# Bench of anisotropic problems
# Numerical experiments with the DDFV method

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## Description of the scheme

We present the numerical results we have obtained with the finite volume method introduced in[a] for approximating diffusion operators with *variable* (continuous or discontinuous, linear or non-linear), *full* tensor coefficients on *arbitrary* meshes. In[b] this type of method has been called Discrete-Duality Finite Volume (DDFV), in order to emphasize that it satisfies a discrete integration by parts.
The main idea lies in using two different meshes, namely an arbitrary (given) *primal* mesh and a *dual* mesh that is made up from the primal mesh. Here we have chosen the *median* dual mesh whose vertices are the centers of gravity of the primal cells and the middle of the primal sides. The diffusion equation to be dealt with is integrated both on the cells of the primal mesh and on those of the dual mesh while the degrees of freedom are the values of the unknown function at the centers of gravity of the primal cells, the middle of the boundary sides and the vertices of the primal mesh.
The method provides symmetric positive definite matrices. Moreover, it provides natural, accurate, *a posteriori* approximations of both the gradient and the Hessian of the solution. The approximated gradient is calculated at the middle of the cell sides while the approximated Hessian is calculated both at the centers of gravity and vertices of the primal cells.

[a] F. Hermeline, *C.R. Acad. Sci. Paris*, Ser. 1 326, p 1433-1436, 1998, *J. Comp. Phys.*, Vol. 160, p 481-499, 2000, *Comp. Methods Appl. Mech. Engrg*, Vol. 192, p 1939-1959, 2003.
[b] S. Delcourte, K. Domelevo, P. Omnes, *Finite Volumes for Complex Applications* (4), F. Benkhaldoun, D. Ouazar and S. Raghay Eds, Hermes, p 447-458, 2005.

## Results for Test 1.1

**umin**= 0.0, **umax**=1.0.

- Triangular mesh **mesh1** **ocvl2**= 2.00, **ocvgradl2**= 2.00.

| i | nunkw | nnmat | sumflux | erl2 | ergrad | ratiol2 | ratiograd |
|---|---|---|---|---|---|---|---|
| 1 | 109 | 864 | -1.79e-10 | 2.13e-2 | 2.33e-2 | | |
| 2 | 385 | 3252 | -1.89e-11 | 5.77e-3 | 5.92e-3 | 2.07 | 2.17 |
| 3 | 1441 | 12564 | 2.60e-11 | 1.47e-3 | 1.48e-3 | 2.07 | 2.09 |
| 4 | 5569 | 49332 | 7.57e-11 | 3.68e-4 | 3.72e-4 | 2.04 | 2.05 |
| 5 | 21889 | 195444 | 9.47e-11 | 9.21e-5 | 9.32e-5 | 2.02 | 2.02 |
| 6 | 86785 | 777972 | 6.36e-11 | 2.30e-5 | 2.33e-5 | 2.01 | 2.01 |
| 7 | 345601 | 3104244 | -3.93e-11 | 5.76e-6 | 5.82e-6 | 2.01 | 2.00 |

| i | erflx0 | erflx1 | erfly0 | erfly1 | erflm | umin | umax |
|---|---|---|---|---|---|---|---|
| 1 | 1.06e-2 | 1.06e-2 | 1.06e-2 | 1.06e-2 | 1.37e-1 | 0.0 | 0.99101 |
| 2 | 2.65e-3 | 2.65e-3 | 2.65e-3 | 2.65e-3 | 3.83e-2 | 0.0 | 0.99785 |
| 3 | 6.63e-4 | 6.63e-4 | 6.63e-4 | 6.63e-4 | 9.99e-3 | 0.0 | 0.99946 |
| 4 | 1.65e-4 | 1.65e-4 | 1.65e-4 | 1.65e-4 | 2.54e-3 | 0.0 | 0.99986 |
| 5 | 4.14e-5 | 4.14e-5 | 4.14e-5 | 4.14e-5 | 6.42e-4 | 0.0 | 0.99997 |
| 6 | 1.03e-5 | 1.03e-5 | 1.03e-5 | 1.03e-5 | 1.62e-4 | 0.0 | 0.99999 |
| 7 | 2.60e-6 | 2.60e-6 | 2.60e-6 | 2.60e-6 | 4.07e-5 | 0.0 | 0.99999 |

- Distorted quadrangular mesh **mesh4_j_i** **ocvl2**= 2.00, **ocvgradl2**= 1.99.

| i | nunkw | nnmat | sumflux | erl2 | ergrad | ratiol2 | ratiograd |
|---|---|---|---|---|---|---|---|
| 1 | 681 | 5707 | -5.85e-11 | 2.04e-2 | 1.95e-2 | | |
| 2 | 2517 | 21820 | -2.73e-11 | 5.16e-3 | 5.08e-3 | 2.10 | 2.06 |
| 3 | 5509 | 48340 | 3.90e-11 | 2.29e-3 | 2.27e-3 | 2.07 | 2.05 |
| 4 | 9657 | 85264 | 1.89e-12 | 1.29e-3 | 1.28e-3 | 2.05 | 2.04 |
| 5 | 14961 | 132592 | -3.38e-12 | 8.25e-4 | 8.21e-4 | 2.04 | 2.03 |
| 6 | 21421 | 190324 | 1.31e-11 | 5.73e-4 | 5.71e-4 | 2.03 | 2.02 |

| i | erflx0 | erflx1 | erfly0 | erfly1 | erflm | umin | umax |
|---|---|---|---|---|---|---|---|
| 1 | 1.53e-3 | 1.10e-2 | 3.87e-3 | 6.90e-3 | 1.47e-1 | 0.0 | 1.026 |
| 2 | 3.01e-4 | 2.86e-3 | 9.34e-4 | 1.61e-3 | 6.28e-2 | 0.0 | 1.005 |
| 3 | 1.36e-4 | 1.27e-3 | 4.13e-4 | 7.18e-4 | 3.77e-2 | 0.0 | 1.002 |
| 4 | 7.77e-5 | 7.19e-4 | 2.32e-4 | 4.06e-4 | 2.50e-2 | 0.0 | 1.001 |
| 5 | 5.02e-5 | 4.60e-4 | 1.48e-4 | 2.61e-4 | 1.77e-2 | 0.0 | 1.001 |
| 6 | 3.50e-5 | 3.20e-4 | 1.03e-4 | 1.81e-4 | 1.32e-2 | 0.0 | 1.000 |

- **Comments**

  The method is second-order accurate in the $L_2-$norm for the solution *and* its gradient and first-order accurate in the $L_2-$norm for its Hessian.

## Results for Test 1.2

**umin**= 0.0, **umax**=$1 + \sin(1)$.

- Triangular mesh **mesh1** **ocvl2**= 2.00, **ocvgradl2**= 1.92.

| i | nunkw | nnmat | sumflux | erl2 | ergrad | ratiol2 | ratiograd |
|---|---|---|---|---|---|---|---|
| 1 | 109 | 864 | -2.20e-10 | 2.65e-3 | 7.93e-3 | | |
| 2 | 385 | 3252 | -8.63e-10 | 6.55e-4 | 2.22e-3 | 2.21 | 2.01 |
| 3 | 1441 | 12564 | -6.95e-10 | 1.65e-4 | 6.11e-4 | 2.09 | 1.95 |
| 4 | 5569 | 49332 | -8.17e-9 | 4.14e-5 | 1.66e-4 | 2.04 | 1.93 |
| 5 | 21889 | 195444 | -9.93e-9 | 1.04e-5 | 4.44e-5 | 2.02 | 1.92 |
| 6 | 86785 | 777972 | 3.09e-8 | 2.59e-6 | 1.18e-5 | 2.01 | 1.92 |
| 7 | 345601 | 3104244 | 3.76e-8 | 6.47e-7 | 3.12e-6 | 2.00 | 1.92 |

| i | erflx0 | erflx1 | erfly0 | erfly1 | erflm | umin | umax |
|---|---|---|---|---|---|---|---|
| 1 | 4.08e-3 | 4.86e-3 | 5.36e-3 | 1.60e-2 | 4.83e-2 | 0.0 | 1.84147 |
| 2 | 1.20e-3 | 1.32e-3 | 2.03e-3 | 5.13e-3 | 2.47e-2 | 0.0 | 1.84147 |
| 3 | 3.51e-4 | 3.37e-4 | 6.69e-4 | 1.57e-3 | 1.30e-2 | 0.0 | 1.84147 |
| 4 | 1.08e-4 | 8.48e-5 | 2.11e-4 | 4.74e-4 | 6.75e-3 | 0.0 | 1.84147 |
| 5 | 2.82e-5 | 2.14e-5 | 6.35e-5 | 1.37e-4 | 3.44e-3 | 0.0 | 1.84147 |
| 6 | 7.79e-6 | 5.33e-6 | 1.85e-5 | 3.87e-5 | 1.73e-3 | 0.0 | 1.84147 |
| 7 | 2.07e-6 | 1.34e-6 | 5.42e-6 | 1.09e-5 | 8.73e-4 | 0.0 | 1.84147 |

- Locally refined mesh **mesh3** **ocvl2**= 1.99, **ocvgradl2**= 1.52.

| i | nunkw | nnmat | sumflux | erl2 | ergrad | ratiol2 | ratiograd |
|---|---|---|---|---|---|---|---|
| 1 | | | | | | | |
| 1 | 121 | 929 | -1.67e-10 | 7.83e-3 | 2.57e-2 | | |
| 2 | 401 | 3304 | -8.58e-10 | 1.94e-3 | 8.96e-3 | 2.33 | 1.76 |
| 3 | 1441 | 12376 | 3.86e-10 | 4.84e-4 | 3.10e-3 | 2.16 | 1.66 |
| 4 | 5441 | 47800 | 8.43e-9 | 1.21e-4 | 1.08e-3 | 2.08 | 1.59 |
| 5 | 21121 | 187768 | 9.12e-9 | 3.04e-5 | 3.77e-4 | 2.04 | 1.55 |

| i | erflx0 | erflx1 | erfly0 | erfly1 | erflm | umin | umax |
|---|---|---|---|---|---|---|---|
| 1 | 7.58e-3 | 1.26e-3 | 9.86e-4 | 1.44e-2 | 1.27e-1 | 0.0 | 1.84147 |
| 2 | 1.23e-3 | 1.01e-4 | 9.57e-5 | 6.70e-3 | 6.90e-2 | 0.0 | 1.84147 |
| 3 | 1.30e-4 | 1.41e-5 | 4.51e-5 | 2.50e-3 | 3.59e-2 | 0.0 | 1.84147 |
| 4 | 1.29e-5 | 3.30e-6 | 2.07e-5 | 8.34e-4 | 1.84e-2 | 0.0 | 1.84147 |
| 5 | 1.48e-5 | 6.49e-7 | 7.54e-6 | 2.61e-4 | 9.37e-3 | 0.0 | 1.84147 |

- **Comments**

  The method is second-order accurate in the $L_2-$norm for the solution, second-order (1.5-order) accurate in the $L_2-$norm for its gradient and first-order (0.5-order) accurate in the $L_2-$norm for its Hessian, for *triangle* meshes (*non-conforming* rectangle meshes).

## Results for Test 2 Numerical locking

Triangular mesh **mesh1**. **umin**= −1, **umax**= 1.

- $\delta = 10^5$.

| i | nunkw | nnmat | sumflux | erl2 | ergrad | ratiol2 | ratiograd |
|---|---|---|---|---|---|---|---|
| 1 | 109 | 864 | -7.09e+1 | 7.19e-1 | 5.27e-1 | | |
| 2 | 385 | 3252 | -3.25e+0 | 5.92e-1 | 5.34e-1 | 0.31 | -0.02 |
| 3 | 1441 | 12564 | -1.53e+1 | 5.12e-1 | 3.93e-1 | 0.22 | 0.44 |
| 4 | 5569 | 49332 | 2.88e+0 | 4.36e-1 | 2.69e-1 | 0.23 | 0.54 |
| 5 | 21889 | 195444 | 1.18e+0 | 2.85e-1 | 1.07e-1 | 0.62 | 1.33 |
| 6 | 86785 | 777972 | -7.92e-1 | 8.10e-2 | 3.60e-2 | 1.81 | 1.57 |
| 7 | 345601 | 3104244 | -6.50e-7 | 1.19e-4 | 3.14e-4 | 9.44 | 6.84 |

| i | erflx0 | erflx1 | fluy0 | fluy1 | erflm | umin | umax |
|---|---|---|---|---|---|---|---|
| 1 | 5.34e-3 | 1.89e-3 | -1.07e+2 | -5.35e+1 | 6.80e-1 | -0.47 | 0.77 |
| 2 | 6.92e-4 | 1.12e-3 | 1.00e+0 | -2.14e+0 | 1.24e+0 | -0.42 | 1.05 |
| 3 | 2.16e-5 | 2.57e-5 | 3.42e+0 | -1.58e+1 | 8.31e-1 | -0.64 | 0.89 |
| 4 | 2.84e-5 | 1.25e-4 | 5.02e+0 | 7.68e-1 | 6.34e-1 | -0.75 | 9.12 |
| 5 | 9.03e-7 | 1.29e-6 | -1.85e-1 | -1.11e+0 | 2.50e-1 | -0.74 | 1.12 |
| 6 | 7.49e-7 | 2.65e-7 | 6.47e-3 | 9.74e-1 | 1.72e-1 | -0.93 | 1.03 |
| 7 | 2.50e-10 | 2.52e-10 | 1.16e-7 | -1.59e-7 | 7.16e-3 | -1.00 | 1.00 |

- $\delta = 10^6$.

| i | nunkw | nnmat | sumflux | erl2 | ergrad | ratiol2 | ratiograd |
|---|---|---|---|---|---|---|---|
| 1 | 109 | 864 | 4.67e+1 | 9.24e-1 | 9.20e-1 | | |
| 2 | 385 | 3252 | -1.17e+1 | 7.99e-1 | 9.41e-1 | 0.23 | -0.02 |
| 3 | 1441 | 12564 | -3.32e+1 | 7.06e-1 | 7.13e-1 | 0.18 | 0.39 |
| 4 | 5569 | 49332 | -6.21e-1 | 5.52e-1 | 4.61e-1 | 0.36 | 0.62 |
| 5 | 21889 | 195444 | 1.40e+0 | 4.56e-1 | 3.07e-1 | 0.28 | 0.59 |
| 6 | 86785 | 777972 | 3.31e-1 | 3.79e-1 | 2.18e-1 | 0.27 | 0.49 |
| 7 | 345601 | 3104244 | 7.72e-1 | 2.64e-1 | 7.60e-2 | 0.52 | 1.52 |

| i | erflx0 | erflx1 | fluy0 | fluy1 | erflm | umin | umax |
|---|---|---|---|---|---|---|---|
| 1 | 9.74e-3 | 4.19e-3 | 4.23e+1 | 1.18e+2 | 1.20e+0 | -0.33 | 0.71 |
| 2 | 9.54e-4 | 1.06e-3 | -3.58e-1 | -1.06e+2 | 1.47e+0 | -0.21 | 0.91 |
| 3 | 1.09e-4 | 1.86e-4 | 9.75e-1 | -2.27e+1 | 1.77e+0 | -0.35 | 0.67 |
| 4 | 6.59e-9 | 4.61e-4 | 5.72e+1 | 2.19e+0 | 8.18e-1 | -0.60 | 0.87 |
| 5 | 4.87e-6 | 7.01e-6 | -1.36e+0 | -6.76e+0 | 8.13e-1 | -0.74 | 0.86 |
| 6 | 2.63e-6 | 3.98e-6 | 1.15e+0 | -1.25e+0 | 5.36e-1 | -0.70 | 0.97 |
| 7 | 7.33e-8 | 1.72e-7 | -8.00e-2 | 7.83e-1 | 1.64e-1 | -0.78 | 1.11 |

- **Comments**

  The conjugate gradient method does not converge (no attempt has been made to optimize the preconditioner).

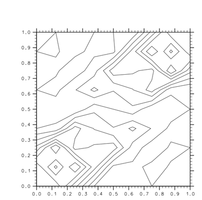
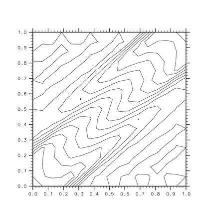
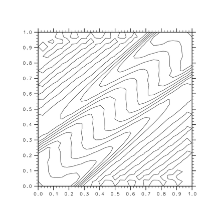
## Results for Test 3 : Oblique flow

- Uniform rectangular mesh **mesh2**. **umin**= 0.0, **umax**= 1.0.

| i | nunkw | nnmat | sumflux | umin | umax |
|---|---|---|---|---|---|
| 1 | 57 | 403 | 4.99e-16 | -4.72e-3 | 1.00472 |
| 2 | 177 | 1387 | 2.38e-10 | -5.63e-2 | 1.05634 |
| 3 | 609 | 5083 | 2.41e-10 | -1.52e-2 | 1.01528 |
| 4 | 2241 | 19387 | 1.95e-10 | -1.82e-2 | 1.01824 |
| 5 | 8577 | 75643 | 2.76e-10 | -1.07e-3 | 1.00106 |
| ref | 52161 | 465595 | 8.74e-10 | -5.96e-8 | 1.00000 |

| i | flux0 | flux1 | fluy0 | fluy1 |
|---|---|---|---|---|
| 1 | -1.82e-1 | 1.82e-1 | -1.21e-1 | 1.21e-1 |
| 2 | -1.91e-1 | 1.91e-1 | -1.03e-1 | 1.03e-1 |
| 3 | -1.94e-1 | 1.94e-1 | -9.83e-2 | 9.83e-2 |
| 4 | -1.94e-1 | 1.94e-1 | -9.79e-2 | 9.79e-2 |
| 5 | -1.93e-1 | 1.93e-1 | -9.82e-2 | 9.82e-2 |
| ref | -1.93e-1 | 1.93e-1 | -9.85e-2 | 9.85e-2 |

- Solution on mesh2_i for i=2 (left), i=3 (center), i=4 (right)

- **Comments**

  The method provides satisfactory results but it does not satisfy the discrete maximum principle.

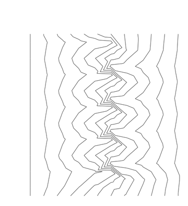
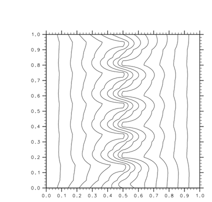
## Results for Test 4 : Vertical fault

- Non conforming rectangular mesh **mesh5**. **umin**= 0.0, **umax**= 1.0.

| i | nunkw | nnmat | sumflux | umin | umax |
|---|---|---|---|---|---|
| 1 | 282 | 2278 | 7.60e-9 | 0.0 | 1.0 |
| ref | 52161 | 465595 | 3.15e-8 | 0.0 | 1.0 |

| i | flux0 | flux1 | fluy0 | fluy1 |
|---|---|---|---|---|
| 1 | -4.00e+1 | 4.18e+1 | -1.81e+0 | 9.08e-4 |
| ref | -4.20e+1 | 4.43e+1 | -2.35e+0 | 7.97e-4 |

- Solution for the vertical fault on the meshes: (Left)**mesh5** (Right) **mesh5_ref**.

- **Comments**

  In this case the method satisfies the discrete maximum principle.

## Results for Test 5 : Heterogeneous rotating anisotropy

- Rectangular mesh **mesh5**. **umin**= 0.0, **umax**= 1.0.
  **ocvl2**= 2.00, **ocvgradl2**= 1.95.

| i | nunkw | nnmat | sumflux | erl2 | ergrad | ratiol2 | ratiograd |
|---|---|---|---|---|---|---|---|
| 1 | 57 | 403 | 3.21e-10 | 4.83e-2 | 5.86e-2 | | |
| 2 | 177 | 1387 | 3.65e-11 | 1.11e-2 | 1.83e-2 | 2.59 | 2.05 |
| 3 | 609 | 5083 | 2.02e-12 | 2.68e-3 | 5.14e-3 | 2.30 | 2.05 |
| 4 | 2241 | 19387 | 7.62e-12 | 6.62e-4 | 1.39e-3 | 2.14 | 2.00 |
| 5 | 8577 | 75643 | -8.70e-13 | 1.65e-4 | 3.68e-4 | 2.07 | 1.98 |
| 6 | 33537 | 298747 | 1.56e-13 | 4.12e-5 | 9.64e-5 | 2.03 | 1.97 |
| 7 | 132609 | 1187323 | 3.37e-14 | 1.03e-5 | 2.50e-5 | 2.02 | 1.96 |
| 8 | 527361 | 4733947 | -3.62e-13 | 2.58e-6 | 6.47e-6 | 2.00 | 1.96 |
| 9 | 2103297 | 18905083 | -3.74e-13 | 6.44e-7 | 1.67e-6 | 2.00 | 1.96 |

| i | erflx0 | erflx1 | erfly0 | erfly1 | erflm | umin | umax |
|---|---|---|---|---|---|---|---|
| 1 | 1.25e-1 | 4.48e-2 | 1.25e-1 | 4.48e-2 | 4.73e-1 | 0.0 | 1.01089 |
| 2 | 4.07e-2 | 1.04e-2 | 4.07e-2 | 1.04e-2 | 2.37e-1 | 0.0 | 1.00473 |
| 3 | 1.24e-2 | 2.56e-3 | 1.24e-2 | 2.56e-3 | 1.19e-1 | 0.0 | 1.00158 |
| 4 | 3.67e-3 | 6.35e-4 | 3.67e-3 | 6.35e-4 | 5.95e-2 | 0.0 | 1.00045 |
| 5 | 1.06e-3 | 1.58e-4 | 1.06e-3 | 1.58e-4 | 2.97e-2 | 0.0 | 1.00012 |
| 6 | 2.98e-4 | 3.90e-5 | 2.98e-4 | 3.95e-5 | 1.49e-2 | 0.0 | 1.00003 |
| 7 | 8.33e-5 | 9.40e-6 | 8.33e-5 | 9.88e-6 | 7.43e-3 | 0.0 | 1.00000 |
| 8 | 2.30e-5 | 2.02e-6 | 2.30e-5 | 2.50e-6 | 3.71e-3 | 0.0 | 1.00000 |
| 9 | 6.30e-6 | 1.75e-7 | 6.30e-6 | 6.55e-7 | 1.36e-3 | 0.0 | 1.00000 |

- **Comments**

  The method is second-order accurate in the $L_2-$norm for the solution *and* its gradient and first-order accurate in the $L_2-$norm for its Hessian. Similar results have been obtained for the anisotropic ratio: $\epsilon = 10^{-12}$.

## Results for Test 6 and Test 7

- **Test 6** Oblique drain, min $= -1.2$, max $= 0$, coarse (C) and fine (F) oblique meshes, **mesh6** and **mesh7**

| nunkw | nnmat | sumflux | erl2 | ergrad |
|---|---|---|---|---|
| 514 | 4237 | 9.32e-14 | 7.31e-16 | 1.22e-14 |
| 574 | 4777 | 3.97e-13 | 1.12e-15 | 1.54e-14 |

| erflx0 | erflx1 | erfly0 | erfly1 | erflm | umin | umax |
|---|---|---|---|---|---|---|
| 2.98e-16 | 1.53e-14 | 4.02e-15 | 5.82e-15 | 6.88e-14 | -1.20 | 0 |
| 6.86e-15 | 6.02e-14 | 4.86e-15 | 1.41e-14 | 7.35e-14 | -1.20 | 0 |

- **Test 7** Oblique barrier, min $= -5.575$, max $= 0.575$, coarse oblique mesh **mesh6**

| nunkw | nnmat | sumflux | erl2 | ergrad |
|---|---|---|---|---|
| 514 | 4237 | 2.89e-14 | 8.00e-16 | |
| 574 | 4777 | 5.05e-14 | 1.46e-15 | |

| erflx0 | erflx1 | erfly0 | erfly1 | erflm | umin | umax |
|---|---|---|---|---|---|---|
| 6.28e-14 | 6.24e-14 | 2.55e-15 | 4.13e-14 | 1.03e-14 | -5.575 | 0.575 |
| 7.77e-15 | 1.37e-13 | 3.55e-15 | 7.59e-14 | 3.87e-14 | -5.575 | 0.575 |

- **Comments**

  The method is exact.

## Results for Test 8 and Test 9

- **Test 8** Perturbed parallelograms mesh **mesh8**, **umin**= 0.0, **umax**= 1.0.

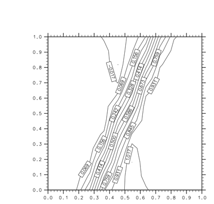
| nunkw | nnmat | sumflux | umin | umax |
|---|---|---|---|---|
| 309 | 2503 | -1.32e-10 | -1.61e-3 | 8.99e-2 |

| flux0 | flux1 | fluy0 | fluy1 |
|---|---|---|---|
| 2.45e-10 | -1.83e-10 | 4.80e-1 | 5.11e-1 |

- **Test 9** Anisotropy with wells. Square uniform grid **mesh9**. **umin**= 0.0, **umax**= 1.0,

| i | nunkw | nnmat | sumflux | umin | umax |
|---|---|---|---|---|---|
| 1 | 309 | 2503 | 2.38e-5 | -1.03e-1 | 1.100e+0 |
| ref | 52161 | 465595 | -1.82e-11 | -1.47e-3 | 1.001e+0 |

- Solution of Test 9



- **Comments**

  The method does not satisfy the discrete maximum principle.