The m-DDFV Method for Heterogeneous Linear and Nonlinear Elliptic Problems

UNIVERSITÉ DE PROVENCE Aix-Marseille I
UNIVERSITÉ PAUL CÉZANNE Aix-Marseille III

Franck Boyer, Florence Hubert

Laboratoire d'Analyse Topologie Probabilités, UMR 6632, Marseille, FRANCE [fboyer,fhubert]@cmi.univ-mrs.fr

We propose a DDFV (Discrete Duality Finite Volume) scheme for the nonlinear elliptic equation

$$\begin{cases} -\mathrm{div}\,(\varphi(z, \nabla ue(z))) = f(z), & \text{in } \Omega, \\ ue = 0, \ \text{sur } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a polygonal open set of $\mathbb{R}^2$ and the operator $u \mapsto -\mathrm{div}(\varphi(\cdot, \nabla u))$ is monotonic and coercive (of Leray-Lions type). We are particularly interested in the case where φ is discontinuous with respect to the variable z (transmission problems).

The scheme is constructed in a way to ensure the consistency of the numerical fluxes where φ is discontinuous.

We obtain error estimates for sufficiently piecewise smooth solutions of the same order than in the case of regular fluxes. Numerical results confirm the gain obtained with these new scheme.

1. Assumptions

- Let $p \in]1, \infty[$, $p' = \frac{p}{p-1}$ and $f \in L^{p'}(\Omega)$. Assume here $p \geq 2$ to simplify.
- $\varphi : \Omega \times \mathbb{R}^2 \to \mathbb{R}^2$ is a Caratheodory function such that $\xi, \eta \in \mathbb{R}^2$:

$$(\varphi(z,\xi), \xi) \geq C_\varphi (|\xi|^p - 1), \tag{$\mathcal{H}_1$}$$

$$|\varphi(z,\xi)| \leq C_\varphi \left(|\xi|^{p-1} + 1\right). \tag{$\mathcal{H}_2$}$$

$$(\varphi(z,\xi) - \varphi(z,\eta), \xi - \eta) \geq \frac{1}{C_\varphi}|\xi - \eta|^p. \tag{$\mathcal{H}_3$}$$

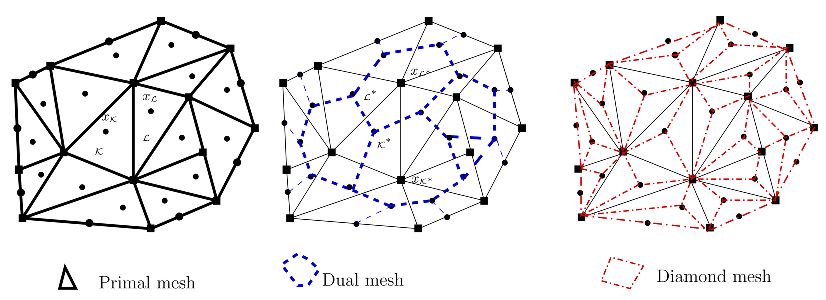
$$|\varphi(z,\xi) - \varphi(z,\eta)| \leq C_\varphi \left(1 + |\xi|^{p-2} + |\eta|^{p-2}\right)|\xi - \eta|. \tag{$\mathcal{H}_4$}$$

• Remark : More general cases can be studied. For instance if φ is *nonlinear* on Ω_1 and *linear* on $\Omega_2 = \Omega \setminus \Omega_1$ (See [3]).

2. Back to the DDFV schemes

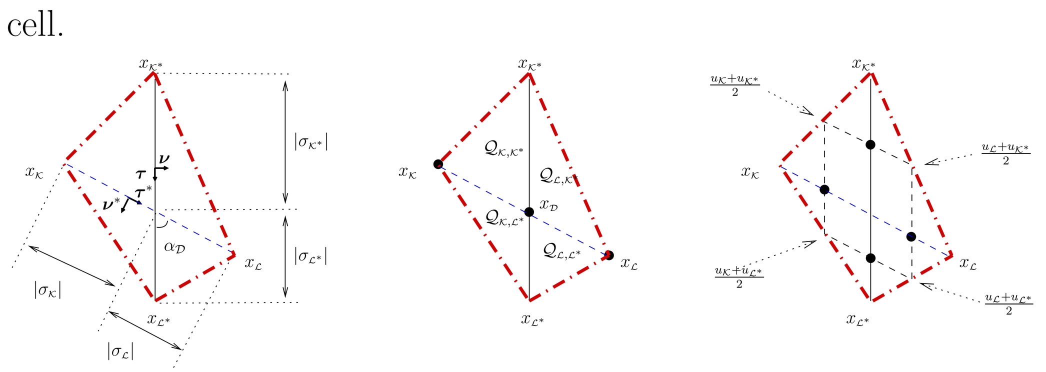
See [4], for the Laplace operator.
See [1] and [2] for the nonlinear case.

- **The DDFV meshes** primal, dual and "diamond".



- **Zoom on the diamond cells**

Diamond cells are supposed to be convex. Let us note $\mathcal{Q}$ the quarters of this cell.



- **The discrete unkhowns**

$$u^{\mathcal{T}} = \left(u^{\mathfrak{M}}, u^{\mathfrak{M}^*}\right) \text{ where } u^{\mathfrak{M}} = (u_{\mathcal{K}})_{\mathcal{K} \in \mathfrak{M}} \ u^{\mathfrak{M}^*} = (u_{\mathcal{K}^*})_{\mathcal{K}^* \in \mathfrak{M}^*}$$

- **The discrete gradient** : $\nabla^{\mathcal{T}} u^{\mathcal{T}}$ constant on each diamond cell $\mathcal{D}$

$$\nabla^{\mathcal{T}}_{\mathcal{D}} u^{\mathcal{T}} = \frac{1}{\sin \alpha_{\mathcal{D}}} \left(\frac{u_{\mathcal{L}} - u_{\mathcal{K}}}{|\sigma^*|}\boldsymbol{\nu} + \frac{u_{\mathcal{L}^*} - u_{\mathcal{K}^*}}{|\sigma|}\boldsymbol{\nu}^* \right), \quad \forall \mathcal{D}. \tag{2}$$

- **The DDFV scheme**

$$\boxed{\begin{aligned} -\sum_{\mathcal{D}_{\sigma,\sigma^*} \cap \mathcal{K} \neq \emptyset} |\sigma| \left(\varphi_{\mathcal{D}}(\nabla^{\mathcal{T}}_{\mathcal{D}} u^{\mathcal{T}}), \boldsymbol{\nu}\right) &= \int_{\mathcal{K}} f(z)\, dz, \quad \forall \mathcal{K} \in \mathfrak{M} \\ -\sum_{\mathcal{D}_{\sigma,\sigma^*} \cap \mathcal{K}^* \neq \emptyset} |\sigma^*| \left(\varphi_{\mathcal{D}}(\nabla^{\mathcal{T}}_{\mathcal{D}} u^{\mathcal{T}}), \boldsymbol{\nu}^*\right) &= \int_{\mathcal{K}^*} f(z)\, dz, \ \forall \mathcal{K}^* \in \mathfrak{M}^*, \end{aligned}} \tag{3}$$

with

$$\varphi_{\mathcal{D}}(\xi) = \frac{1}{|\mathcal{D}|}\int_{\mathcal{D}} \varphi(z,\xi)\, dz.$$

- **Error estimates obtained in [1] :**

Assume that φ is Lipschitz continuous with respect to z with

$$\left|\frac{\partial \varphi}{\partial z}(z,\xi)\right| \leq C_\varphi \left(1 + |\xi|^{p-1}\right), \quad \forall \xi \in \mathbb{R}^2. \tag{$\mathcal{H}_5$}$$

If $ue \in W^{2,p}(\Omega)$ we have

$$\|ue - u^{\mathfrak{M}}\|_{L^p} + \|ue - u^{\mathfrak{M}^*}\|_{L^p} + \|\nabla ue - \nabla^{\mathcal{T}} u^{\mathcal{T}}\|_{L^p} \leq C\, \mathrm{size}(\mathcal{T})^{\frac{1}{p-1}}.$$

3. Discontinuities in the flux φ

In this case $(\mathcal{H}_5)$ is not satisfied since $ue \notin W^{2,p}(\Omega)$, as the gradient of ue is discontinuous. The DDFV still converges (see [1]) but the fluxes are no more consistent along the interface of discontinuities.

> Assume that on each $\mathcal{Q}$, φ is Lip. and satisfies $(\mathcal{H}_5)$.
> AIM : obtain the consistency along the discontinuity!

We construct a new approximation $\varphi^{\mathcal{N}}_{\mathcal{D}}$ on each diamond cell to obtain the consistency of the discrete fluxes (see Section 5)

$$\boxed{\begin{aligned} -\sum_{\mathcal{D}_{\sigma,\sigma^*} \cap \mathcal{K} \neq \emptyset} |\sigma| \left(\varphi^{\mathcal{N}}_{\mathcal{D}}(\nabla^{\mathcal{T}}_{\mathcal{D}} u^{\mathcal{T}}), \boldsymbol{\nu}\right) &= \int_{\mathcal{K}} f(z)\, dz, \quad \forall \mathcal{K} \in \mathfrak{M} \\ -\sum_{\mathcal{D}_{\sigma,\sigma^*} \cap \mathcal{K}^* \neq \emptyset} |\sigma^*| \left(\varphi^{\mathcal{N}}_{\mathcal{D}}(\nabla^{\mathcal{T}}_{\mathcal{D}} u^{\mathcal{T}}), \boldsymbol{\nu}^*\right) &= \int_{\mathcal{K}^*} f(z)\, dz, \ \forall \mathcal{K}^* \in \mathfrak{M}^* \end{aligned}} \tag{5}$$

4. The method in 1D

Let us consider the problem

$$\Omega =]-1, 1[, \ \varphi(x,\cdot) = \varphi_-(\cdot), \text{if } x < 0, \ \varphi(x,\cdot) = \varphi_+(\cdot), \text{if } x > 0.$$

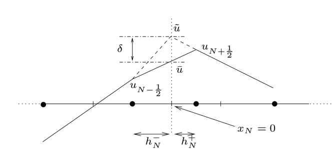
Let $x_0 = -1 < \ldots < x_N = 0 < \ldots < x_{N+M} = 1$ be a discretization of $[-1, 1]$.
The FV scheme writes in 1D:

$$-F_{i+1} + F_i = \int_{x_i}^{x_{i+1}} f(x)\, dx, \forall i \in \{0, N+M-1\}. \tag{6}$$

with

$$F_i = \varphi(x_i, \nabla_i u^{\mathcal{T}}), \quad \nabla_i u^{\mathcal{T}} = \frac{u_{i+\frac{1}{2}} - u_{i-\frac{1}{2}}}{x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}}, \quad \forall i \neq N, \tag{7}$$

> QUESTION : How to define F_N ?



To ensure the consistency and the conservativity of the discrete flux at $x_N = 0$, we look for $\tilde{u}$ such that

$$\nabla^+_N u^{\mathcal{T}} = \frac{u_{N+\frac{1}{2}} - \tilde{u}}{h^+_N}, \quad \nabla^-_N u^{\mathcal{T}} = \frac{\tilde{u} - u_{N-\frac{1}{2}}}{h^-_N},$$

and impose that

$$\varphi_-(\nabla^-_N u^{\mathcal{T}}) = \varphi_+(\nabla^-_N u^{\mathcal{T}}).$$

In fact we prefer to look for $\tilde{u}$ under the form

$$\tilde{u} = \bar{u} + \delta, \quad \text{with } \bar{u} = \frac{h^-_N u_{N+\frac{1}{2}} + h^+_N u_{N-\frac{1}{2}}}{h^-_N + h^+_N}.$$

that is

$$\nabla^+_N u^{\mathcal{T}} = \nabla_N u^{\mathcal{T}} - \frac{\delta}{h^+_N}, \quad \text{and } \nabla^-_N u^{\mathcal{T}} = \nabla_N u^{\mathcal{T}} + \frac{\delta}{h^-_N}. \tag{8}$$

> **Proposition.**
> - For any $u^{\mathcal{T}} \in \mathbb{R}^N$, there exists a unique $\delta_N(\nabla_N u^{\mathcal{T}})$ such that
>
> $$F_N \stackrel{\text{def}}{=} \varphi_- \left(\nabla_N u^{\mathcal{T}} + \frac{\delta_N(\nabla_N u^{\mathcal{T}})}{h^-_N}\right) = \varphi_+ \left(\nabla_N u^{\mathcal{T}} - \frac{\delta_N(\nabla_N u^{\mathcal{T}})}{h^+_N}\right), \tag{10}$$
>
> - The scheme (6), (7), (10) admits a unique solution.

• Example :
Let us consider two fluxes of p-laplacian type

$$\varphi_-(\xi) = k_-|\xi + G_-|^{p-2}(\xi + G_-), \quad \text{and} \quad \varphi_+(\xi) = k_+|\xi + G_+|^{p-2}(\xi + G_+),$$

where $k_-, k_+ \in \mathbb{R}^+$ and $G_-, G_+ \in \mathbb{R}^2$. We obtain

$$\boxed{F_N = \left(\frac{k_-^{\frac{1}{p-1}} k_+^{\frac{1}{p-1}}(h^-_N + h^+_N)}{h^+_N k_-^{\frac{1}{p-1}} + h^-_N k_+^{\frac{1}{p-1}}}\right)^{p-1} \left|\nabla_N u^{\mathcal{T}} + \overline{G}\right|^{p-2} \left(\nabla_N u^{\mathcal{T}} + \overline{G}\right),}$$

where $\overline{G}$ is the weighted arytmmetic mean value of G_- and G_+ defined by

$$\overline{G} = \frac{h^-_N G_- + h^+_N G_+}{h^-_N + h^+_N}.$$

5. The method in 2D

We define a new gradient on each quarter of diamond cell

$$\nabla^{\mathcal{N}}_{\mathcal{D}} u^{\mathcal{T}} = \sum_{\mathcal{Q} \in \mathfrak{Q}_{\mathcal{D}}} 1_{\mathcal{Q}} \nabla^{\mathcal{N}}_{\mathcal{Q}} u^{\mathcal{T}}, \ \text{avec } \nabla^{\mathcal{N}}_{\mathcal{Q}} u^{\mathcal{T}} = \nabla^{\mathcal{T}}_{\mathcal{D}} u^{\mathcal{T}} + B_{\mathcal{Q}} \delta^{\mathcal{D}}, \ \delta^{\mathcal{D}} \in \mathbb{R}^4,$$

where $\delta^{\mathcal{D}}$ is a set of 4 artificial unknowns

$$B_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}} = \frac{1}{|\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}|}\left(|\sigma_{\mathcal{K}}|\boldsymbol{\nu}^*, 0, |\sigma_{\mathcal{K}^*}|\boldsymbol{\nu}, 0\right), \ B_{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}} = \frac{1}{|\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}|}\left(-|\sigma_{\mathcal{K}}|\boldsymbol{\nu}^*, 0, 0, |\sigma_{\mathcal{L}^*}|\boldsymbol{\nu}\right),$$

$$B_{\mathcal{Q}_{\mathcal{L},\mathcal{L}^*}} = \frac{1}{|\mathcal{Q}_{\mathcal{L},\mathcal{L}^*}|}\left(0, -|\sigma_{\mathcal{L}}|\boldsymbol{\nu}^*, 0, -|\sigma_{\mathcal{L}^*}|\boldsymbol{\nu}\right), \ B_{\mathcal{Q}_{\mathcal{L},\mathcal{K}^*}} = \frac{1}{|\mathcal{Q}_{\mathcal{L},\mathcal{K}^*}|}\left(0, |\sigma_{\mathcal{L}}|\boldsymbol{\nu}^*, -|\sigma_{\mathcal{K}^*}|\boldsymbol{\nu}, 0\right).$$

We impose the conservativity of the fluxes to determine $\delta^{\mathcal{D}} \in \mathbb{R}^4$.

$$\begin{aligned} \left(\varphi_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}(\nabla^{\mathcal{T}}_{\mathcal{D}} u^{\mathcal{T}} + B_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}\delta^{\mathcal{D}}), \boldsymbol{\nu}^*\right) &= \left(\varphi_{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}}(\nabla^{\mathcal{T}}_{\mathcal{D}} u^{\mathcal{T}} + B_{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}}\delta^{\mathcal{D}}), \boldsymbol{\nu}^*\right), \\ \left(\varphi_{\mathcal{Q}_{\mathcal{L},\mathcal{K}^*}}(\nabla^{\mathcal{T}}_{\mathcal{D}} u^{\mathcal{T}} + B_{\mathcal{Q}_{\mathcal{L},\mathcal{K}^*}}\delta^{\mathcal{D}}), \boldsymbol{\nu}^*\right) &= \left(\varphi_{\mathcal{Q}_{\mathcal{L},\mathcal{L}^*}}(\nabla^{\mathcal{T}}_{\mathcal{D}} u^{\mathcal{T}} + B_{\mathcal{Q}_{\mathcal{L},\mathcal{L}^*}}\delta^{\mathcal{D}}), \boldsymbol{\nu}^*\right), \\ \left(\varphi_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}(\nabla^{\mathcal{T}}_{\mathcal{D}} u^{\mathcal{T}} + B_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}\delta^{\mathcal{D}}), \boldsymbol{\nu}\right) &= \left(\varphi_{\mathcal{Q}_{\mathcal{L},\mathcal{K}^*}}(\nabla^{\mathcal{T}}_{\mathcal{D}} u^{\mathcal{T}} + B_{\mathcal{Q}_{\mathcal{L},\mathcal{K}^*}}\delta^{\mathcal{D}}), \boldsymbol{\nu}\right), \\ \left(\varphi_{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}}(\nabla^{\mathcal{T}}_{\mathcal{D}} u^{\mathcal{T}} + B_{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}}\delta^{\mathcal{D}}), \boldsymbol{\nu}\right) &= \left(\varphi_{\mathcal{Q}_{\mathcal{L},\mathcal{L}^*}}(\nabla^{\mathcal{T}}_{\mathcal{D}} u^{\mathcal{T}} + B_{\mathcal{Q}_{\mathcal{L},\mathcal{L}^*}}\delta^{\mathcal{D}}), \boldsymbol{\nu}\right). \end{aligned} \tag{11}$$

> **Proposition.** For all $u^{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$ and all diamond cell $\mathcal{D}$, there exists a unique $\delta^{\mathcal{D}}(\nabla^{\mathcal{T}}_{\mathcal{D}} u^{\mathcal{T}}) \in \mathbb{R}^4$ such that (11) is fullfilled or equivalently
>
> $$\sum_{\mathcal{Q} \in \mathfrak{Q}_{\mathcal{D}}} |\mathcal{Q}| B^t_{\mathcal{Q}} \varphi_{\mathcal{Q}}(\nabla^{\mathcal{T}}_{\mathcal{D}} u^{\mathcal{T}} + B_{\mathcal{Q}} \delta^{\mathcal{D}}(\nabla^{\mathcal{T}}_{\mathcal{D}} u^{\mathcal{T}})) = 0.$$

Let $\varphi_{\mathcal{Q}}$ be the mean value of φ on $\mathcal{Q}$, the numerical flux is then given by:

$$\varphi^{\mathcal{N}}_{\mathcal{D}}(\nabla^{\mathcal{T}}_{\mathcal{D}} u^{\mathcal{T}}) = \frac{1}{|\mathcal{D}|}\sum_{\mathcal{Q} \in \mathfrak{Q}_{\mathcal{D}}} |\mathcal{Q}| \varphi_{\mathcal{Q}}(\nabla^{\mathcal{T}}_{\mathcal{D}} u^{\mathcal{T}} + B_{\mathcal{Q}} \delta^{\mathcal{D}}(\nabla^{\mathcal{T}}_{\mathcal{D}} u^{\mathcal{T}})), \tag{12}$$

• Remark : If φ is **linear** and **constant on the control volumes**, we get the scheme proposed by [5] for which the calculations are made **explicitly**.

> **Theorem.** Assume that φ is discontinuous along some curves in Ω and that φ satisfies $(\mathcal{H}_5)$ on each quarter of diamond. The scheme defined by (5), (12) admits a **unique** solution $u^{\mathcal{T}}$. Moreover if $ue_{|\mathcal{Q}} \in W^{2,p}(\mathcal{Q}), \forall \mathcal{Q}$, we have
>
> $$\|ue - u^{\mathfrak{M}}\|_{L^p} + \|ue - u^{\mathfrak{M}^*}\|_{L^p} + \|\nabla ue - \nabla^{\mathcal{N}} u^{\mathcal{T}}\|_{L^p} \leq C\, \mathrm{size}(\mathcal{T})^{\frac{1}{(p-1)}}.$$

• Key point : Obtain the consistency of the new gradient $\nabla^{\mathcal{N}}$!

6. Numerical results

The scheme (5) is solved by the following iterative algorithm ($r > 0$ is given):

- *Step 1 :* Find $(u^{\mathcal{T},n}, \delta^n_{\mathcal{D}})$ solution of

$$r \sum_{\mathcal{Q} \in \mathfrak{Q}} |\mathcal{Q}|(\nabla^{\mathcal{T}}_{\mathcal{D}} u^{\mathcal{T},n} + B_{\mathcal{Q}}\delta^n_{\mathcal{D}} - g^{n-1}_{\mathcal{Q}}, \nabla^{\mathcal{T}}_{\mathcal{D}} v^{\mathcal{T}})$$
$$= \sum_{\mathcal{K}} |\mathcal{K}| f_{\mathcal{K}} v_{\mathcal{K}} + \sum_{\mathcal{K}^*} |\mathcal{K}^*| f_{\mathcal{K}^*} v_{\mathcal{K}^*} + \sum_{\mathcal{Q} \in \mathfrak{Q}} |\mathcal{Q}|(\lambda^{n-1}_{\mathcal{Q}}, \nabla^{\mathcal{T}}_{\mathcal{D}} v), \quad \forall v^{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}.$$
$$r \sum_{\mathcal{Q} \in \mathfrak{Q}_{\mathcal{D}}} |\mathcal{Q}|^t B_{\mathcal{Q}}(B_{\mathcal{Q}}\delta^n_{\mathcal{D}} + \nabla^{\mathcal{T}}_{\mathcal{D}} u^{\mathcal{T},n} - g^{n-1}_{\mathcal{Q}}) - \sum_{\mathcal{Q} \in \mathfrak{Q}_{\mathcal{D}}} |\mathcal{Q}|^t B_{\mathcal{Q}} \lambda^{n-1}_{\mathcal{Q}} = 0, \quad \forall \mathcal{D} \in \mathfrak{D}.$$

- *Step 2 :* On each $\mathcal{Q}$, find $g^n_{\mathcal{Q}}$ solution of

$$\varphi_{\mathcal{Q}}(g^n_{\mathcal{Q}}) + \lambda^{n-1}_{\mathcal{Q}} + r(g^n_{\mathcal{Q}} - \nabla^{\mathcal{T}}_{\mathcal{D}} u^{\mathcal{T},n} - B_{\mathcal{Q}}\delta^n_{\mathcal{D}}) = 0.$$

- *Step 3 :* On each $\mathcal{Q}$ calculate $\lambda^n_{\mathcal{Q}}$ defined by

$$\lambda^n_{\mathcal{Q}} = \lambda^{n-1}_{\mathcal{Q}} + r(g^n_{\mathcal{Q}} - \nabla^{\mathcal{T}}_{\mathcal{D}} u^{\mathcal{T},n} - B_{\mathcal{Q}}\delta^n_{\mathcal{D}}).$$

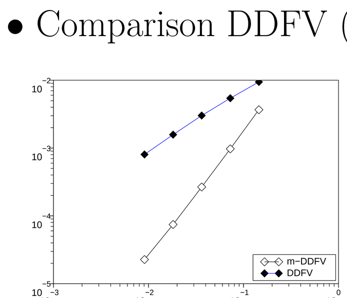
> **Theorem.** $\forall r > 0$, the algorithm converges to the unique solution of (5).

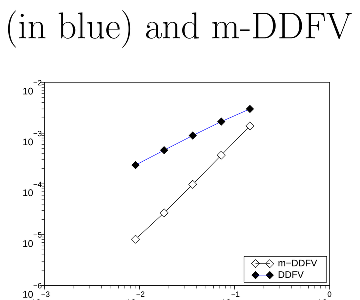
- **Results :** see http://www.cmi.univ-mrs.fr/~fhubert/Numerique

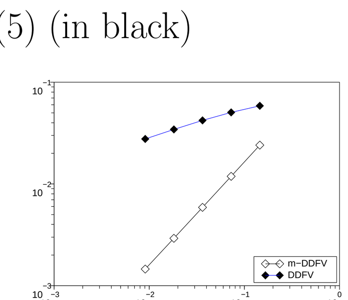
$\Omega = [0,1] \times [0,1]$, triangular mesh, ue polynomial, $p = 3$

$$\text{si } z_1 < 0.5, \quad \varphi(z,\xi) = |\xi|^{p-2}\xi,$$
$$\text{si } z_1 > 0.5, \quad \varphi(z,\xi) = (A\xi,\xi)^{\frac{p-2}{2}} A\xi, \ \text{avec } A = \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}.$$

- Comparison DDFV (3) (in blue) and m-DDFV (5) (in black)


L^∞ error


L^p error


$W^{1,p}$ error

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