FV schemes for hyperelastic-plastic models in finite deformations.

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Contents

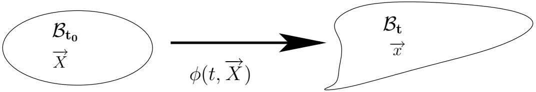
Context of the hyperelastic-plastic model

The hyperelastic system is well-known in elastodynamics. *Godunov and Romenskii* (1973), *Plohr and Sharp* (1988), *Trangenstein and Colella* (1991) have pointed out its importance. Since this rediscovery, many Finite Volume schemes studies have recently been published : *Miller and Colella* (2002), *Titarev and Romenskii and Toro* (2006), *Gavrilyuck and Favrie and Saurel* (2008)...

Plasticity is easily introduced in **hypoelastic models**, extensively used since Wilkins (1964). The projection of the stress on a convex plastic domain has been described by *Nouri and Rascle* (1996), *Després* (2007) in a rigorous mathematical frame, and the difficulty of the non-conservative term of the objective derivatives by *Colombeau and LeRoux* (1986).

In our model, *Kluth and Després* (2008), we introduce multidimensional plasticity by using a specific Equation of State. It is an easy way to introduce perfect plasticity : we use the same system of PDEs as in elastodynamics.

The hyperelastic-plastic model



$$\hat{F} = \nabla_{\overrightarrow{X}} \overrightarrow{x}$$
$$\overrightarrow{v} = D_t \overrightarrow{x}$$

PDEs : the hyperelastic system of conservation laws

It is hyperbolic, frame-indifferent (even under rigid rotations) and in a conservative form. This allows us :

- to avoid the difficulties faced when studying mathematical solutions of usual models,
- to use standard FV schemes, projection methods and high-order methods.

$$\left\{ \begin{array}{lcl} D_t(F_{ij}) & = & \partial_{X_j} v_i , \\ D_t(\rho_0 v_i) & = & \partial_{X_k} \left(\rho_0 \, \frac{\partial \epsilon}{\partial F_{ik}} \right) , \\ D_t(\rho_0 \, e) & = & \partial_{X_k} \left(\rho_0 \, v_l \, \frac{\partial \epsilon}{\partial F_{lk}} \right) . \end{array} \right. \quad (1)$$

A specific Equation of State

For introducing perfect plasticity in the hyperelastic system, we use the EOS (2) which guarantees the Von Mises criteria :

$$\|dev(\sigma)\|^2 \leq \frac{2}{3} Y^2.$$

$$\epsilon = \epsilon_{hydro}(S, \tau) + \rho_0 \tau \; \psi(\hat{F}). \qquad (2)$$

An involutive constraint

$$div(\rho^t \hat{F}) = 0$$

It gives the existence of the motion and the equivalence between Eulerian and Lagrangian formulations.

The Hugoniot experiment

Description

Two plates of the same material are impacted with opposite velocities.

This is a 1D uniaxial problem :

$$\hat{F} = \begin{bmatrix} \rho_0 \tau & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \overrightarrow{v} = \begin{bmatrix} u \\ 0 \\ 0 \end{bmatrix}$$

and the initial condition is :

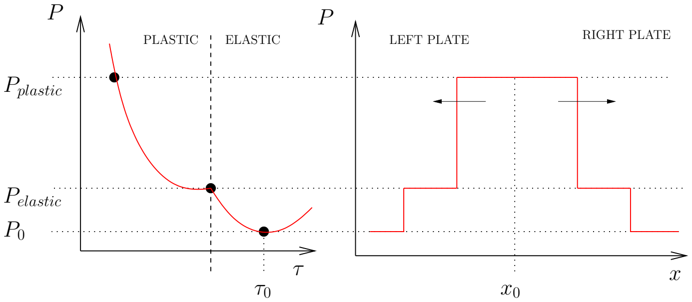
$$\tau = \tau_0, \quad e = e_0, \quad u = \begin{cases} V > 0 & \text{on the left} \\ -V & \text{on the right} \end{cases}$$

Then

- The hyperelastic system is **the Euler hydrodynamic system**.
- The specific EOS gives a **non-convex EOS** :
  $\epsilon(\tau)$ is strictly convex, but not $P(\tau)$.

In this case

- we demonstrate the unicity of the viscous solution,
- and calculate the exact solution.



We see on the right the split shock : on both sides, the elastic precursor is followed by the plastic shock.

The 1D case

In the 1D case (provided $\hat{F} = \hat{I}_d$ initially) :

$$\hat{F} = \begin{bmatrix} F_{11} = \rho_0 \tau & 0 & 0 \\ F_{21} & 1 & 0 \\ F_{31} & 0 & 1 \end{bmatrix}$$

and the 7 equations hyperelastic system in mass variable $dm = \rho dx = \rho_0 dX$ is :

$$D_t \begin{pmatrix} \rho_0^{-1} F_{i1} \\ v_i \\ e \end{pmatrix} - \partial_m \begin{pmatrix} v_i \\ \sigma_{i1} \\ v_1\sigma_{11} + v_2\sigma_{21} + v_3\sigma_{31} \end{pmatrix} = 0$$

with

$$e = \epsilon + \frac{\overrightarrow{v}^2}{2} \quad \text{et} \quad \sigma_{i1} = \frac{\partial \epsilon}{\partial F_{i1}}.$$

FV scheme

We use a FV lagrangian scheme, conservative and consistent. We actualize the positions of the mesh at each time step by solving $\overrightarrow{v} = D_t \overrightarrow{x}$.

Determination of the numerical flux

We freeze the jacobian matrix at each interface and approximate the 6 characteristic fields ($\lambda_k \in \mathbb{R}, R_k \in \mathbb{R}^6$). These aproximation gives for the Riemann invariants

$$J_k = \sigma_{i1} - \lambda_k v_i.$$

They are advected :

$$D_t J_k + \lambda_k \partial_m v_k = 0,$$

which we solve with the following methods :

Ordrer	Scheme	
1	Upwind	Gives an entropic FV scheme
2	Lax-Wendroff	
2	Beam-Warming	
3	O3 de [1]	L^1 and L^∞ stable (asymptotic)

[1] See presentation of **B. Després**, FVCA5.

Determination of the flux

The Lagrangian formalism

has the following consequences :

- we manipulate symmetric matrix of size 3 (the acoustic tensor in particular),
- the characteristic spectrum is definitively signed,
- we can construct entropic schemes (see [2]).

[2] **B. Després**, *Lagrangian systems of conservation laws*, Numerische Mathematik 89, 2001.

We used **approximate** characteristic fields so as to avoid discontinuity of eigenvectors when eigenvalues are crossing.

In the following, we don't use any limiters. Thus, we observe the **Gibbs phenomenon** for high-order methods. It is clearly reduced with order 3 because of the L^1 and L^∞ stability.

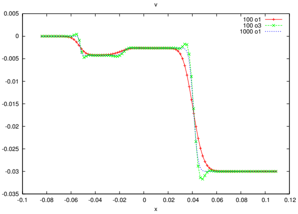
Non-linear elasticity

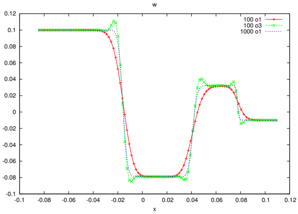
We validate the FV schemes with Finite Deformation Elastic test cases, taken from :

[3] **G.H.Miller**, *An iterative Riemann solver for systems of hyperbolic conservation laws with application to hyperelastic solid mechanics*, JCP 193, 198-225 (2003),

[4] **V.A.Titarev, E.Romenski, E.F.Toro**, *Exact Riemann problem solutions and upwind fluxes for nonlinear elasticity*, Isaac Newton Institute for Mathematical Sciences, preprint NI06018, 2006.

We calculate the entropic solution with each method, for all variables, for each test case.





Second test case of Miller, a collision with shear and crossing eigenvalues.

The Hugoniot experiment

The order 1 method (which is entropic) calculates the viscous solution (whose unicity has been proved).

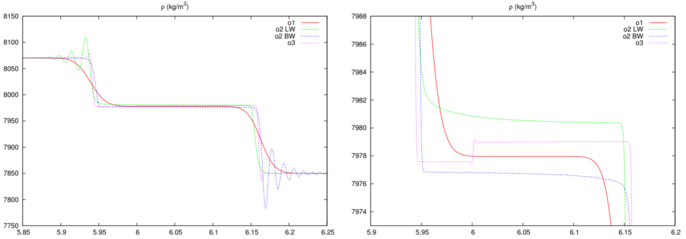
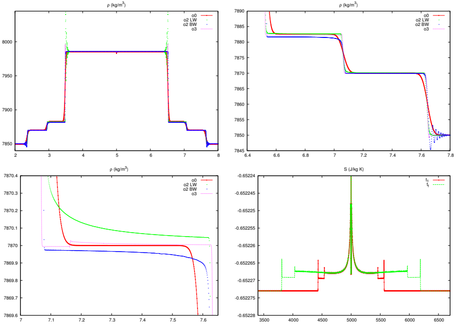


FIG.: Right-facing split shock in the Hugoniot experiment on steel for a velocity of $150m/s$.

But the high-order methods calculate other solutions. We check that entropy increases in each cell nearly everywhere.

With phase transformation

In the EOS $\epsilon = \epsilon_{hydro} + \rho_0 \tau \psi(\hat{F})$, we modify ϵ_{hydro} to simulate a phase transformation, which occurs in the elastic part.



As before, the order 1 method calculates the viscous solution, and the high-order methods calculates other solutions.

Conclusion and future prospects

Our elastic-perfectly plastic model :

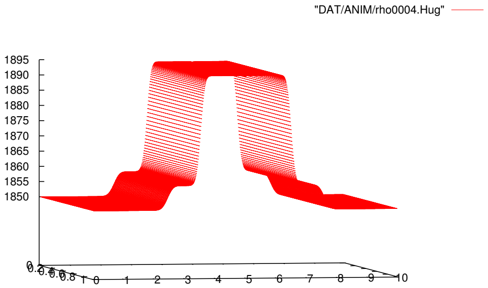
- gives a rigorous frame to the mathematical study of the Hugoniot experiment,
- opens elastic-plasticity in finite deformation to the usual Finite Volume Methods, such as projection or high-order methods,
- but raises the question of capturing viscous solutions with high-order methods, as did [5], or the question of knowing the calculated solution associated with such methods.

[5] **O. Heuzé and S. Jaouen and H. Jourdren**,

Dissipative issue of high order shock-capturing schemes with non-convex equations of state, JCP, 2008.

Our major prospect is to implement a 2D lagrangian scheme based on the hydrodynamic scheme with nodal solvers developed by C. Mazeran (phd 2007). The first results are promising : we reproduce 1D experiments, hydrodynamic cases and the Taylor test are qualitatively good.

The Hugoniot experiment in 2D



We calculate the 1D exact solution with the 2D scheme.

Taylor test

Impact of a cylinder on a rigid wall. $\rho(x, y)$