned} M &= \alpha_2 \rho_2 \frac{3\dot{r}}{r} \\ \dot{r} &= 5 \times 10^{-3} m/s \text{ (combustion velocity of the grains)} \\ r &= 10^{-3} m \text{ (radius of the grains)} \end{aligned} \tag{3}$$

The momentum source term is given by

$$\begin{aligned} Q &= Mu_2 - D \\ D &= C\alpha_1\alpha_2\rho_2(u_1 - u_2)\,|u_1 - u_2| \text{ (drag force)} \\ C &= \frac{3}{4r} \text{ (simplified shape factor)} \end{aligned} \tag{4}$$

The energy source terms are

$$\begin{aligned} S_1 &= -u_2 D + MQ_{ex} \\ S_2 &= u_2 D \\ Q_{ex} &= 37.3839 \times 10^6 \text{ J/kg (chemical combustion energy)} \end{aligned} \tag{5}$$

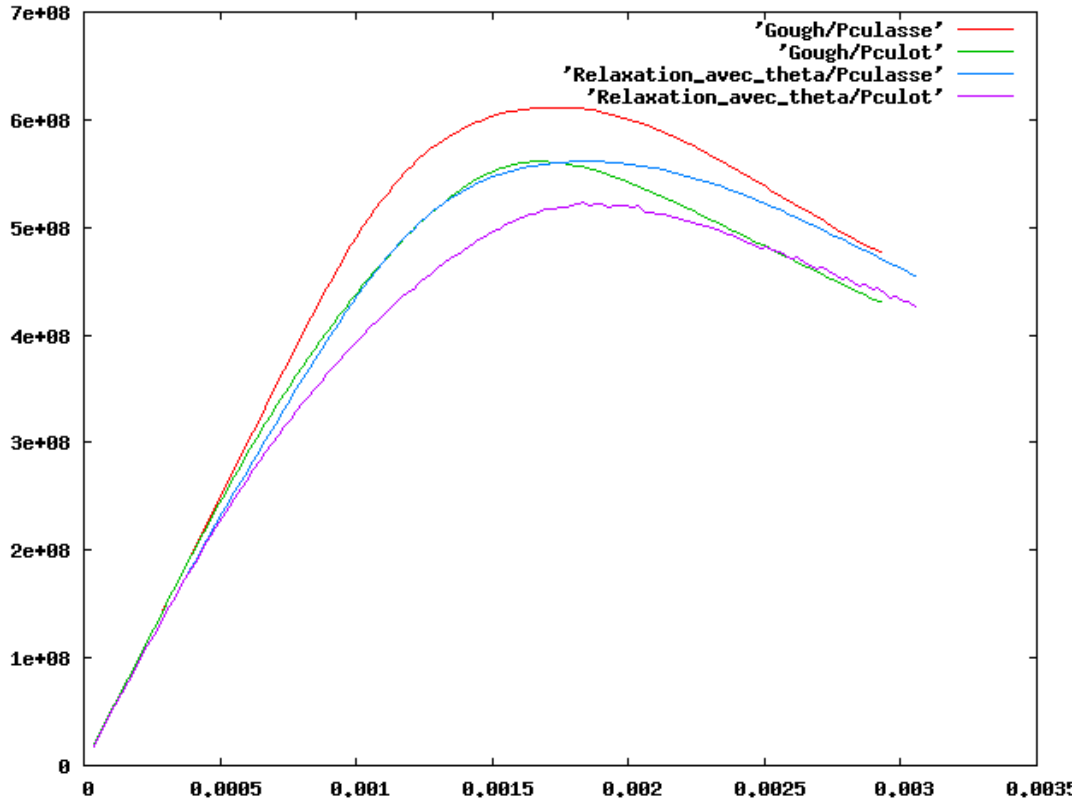


Figure: Pressure evolution at the breech and the shot base during time. Comparison between the Gough and the relaxation model.

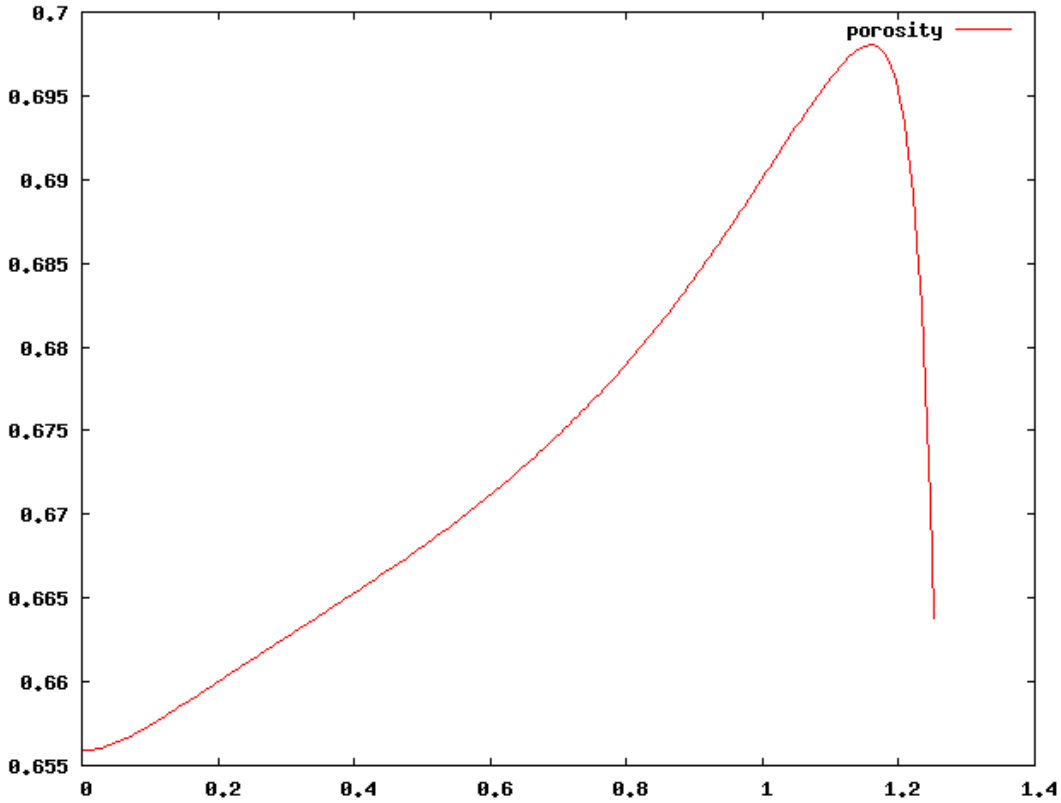


Figure: Porosity at the final time. Relaxation model with granular stress.

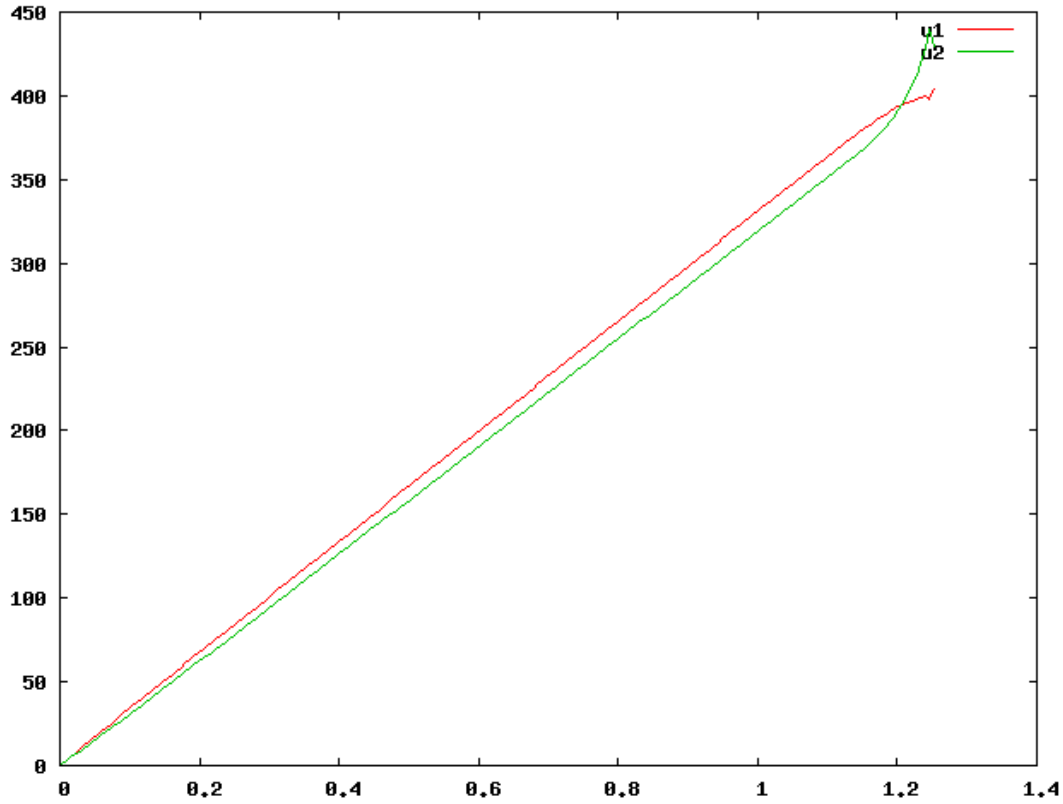


Figure: Velocities at the final time. Relaxation model with granular stress.

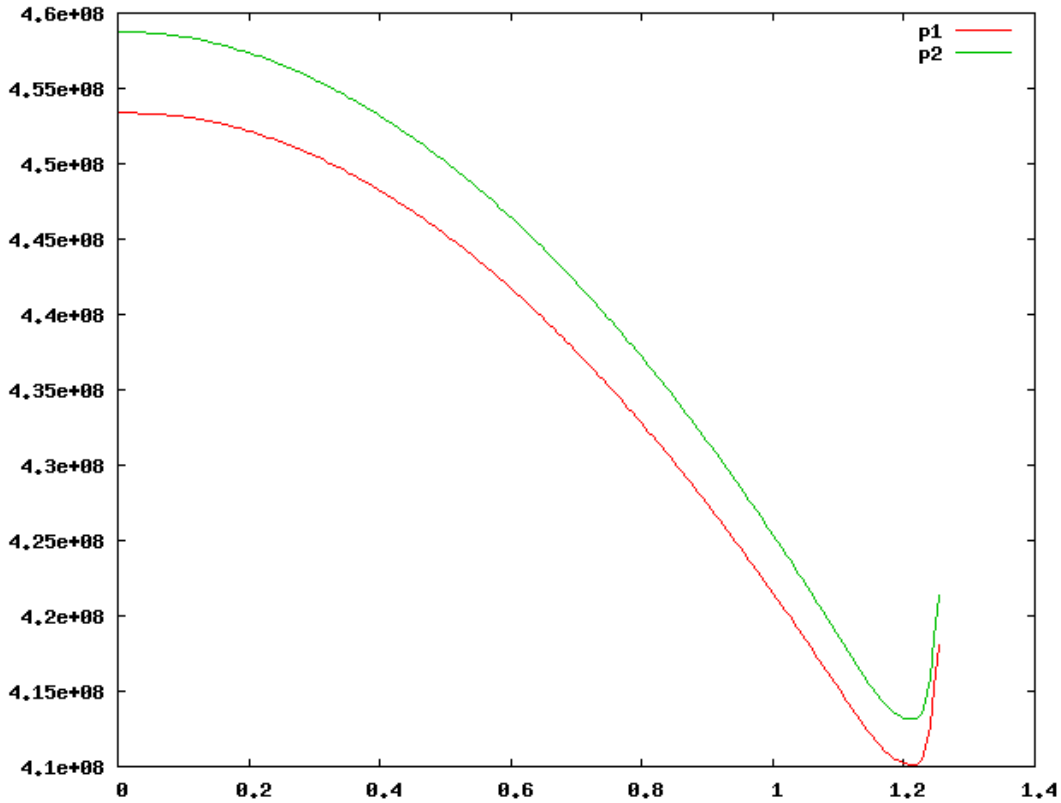


Figure: Pressures at the final time. Relaxation model with granular stress.

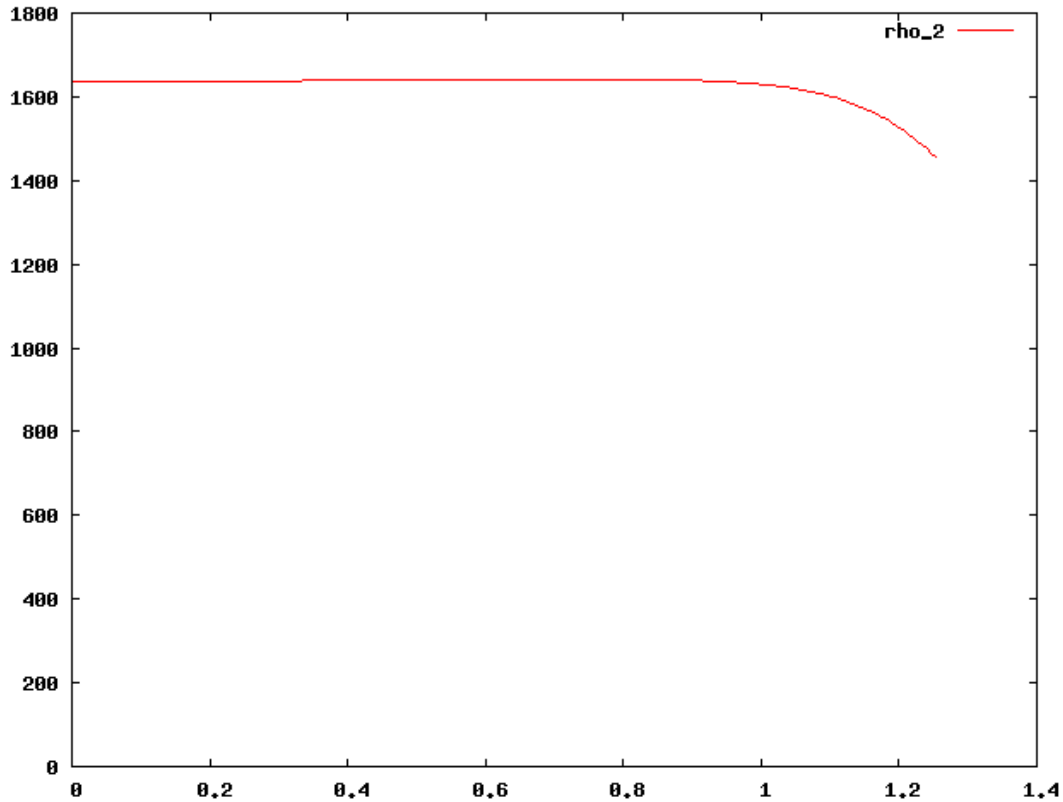


Figure: Density of the solid phase at the final time. Relaxation model with granular stress.

Conclusion

- Good generalization of the one pressure models;
- Extension of the relaxation approach to flows with granular stress;
- Rigorous entropy dissipation and maximum principle on the volume fraction;
- Stability for a finite relaxation time;
- The instability is (fortunately) preserved by the scheme for fast pressure equilibrium;
- The model can be used in practical configurations (the solid phase remains almost incompressible).

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