# Benchmark on Discretization Schemes for Anisotropic Diffusion Problems on General Grids
# Overview of the results


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**Finite volumes for complex applications V**
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# Outlines

# Outlines

# Acknowledgments

**Many thanks to**

- I. Aavatsmark, F. Boyer, R. Eymard, T. Gallouët, C. Le Potier, G. Manzini and R. Masson for their help in the benchmark design.
- All the participants to the benchmark.

# Introduction

**Aim : to test existing discretizations for diffusion problems**

$$
\begin{aligned}
-\quad \nabla\cdot(\mathbf{K}\nabla u) &= f \text{ on } \Omega,\\
u &= \bar{u} \text{ on } \Gamma_D,\\
\mathbf{K}\nabla u\cdot\mathbf{n} &= g \text{ on } \Gamma_N,
\end{aligned}
$$

with

- Anisotropic $\mathbf{K} = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix}$ and/or heterogeneous medium $\mathbf{K} = k(x)Id$.
- Distorted meshes  and/or non conformal meshes .

**Principal properties under study :**

- Rate of convergence.
- Positivity of the schemes : $f \geq 0 \Rightarrow u^{\tau} \geq 0$.
- Maximum principle : if $f = 0$, $u_e \in [a, b] \Rightarrow u^{\tau} \in [a, b]$.
- Approximation of physical quantities such as energy, boundary fluxes,. . .

# Outlines

# The participating schemes and teams

**Cell centred schemes**

- CMPFA, by S. Mundal, D. A. Di Pietro and I. Aavatsmark.
- FVHYB, by L. Agelas and D. A. Di Pietro.
- FVSYM, by C. Le Potier.
- SUSHI-P or SUSHI-NP (full barycentric version), by R. Eymard, T. Gallouët and R. Herbin.

**Control volume finite element schemes**

- CVFE, by M. Afif and B. Amaziane.

**Discontinous Galerkin schemes**

- DG-C, by A. Dedner and R. Klöfkorn.
- DG-W, by D. A. Di Pietro and A. Ern.

# The participating schemes and teams

**Discrete duality finite volume schemes**

- DDFV-BHU, by F. Boyer and F. Hubert.
- DDFV-HER, by F. Hermeline.
- DDFV-MNI, by I. Moukouop Nguena and A. Njifenjou.
- DDFV-OMN, by P. Omnes.

**Finite element schemes**

- FEP1, FEP2, FEQ1, FEQ2, by G. Ansanay-Alex, B. Piar and D. Vola.

**Lattice Boltzmann schemes**

- LATTB, by F. Dubois, P. Lallemand and M. M. Tekitek.

# The participating schemes and teams

## Mixed or hybrid methods

- MFD-BLS, by K. Lipnikov.
- MFD-FHE, by B. Flemisch and R. Helmig.
- MFD-MAN, by G. Manzini.
- MFD-MAR, by S. Marnach.
- MFE, programmed by the benchmark organizers for lack of a submission.
- MFV, by C. Chainais-Hillairet, J. Droniou and R. Eymard.
- SUSHI-P or SUSHI-NP (full hybrid version),by R. Eymard, T. Gallouët and R. Herbin.

## Nonlinear schemes

- FVPMMD, by C. Le Potier.
- NMFV, by D. Svyatskiy.

# Outlines

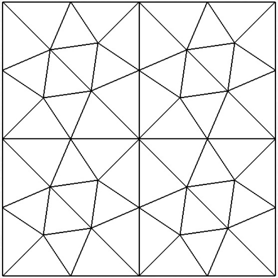
# Test 1 : mild anisotropy

## Test 1.1

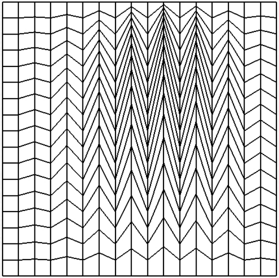
$$-\mathrm{div}(\mathbf{K}\nabla u) = f \text{ in } \Omega$$

Non homogeneous Dirichlet boundary conditions

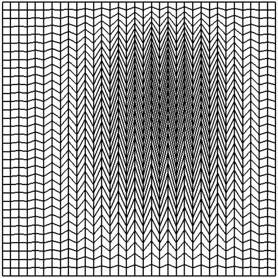
with $\mathbf{K} = \begin{pmatrix} 1.5 & 0.5 \\ 0.5 & 1.5 \end{pmatrix}$ and $u(x,y) = 16x(1-x)y(1-y)$.



Mesh1



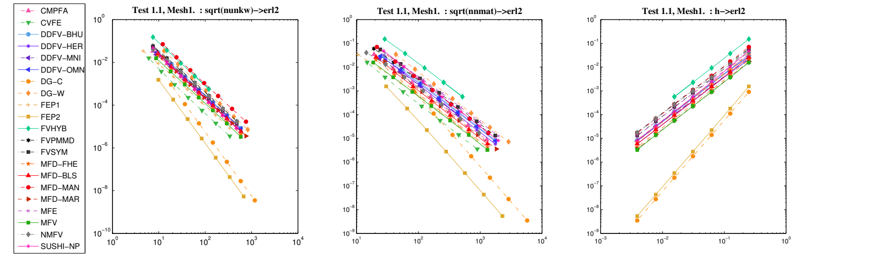
Mesh4_1



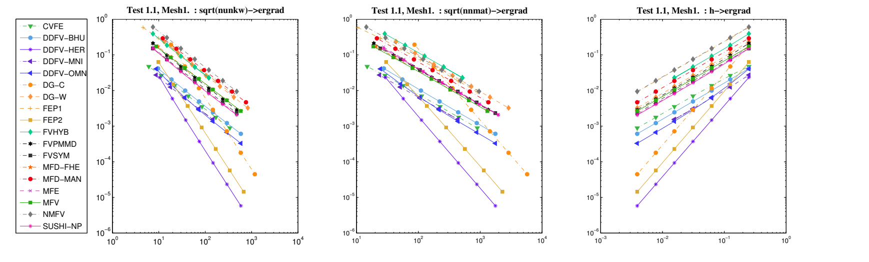
Mesh4_2

# Test 1 : mild anisotropy. Test 1.1 - mesh1_1

## Comparison of $L^2$ norm of the solution (order in $\{2,3\}$)




## and its gradient (order in $\{1,2\}$)



# Test 1 : mild anisotropy. Test 1.1 - mesh4_i

## Minimum and maximum of the approximate solutions

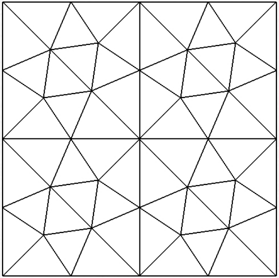
| | mesh 4_1 | | mesh 4_2 | |
|---|---|---|---|---|
| | umin | umax | umin | umax |
| CMPFA | 9.95E-03 | 1.00E+00 | 2.73E-03 | 9.99E-01 |
| CVFE | 0.00E+00 | 8.43E-01 | 0.00E+00 | 9.14E-01 |
| DDFV-BHU | 1.33E-02 | 9.96E-01 | 3.63E-03 | 9.99E-01 |
| DDFV-HER | 0.00E+00 | **1.03E+00** | 0.00E+00 | **1.01E+00** |
| DDFV-MNI | **-3.09E-01** | **1.03E+00** | 0.00E+00 | 1.00E+00 |
| DDFV-OMN | 1.34E-02 | **1.03E+00** | 3.65E-03 | **1.01E+00** |
| DG-C | **-2.33E-03** | 9.96E-01 | **-3.24E-04** | 9.99E-01 |
| DG-W | **-7.90E-05** | 9.22E-01 | **-8.18E-06** | 9.66E-01 |
| FEQ1 | 0.00E+00 | 8.61E-01 | 0.00E+00 | 9.37E-01 |
| FEQ2 | 0.00E+00 | 9.99E-01 | 0.00E+00 | 1.00E+00 |
| FVHYB | 2.14E-03 | 9.84E-01 | 7.16E-04 | 9.93E-01 |
| FVSYM | 7.34E-03 | 9.59E-01 | 2.33E-03 | 9.89E-01 |
| MFD-BLS | 8.54E-03 | 9.55E-01 | 2.44E-03 | 9.87E-01 |
| MFD-FHE | 9.73E-03 | 9.45E-01 | 2.90E-03 | 9.83E-01 |
| MFD-MAN | 6.64E-03 | 9.71E-01 | 1.50E-03 | 9.93E-01 |
| MFD-MAR | 8.82E-03 | 9.60E-01 | 2.47E-03 | 9.88E-01 |
| MFV | 1.08E-02 | 9.42E-01 | 3.34E-03 | 9.82E-01 |
| NMFV | 1.30E-02 | **1.11E+00** | 3.61E-03 | **1.04E+00** |
| SUSHI-NP | 7.64E-03 | 8.88E-01 | 2.33E-03 | 9.61E-01 |

# Test 1 : mild anisotropy. Test 1.2

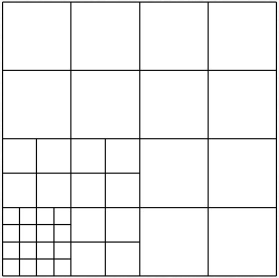
$$-\mathrm{div}(\mathbf{K}\nabla u) = f \text{ in } \Omega$$

Non homogeneous Dirichlet boundary conditions

with $\mathbf{K} = \begin{pmatrix} 1.5 & 0.5 \\ 0.5 & 1.5 \end{pmatrix}$ and $u(x,y) = \sin\left((1-x)(1-y)\right) + (1-x)^3(1-y)^2$.



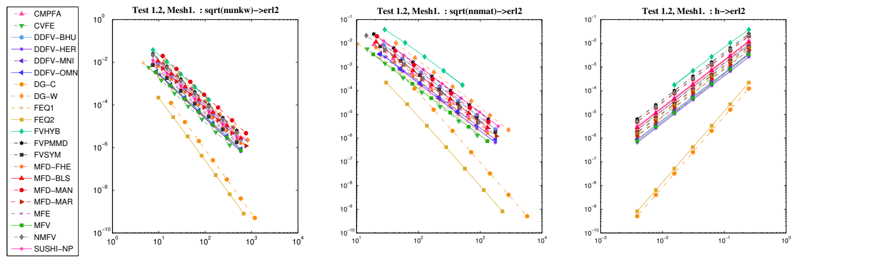
Mesh1



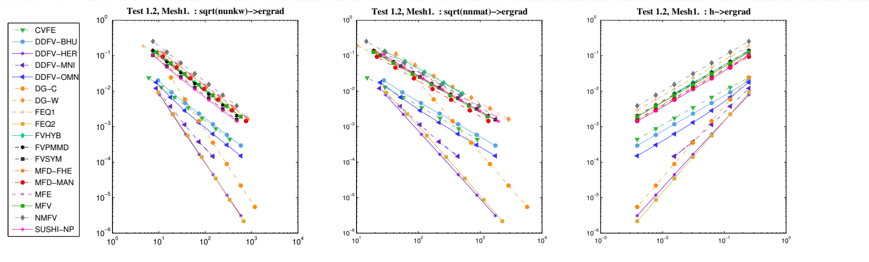
Mesh3

# Test 1 : mild anisotropy. Test 1.2 - mesh1

## Comparison of $L^2$ norm of the solution (order in $\{2, 3\}$)

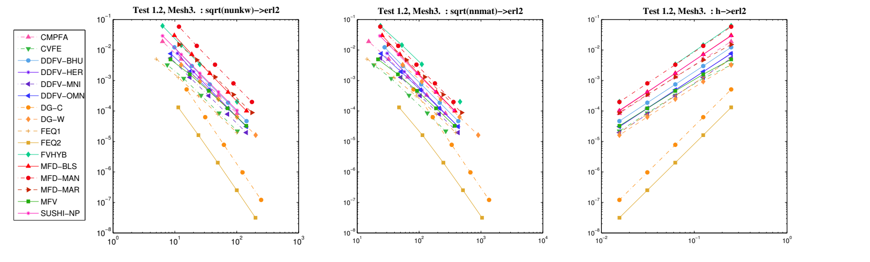



## and its gradient (order in $\{1, 2\}$)

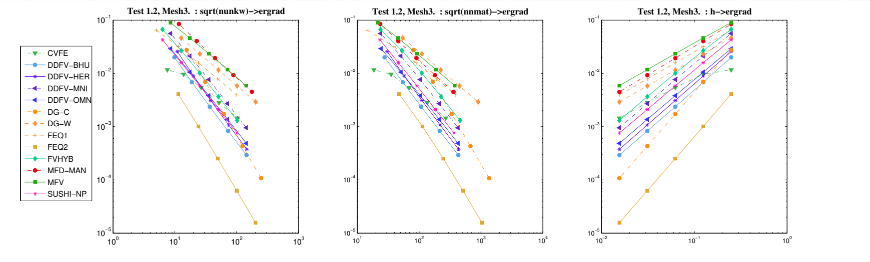


# Test 1 : mild anisotropy. Test 1.2 - mesh3

## Comparison of $L^2$ norm of the solution (order in $\{2,3\}$)

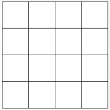



## and its gradient (order in $\{1, 1.5, 2\}$)



# Outlines

# Test 2 : numerical locking



$$-\mathrm{div}(\mathbf{K}\nabla u) = f \text{ in } \Omega$$

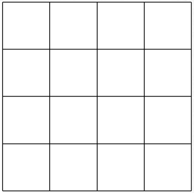
Neumann boundary conditions and $\int_\Omega u\,dx = 0$

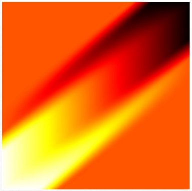
$$\text{with } \mathbf{K} = \begin{pmatrix} 1 & 0 \\ 0 & 10^5 \end{pmatrix} \text{ and } u(x,y) = \sin(2\pi x)e^{-2\pi 10^{-2.5}y}.$$

| | min(umin), i | max(umax), i | ocvl2 | ocvgrad | erflm |
|---|---|---|---|---|---|
| CMPFA | **-1.10E+00, 1** | 1.04E+00, 3 | **1.09** | / | 6.36E+02 |
| CVFE | **-1.01E-00, 2** | **1.01E-01, 2** | 2.00 | 1.00 | 1.55E+01 |
| DDFV-BHU | **-9.27E-01, 1** | **1.17E+00, 1** | 1.76 | 1.21 | 7.51E+00 |
| DDFV-HER | **-4.20E-01, 2** | 9.12E+00, 4 | / | / | 7.16E-03 |
| DDFV-OMN | **-8.24E-01, 1** | **7.76E-01, 1** | 2.00 | 1.00 | 2.11E+00 |
| DG-W | **-1.18E-01, 1** | **1.18E-01, 1** | 2.00 | 1.00 | 1.69E+01 |
| FEP1 | **-9.48E-03, 1** | **9.75E-03, 1** | 2.00 | 1.01 | / |
| FEP2 | **-9.56E-01, 1** | **9.56E-01, 1** | **2.97** | **2.00** | / |
| FVSYM | **-1.76E+00, 2** | **1.80E+00, 2** | 2.38 | 1.47 | 7.29E+00 |
| MFD-BLS | **-6.50E+00, 2** | **5.75E+00, 2** | 2.54 | / | 3.59E+01 |
| MFD-FHE | **-6.50E+00, 2** | **5.75E+00, 2** | 2.54 | 1.51 | 3.59E+01 |
| MFD-MAN | **-6.62E+00, 2** | **5.50E+00, 2** | 2.49 | 1.50 | 3.58E+01 |
| MFD-MAR | **-6.50E+00, 2** | **5.75E+00, 2** | 2.53 | / | 3.59E+01 |
| MFE | **-6.50E+00, 2** | **5.75E+00, 2** | 2.53 | 1.47 | 3.59E+01 |
| MFV | **-6.50E+00, 2** | **5.75E+00, 2** | 2.41 | 1.51 | 3.58E+01 |
| SUSHI-P | **-6.50E+00, 2** | **5.75E+00, 2** | 2.53 | 1.47 | 3.59E+01 |
| SUSHI-NP | -1.93E-02, 4 | 1.89E-02, 4 | **0.37** | 1.99 | / |

# Outlines

# Test 3 : oblique flow





$$-\mathrm{div}(\mathbf{K}\nabla u) = 0 \text{ in } \Omega$$

Non homogeneous Dirichlet boundary conditions

$$\text{with } \mathbf{K} = R_\theta \begin{pmatrix} 1 & 0 \\ 0 & 10^{-3} \end{pmatrix} R_\theta^{-1},\ \theta = 40 \text{ degrees}$$

$$\bar{u}(x,y) = \begin{cases} 1 & \text{on } (0,.2) \times \{0.\} \cup \{0.\} \times (0,.2) \\ 0 & \text{on } (.8,1.) \times \{1.\} \cup \{1.\} \times (.8,1.) \\ \frac{1}{2} & \text{on } ((.3,1.) \times \{0\} \cup \{0\} \times (.3,1.) \\ \frac{1}{2} & \text{on } (0.,.7) \times \{1.\} \cup \{1.\} \times (0.,0.7) \end{cases}$$

# Test 3 : oblique flow

## The values of umin, umax

| | umin_i | umax_i | i |
|---|---|---|---|
| CMPFA | 6.90E-02 | 9.31E-01 | 1 |
| | 9.83E-04 | 9.99E-01 | 7 |
| CVFE | 0.00E+00 | 1.00E+00 | 1 |
| | 0.00E+00 | 1.00E+00 | 7 |
| DDFV-BHU | **-4.72E-03** | 1.00E+00 | 1 |
| | **-5.31E-04** | 1.00E+00 | 7 |
| DDFV-HER | **-4.72E-03** | 1.00E+00 | 1 |
| | **-5.96E-08** | 1.00E+00 | 7 |
| DDFV-MNI | **-4.73E-03** | 1.00E+00 | 1 |
| | **-1.07E-03** | 1.00E+00 | 5 |
| DDFV-OMN | 1.04E-01 | 8.96E-01 | 1 |
| | 1.01E-03 | 9.99E-01 | 7 |
| DG-C | **-9.35E-02** | **1.07E+00** | 1 |
| | **-1.32E-03** | 1.00E+00 | 7 |
| DG-W | **-4.11E-02** | **1.04E+00** | 1 |
| | **-3.71E-03** | 1.00E+00 | 7 |
| FEQ1 | 0.00E+00 | 1.00E+00 | 1 |
| | 0.00E+00 | 1.00E+00 | 7 |
| FEQ2 | 0.00E+00 | 1.00E+00 | 1 |
| | 0.00E+00 | 1.00E+00 | 7 |
| FVHYB | **-1.75E-01** | **1.17E+00** | 1 |
| | **-1.00E-03** | 1.00E+00 | 6 |

| | umin_i | umax_i | i |
|---|---|---|---|
| FVSYM | 6.85E-02 | 9.32E-01 | 1 |
| | 4.92E-04 | 9.99E-01 | 8 |
| LATTB | 1.14E-01 | 8.86E-01 | 1 |
| | 9.36E-04 | 9.99E-01 | 7 |
| MFD-BLS | 6.09E-02 | 9.39E-01 | 1 |
| | 1.29E-03 | 9.99E-01 | 7 |
| MFD-FHE | 7.06E-02 | / | 1 |
| | 1.00E-03 | 9.99E-01 | 7 |
| MFD-MAN | 7.56E-02 | 9.24E-01 | 1 |
| | 8.01E-04 | 9.99E-01 | 8 |
| MFD-MAR | 6.09E-02 | 9.39E-01 | 1 |
| | 1.00E-03 | 9.99E-01 | 8 |
| MFE | 3.12E-02 | 9.69E-01 | 1 |
| | 5.08E-04 | 9.99E-01 | 8 |
| MFV | 1.22E-02 | 8.78E-01 | 1 |
| | 7.92E-04 | 9.99E-01 | 7 |
| NMFV | 1.11e-01 | 8.88e-01 | 1 |
| | 1.28E-03 | 9.99E-01 | 7 |
| SUSHI-NP | 6.03E-02 | 9.40E-01 | 1 |
| | 8.52E-04 | 9.99E-01 | 7 |

# Test 3 : oblique flow

## The energies

| | ener1 | eren_i | i |
|---|---|---|---|
| CVFE | 2.24E-01 | 8.42E-02 | 1 |
| | 2.42E-01 | 3.33E-03 | 7 |
| DDFV-BHU | 2.14E-01 | 9.60E-02 | 1 |
| | 2.42E-01 | 7.11E-06 | 7 |
| DDFV-HER | 2.14E-01 | 9.46E-02 | 1 |
| | 2.42E-01 | 1.91E-05 | 7 |
| DDFV-MNI | 2.14E-01 | 9.61E-02 | 1 |
| | 2.42E-01 | 1.86E-04 | 5 |
| DDFV-OMN | 1.81E-01 | 3.68E-03 | 1 |
| | 2.42E-01 | 1.77E-06 | 7 |
| DG-C | **5.04E-01** | 9.88E-02 | 1 |
| | 2.42E-02 | 2.48E-05 | 7 |
| DG-W | **1.90E-01** | 5.67E-01 | 1 |
| | 2.44E-01 | 2.85E-05 | 7 |
| FEQ1 | 2.21E-01 | 3.67E-01 | 1 |
| | 2.44E-01 | 3.17E-02 | 7 |
| FEQ2 | 2.64E-01 | 3.41E-01 | 1 |
| | 2.42E-01 | 0.00E+00 | 7 |
| FVHYB | 2.13E-01 | 2.55E-01 | 1 |
| | 2.42E-01 | 8.19E-03 | 6 |

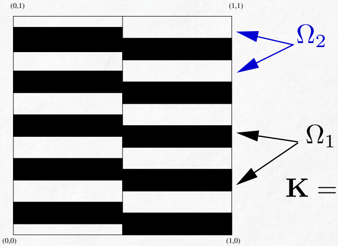
| | ener1 | eren_i | i |
|---|---|---|---|
| FVSYM | 2.20E-01 | 0.00E+00 | 1 |
| | 2.42E-01 | 0.00E+00 | 8 |
| LATTB | 2.42E-01 | 1.64E-02 | 1 |
| | 2.42E-01 | 3.00E-04 | 7 |
| MFD-BLS | 2.38E-01 | 4.44E-15 | 1 |
| | 2.42E-01 | 6.74E-13 | 7 |
| MFD-FHE | 2.19E-01 | 2.09E-01 | 1 |
| | 2.42E-01 | 1.05E-04 | 7 |
| MFD-MAN | **1.91E-01** | 1.87E-14 | 1 |
| | 2.42E-01 | 3.70E-14 | 8 |
| MFD-MAR | 2.38E-01 | 9.85E-13 | 1 |
| | 2.42E-01 | 1.97E-10 | 8 |
| MFE | **1.25E-01** | 2.46E-02 | 1 |
| | 2.41E-01 | 2.91E-03 | 8 |
| MFV | **4.85E-01** | 8.23E-07 | 1 |
| | 2.42E-01 | 9.74E-06 | 7 |
| NMFV | 2.33e-01 | 1.45e-01 | 1 |
| | 2.45E-01 | 1.94E-02 | 7 |
| SUSHI-NP | 2.25E-01 | 3.01E-01 | 1 |
| | 2.43E-01 | 1.28E-02 | 7 |

# Outlines

# Test 4 : vertical fault

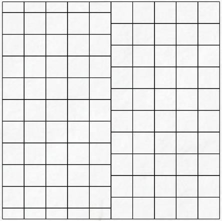
$$-\mathrm{div}(\mathbf{K}\nabla u) = 0 \text{ in } \Omega$$

$$u = \bar{u} \text{ in } \partial\Omega$$

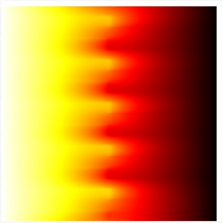



$$\mathbf{K} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \text{ with } \begin{cases} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 10^2 \\ 10 \end{pmatrix} \text{ on } \Omega_1, \\ \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 10^{-2} \\ 10^{-3} \end{pmatrix} \text{ on } \Omega_2 \end{cases}$$

and $\bar{u}(x,y) = 1 - x$.



mesh5



# Test 4 : vertical fault

## Maximum principle

- Problems only with the DG methods.

## The values of the energies

| | ener1 mesh5 | eren mesh5 | ener1 mesh5_ref | eren mesh5_ref |
|---|---|---|---|---|
| CVFE | 45.9 | 1.04E-02 | **43.3** | 6.25E-04 |
| DDFV-BHU | **42.1** | 3.65E-02 | **43.2** | 1.27E-03 |
| DDFV-HER | 49.3 | 1.75E-01 | 43.8 | 1.64E-02 |
| DDFV-MNI | / | / | 43.8 | 6.23E-02 |
| DDFV-OMN | **42.2** | 3.65E-02 | **43.2** | 1.28E-03 |
| DG-W | **43.5** | 1.38E-02 | **43.2** | 7.63E-04 |
| FEQ1 | / | / | **43.3** | 2.31E-03 |
| FEQ2 | / | / | **43.2** | 0.00E+00 |
| FVHYB | **41.4** | 6.12E-02 | / | / |
| MFD-BLS | 33.9 | 7.93E-14 | **43.2** | 2.84E-12 |
| MFD-FHE | / | / | **43.2** | 3.53E-04 |
| MFD-MAN | 31.4 | 1.16E-12 | **43.2** | 4.71E-14 |
| MFD-MAR | **41.1** | 1.30E-13 | **43.2** | 2.69E-12 |
| MFV | 49.9 | 4.21E-05 | **43.2** | 1.88E-05 |
| NMFV | / | / | **43.2** | 5.92E-04 |
| SUSHI-NP | 39.1 | 6.67E-02 | **43.1** | 8.88E-04 |

- The methods that have trouble with PPMax are the most accurate for the energy on coarse meshes.

# Test 4 : vertical fault

## The fluxes

| | flux0 mesh5 | flux0 mesh5_ref | flux1 mesh5 | flux1 mesh5_ref | fluy0 mesh5 | fluy0 mesh5_ref | fluy1 mesh5 |
|---|---|---|---|---|---|---|---|
| CMPFA | -45.2 | **-42.1** | 46.1 | **44.4** | -0.95 | **-2.33** | 4.84E-04 |
| CVFE | -46.6 | **-42.2** | 48.5 | **44.5** | 0.87 | -2.25 | 8.02E-04 |
| DDFV-BHU | -40.0 | **-42.1** | 41.8 | **44.4** | -1.81 | **-2.33** | 9.08E-04 |
| DDFV-HER | -40.0 | **-42.0** | 41.8 | **44.3** | -1.81 | **-2.35** | 9.08E-04 |
| DDFV-MNI | -43.8 | -39.9 | 45.5 | 42.6 | -2.8 | -2.68 | 1.18E+00 |
| DDFV-OMN | -40.0 | **-42.1** | 41.8 | **44.4** | -1.81 | **-2.33** | 9.08E-04 |
| DG-W | **-43.1** | **-42.1** | **45.3** | **44.5** | -2.19 | **-2.32** | 1.50E-03 |
| FEQ1 | / | **-42.2** | / | **44.5** | / | -2.16 | / |
| FEQ2 | / | **-42.1** | / | **44.5** | / | **-2.32** | / |
| FVHYB | -44.3 | / | 46.3 | / | 0.49 | / | 1.55E-04 |
| MFD-BLS | -32.3 | **-42.1** | 36.2 | **44.4** | -3.94 | **-2.33** | 1.22E-03 |
| MFD-FHE | / | **-42.1** | / | **44.5** | / | -2.47 | / |
| MFD-MAN | -29.7 | **-42.1** | 34.1 | **44.4** | -4.37 | **-2.33** | 1.01E-03 |
| MFD-MAR | -39.8 | **-42.1** | 42.5 | **44.4** | **-2.68** | **-2.33** | 9.95E-04 |
| MFE | / | **-42.1** | / | **44.4** | / | **-2.33** | / |
| MFV | -44.0 | **-42.1** | 50.3 | **44.4** | -8.03 | **-2.33** | 1.72E+00 |
| NMFV | **-43.2** | **-42.1** | **44.5** | **44.4** | -1.23 | **-2.33** | 2.32E-04 |
| SUSHI-NP | **-40.9** | **-42.1** | **43.1** | **44.4** | **-2.21** | **-2.33** | 6.94E-04 |

# Outlines

# Test 5 : heterogeneous rotating anisotropy

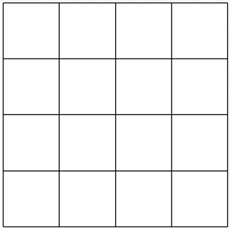
$$-\mathrm{div}(\mathbf{K}\nabla u) = f \text{ in } \Omega$$

Non homogeneous Dirichlet boundary conditions

avec

$$\mathbf{K} = \frac{1}{(x^2+y^2)} \begin{pmatrix} 10^{-3}x^2 + y^2 & (10^{-3}-1)xy \\ (10^{-3}-1)xy & x^2 + 10^{-3}y^2 \end{pmatrix}$$

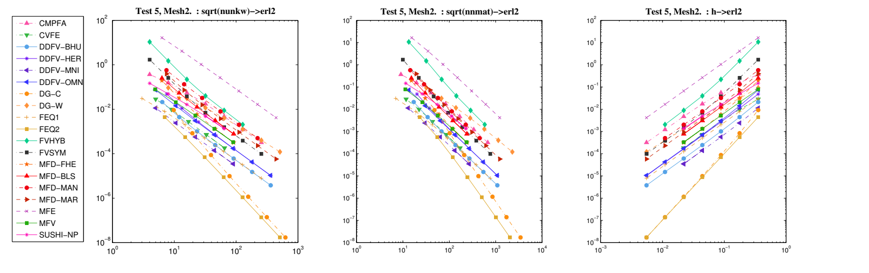
and $u(x,y) = \sin \pi x \sin \pi y$.



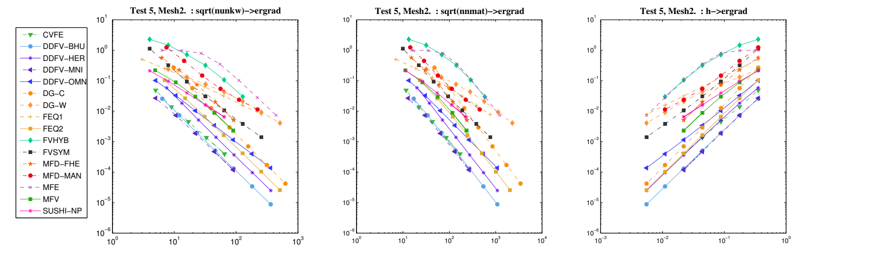
`mesh2`

# Test 5 : heterogeneous rotating anisotropy

## Comparison of $L^2$ norm of the solution (order in $\{2,3\}$)




## and its gradient (order in $[1,2]$)



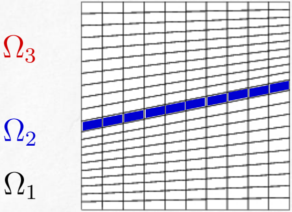
# Test 5 : heterogeneous rotating anisotropy

**Minimum and maximum values on the coarsest mesh for the schemes which do not satisfy the maximum principle**

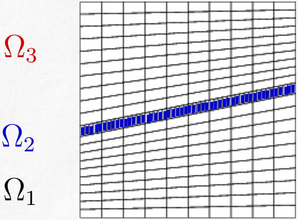
| | umin | umax |
|---|---|---|
| CMPFA | **-1.06E-01** | **1.09E+00** |
| DDFV-HER | 0.00E+00 | **1.01E+00** |
| DG-C | **-7.95E-04** | **1.02E+00** |
| DG-W | **-7.68E-02** | **1.06E+00** |
| FEQ1 | 0.00E+00 | **1.05E+00** |
| FVHYB | **-1.92E+01** | **5.38E+00** |
| FVSYM | **-8.67E-01** | **2.57E+00** |
| MFE | **-1.62E+00** | **1.90E+01** |

# Outlines

# Test 6 : oblique drain




mesh6




mesh7

$$-\mathrm{div}(\mathbf{K}\nabla u) = f \text{ in } \Omega$$

Non homogeneous Dirichlet boundary conditions

with

$$\mathbf{K} = R_\theta \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} R_\theta^{-1},$$

and $u(x, y) = -x - \delta y$ with $\delta = \tan\theta = 0.2$.

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 10^2 \\ 10 \end{pmatrix} \text{ on } \Omega_2$$

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 \\ 10^{-1} \end{pmatrix} \text{ on } \Omega_1 \cup \Omega_3.$$

# Test 6 : oblique drain

- Only a few schemes (CVFE, DDFV-HER, MFV, NMFV) are not exact on such a problem.
- All schemes satisfy the discrete maximum principle.
- The interest of this case lies in the approximation of the interface fluxes $\rightsquigarrow$ approximation of a coupled system of diffusion and convection equations.

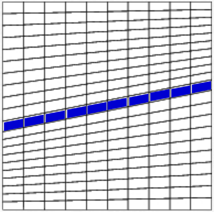
# Outlines

# Test 7 : oblique barrier

$\Omega_3$

$\Omega_2$

$\Omega_1$



mesh6

$$-\mathrm{div}(\mathbf{K}\nabla u) = f \text{ in } \Omega$$

Non homogeneous Dirichlet boundary conditions

with

$$\mathbf{K} = \alpha Id, \text{ for } \alpha = \begin{cases} 1 & \text{on } \Omega_1 \cup \Omega_3, \\ 10^{-2} & \text{on } \Omega_2, \end{cases}$$

and

$$u(x,y) = \begin{cases} -(y - 0.2x - 0.375) \text{ on } \Omega_1, \\ -(y - 0.2x - 0.375)/10^{-2} \text{ on } \Omega_2, \\ -(y - 0.2x - 0.425) - 0.05/10^{-2} \text{ on } \Omega_3. \end{cases}$$

# Test 7 : oblique barrier

| | erl2 | ergrad | umin | umax |
|---|---|---|---|---|
| CMPFA | 1.23E-15 | / | **-5.54** | **5.37E-01** |
| CVFE | **1.83E-05** | **3.86E-05** | **-5.57** | **5.75E-01** |
| DDFV-BHU | 1.38E-14 | 5.86E-14 | **-5.575** | **5.75E-01** |
| DDFV-HER | **6.53E-08** | / | **-5.575** | **5.75E-01** |
| DDFV-MNI | 1.38E-15 | 3.48E-14 | **-5.58** | **5.75E-01** |
| DDFV-OMN | **3.79E-08** | **4.51E-08** | **-5.58** | **5.75E-01** |
| DG-C | 1.39E-13 | 1.88E-12 | **-5.58** | **5.75E-01** |
| DG-W | 1.18E-14 | 1.89E-14 | **-5.58** | **5.75E-01** |
| FEQ1 | 3.79E-15 | 6.49E-14 | **-5.575** | **5.75E-01** |
| FEQ2 | 6.12E-15 | 8.19E-14 | **-5.575** | **5.75E-01** |
| FVHYB | 1.24E-15 | 5.10E-14 | **-5.54** | **5.37E-01** |
| FVSYM | $\epsilon$ | $\epsilon$ | **-5.53** | **5.37E-01** |
| MFD-BLS | $\epsilon$ | / | **-5.54** | **5.37E-01** |
| MFD-FHE | 3.25E-15 | 4.73E-15 | **-5.54** | **5.37E-01** |
| MFD-MAN | 1.39E-15 | 1.92E-15 | **-5.54** | **5.37E-01** |
| MFD-MAR | 8.19E-13 | / | **-5.54** | **5.37E-01** |
| MFV | **1.40E-08** | **4.69E-08** | **-5.54** | **5.37E-01** |
| NMFV | **4.98E-03** | / | **-5.54** | **5.37E-01** |
| SUSHI-NP | 1.30E-15 | 1.35E-14 | **-5.54** | **5.37E-01** |

# Outlines

# Test 8 : perturbed parallelograms

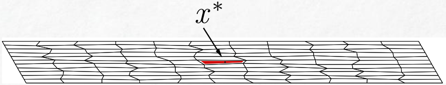
$$-\mathrm{div}(\mathbf{K}\nabla u) = f \text{ in } \Omega$$

Homogeneous Dirichlet boundary conditions

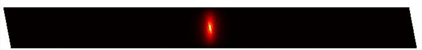
with $\mathbf{K} = Id$.
The source term $f$ is defined by $f = 0$ except on $C^*$

- $C^* = \mathrm{cell}(6,6)$
- $\int_{\mathrm{cell}(6,6)} f(x)\,dx = 1.$

- $C^* = x^*$
- $\int_{\Omega} f(x)\,dx = 1.$




The distorted quadrangle mesh `mesh9`



Obtained on a regular fine grid

# Test 8 : perturbed parallelograms

## Minimum and maximum values

| | umin | umax | | umin | umax |
|---|---|---|---|---|---|
| Fine grid | **1.07E-24** | **4.10E-01** | | | |
| CMPFA | -2.31E-02 | 1.03E-01 | FVHYB | -3.38E-02 | 1.12E-01 |
| CVFE | -1.23E-03 | 4.24E-02 | FVSYM | -7.21E-02 | 1.52E-01 |
| DDFV-BHU | -1.25E-03 | 8.22E-02 | FVPMMD | **1.22E-09** | **3.99E-01** |
| DDFV-HER | -1.61E-03 | 8.99E-02 | MFD-BLS | **-1.03E-01** | 1.85E-01 |
| DDFV-MNI | -1.46E-03 | 6.69E-02 | MFD-FHE | -6.54E-02 | 1.44E-01 |
| DDFV-OMN | -1.77E-03 | 8.36E-02 | MFD-MAR | -2.62E-02 | 9.07E-02 |
| DG-C | -7.33E-03 | 1.05E-01 | MFV | -8.08E-03 | 5.81E-02 |
| DG-W | -9.03E-03 | 6.57E-02 | NMFV | **3.05E-15** | 9.42E-02 |
| FEQ1 | -4.17E-03 | 4.90E-02 | SUSHI-NP | -1.19E-03 | 5.65E-02 |
| FEQ2 | -5.07E-03 | 8.04E-02 | SUSHI-P | **3.26E-06** | **6.77E-03** |

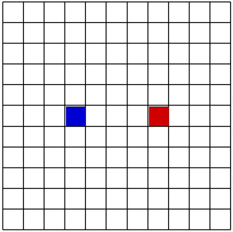
# Test 8 : perturbed parallelograms

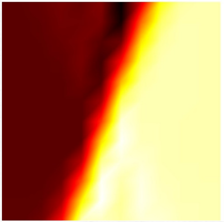
## Boundary fluxes

| | flux0 | flux1 | fluy0 | fluy1 |
|---|---|---|---|---|
| Fine grid | **5.46E-21** | **5.46E-21** | **5.00E-01** | **5.00E-01** |
| CVFE | -1.17E-05 | 2.63E-05 | 2.87E-01 | 5.54E-01 |
| DDFV-BHU | **-5.814E-10** | **-3.35E-10** | **4.97E-01** | **5.02E-01** |
| DDFV-HER | **2.45E-10** | **-1.83E-10** | 4.80E-01 | 5.11E-01 |
| DDFV-MNI | -4.37E-05 | 6.50E-05 | 5.07E-01 | 4.93E-01 |
| DDFV-OMN | **-6.86E-10** | **-4.86E-10** | **4.98E-01** | **5.02E-01** |
| DG-C | -1.66E-07 | 1.41E-07 | **5.08E-01** | **4.92E-01** |
| DG-W | 5.02E-01 | 0.00E+00 | **4.98E-01** | 0.00E+00 |
| FEQ1 | 5.51E-06 | 7.15E-05 | 5.46E-01 | 4.89E-01 |
| FEQ2 | -5.52E-05 | 1.96E-05 | **4.98E-01** | **5.01E-01** |
| FVSYM | 1.37E-04 | -1.15E-04 | **4.96E-01** | **5.04E-01** |
| FVPMMD | 1.76E-06 | 3.5E-06 | 4.55E-01 | 5.44E-01 |
| MFD-BLS | -5.14E-04 | -3.13E-03 | **5.01E-01** | **5.03E-01** |
| MFD-FHE | 4.48E-04 | -4.08E-03 | **5.03E-01** | **5.01E-01** |
| MFD-MAR | -1.79E-02 | 2.61E-03 | **5.05E-01** | **5.10E-01** |
| MFV | -2.30E-02 | 4.95E-02 | 2.74E-01 | 6.99E-01 |
| NMFV | 0.00E+00 | 0.00E+00 | **4.99E-01** | **5.01E-01** |
| SUSHI-NP | 7.35E-04 | 1.29E-04 | **4.99E-01** | **5.00E-01** |
| SUSHI-P | -4.21E-02 | -3.29E-02 | 5.38E-01 | 5.37E-01 |

# Outlines

# Test 9 : anisotropy and wells





$$-\mathrm{div}(\mathbf{K}\nabla u) = 0 \text{ in } \Omega$$

Neumann boundary conditions

with

$$\mathbf{K} = R_\theta^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 10^{-3} \end{bmatrix} R_\theta,$$

where $\theta = 67.5°$ and

$$u = 0 \quad \text{in cell } (4,6),$$

$$u = 1 \quad \text{in cell } (8,6).$$

# Test 9 : anisotropy and wells

**Values of `umin` and `umax`** should be `umin`=0 and `umax`=1

| | umin | umax |
|---|---|---|
| CMPFA | -6.77E-01 | 1.68E+00 |
| CVFE | -1.16E-01 | 1.12E+00 |
| DDFV-BHU | -1.38E-01 | 1.14E+00 |
| DDFV-HER | -1.03E-01 | 1.10E+00 |
| DDFV-OMN | -7.07E-02 | 1.07E+00 |
| DG-C | **-1.02E-03** | **9.98E-01** |
| FEQ1 | -2.36E-02 | 1.02E+00 |
| FEQ2 | -5.94E-03 | 1.01E+00 |
| FVHYB | -3.69E-02 | 1.04E+00 |
| FVSYM | -7.63E-02 | 1.07E+00 |

| | umin | umax |
|---|---|---|
| | | |
| FVPMMD | **0.00E+00** | **1.00E+00** |
| MFD-BLS | -4.30E-02 | 1.04E+00 |
| MFD-FHE | -4.21E-02 | 1.04E+00 |
| MFD-MAR | -4.30E-02 | 1.04E+00 |
| MFE | **0.00E+00** | **1.00E+00** |
| MFV | -1.22E-01 | 1.07E+00 |
| NMFV | **1.83E-02** | **1.01E+00** |
| SUSHI-NP | -1.00E+00 | 2.00E+00 |
| SUSHI-P | **0.00E+00** | **1.00E+00** |

# Outlines

# Concluding remarks

- $\rightsquigarrow$ Relative homogeneity of the results.
- $\rightsquigarrow$ Higher order scheme are more precise in both $L^2$ and $H^1$ norm. DDFV method are more precise in $H^1$ norm.
- $\rightsquigarrow$ Cell centered schemes and mimetic schemes are generally more robust.
- $\rightsquigarrow$ Nonlinear schemes seem necessary to enforce the maximum principle and/or the positiveness (or monotonicity) of the schemes.

# Perspectives

- $\rightsquigarrow$ Positive schemes and schemes satisfying the maximum principle.
- $\rightsquigarrow$ Coupling strongly heterogeneous problems with transport equations :

$$\begin{cases} -\mathrm{div}(\mathbf{K}\nabla p) = 0 \\ c_t + \mathrm{div}(f(c)\mathbf{K}\nabla p) = 0 \end{cases}$$

- $\rightsquigarrow$ 3D tests on distorted nonconforming meshes

# Annoncement : February 6th-March 9th 2009

⇝ **Residential month at CIRM, Marseille France.**

**Winter school ”New trends in scientific computing. February 9th-13th 2009.**

http://www.latp.univ-mrs.fr/cirm09

## ▶ Organizing Commitee

- **Franck Boyer** (Marseille, France),
- **Florence Hubert** (Marseille, France),
- **Jean-Claude Latché** (IRSN, Cadarache, France)

## ▶ Scientific Commitee

- **Christine Bernardi** (Paris, France)
- **Vit Dolesji** (Praha, Czeck Republic)
- **Raphaèle Herbin** (Marseille, France)
- **Christian Röhde** (Bielfeld, Germany)

## ▶ Mini-courses

- **Rémi Abgrall** (Bordeaux - France) : “Residual Schemes for Hyperbolic problems”
- **Alexandre Ern** (Paris, France) : “Discontinuous Galerkin Methods”
- **Jean Luc Guermond** (Texas, USA) ”approximation des équations de Navier-stokes”
- **Frédéric Nataf** (Paris, France) : “Méthodes de Schwarz optimisées”
- **Ulrich Ruede** (Erlangen-Nuremberg, Germany) : “Multilevel methods”

## ▶ Surveys

- **Alfredo Bermudez** (Santiago di Compostela, Spain)
- **Yves Coudière** (Nantes, France)
- **Cédric Galusinski** (Toulon, France)
- **Martin Gander** (Genève, Switzerland)
- **Hervé Guillard** (Sophia-Antipolis, France)
- **Kunibert Siebert** (Augsburg, Germany)
- **Zdenek Strakos** (Praha, Czeck Republic)
- **Martin Vohralik** (Paris, France)

# The discussion

- Others challenging tests (homogeneous, heterogeneous, comparison of other quantities, ...) by Ivar Aavatsmark.
- Some ideas for the maximum principle for the DDFV schemes by Pascal Omnes.
- Curved frontiers by Konstantin Lipnikov.
- Diffusion problems in 3D by François Hermeline.
- Some problems of IFP by Roland Masson.
- Some problems of ANDRA by Clément Chavent.
- ....