Non-overlapping Schwarz algorithm for solving 2D m-DDFV schemes

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joint work with F. Boyer and F. Hubert

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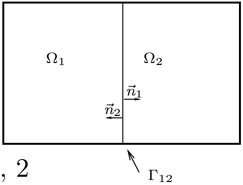
FVCA5 - Aussois

Outline

Outline

Schwarz algorithm

Non-overlapping Schwarz method proposed by Lions 90



To solve

$$\begin{aligned} -\Delta u &= f \text{ in } \Omega = \Omega_1 \cup \Omega_2, \\ u &= h \text{ on } \partial\Omega. \end{aligned}$$

we use the following iterative algorithm for i=1, 2

$$\begin{cases} -\Delta u_i^{(n+1)} = f \;\text{ in }\; \Omega_i, \\ u_i^{(n+1)} = h_i \;\text{ on }\; \partial\Omega \cap \partial\Omega_i, \\ \dfrac{\partial u_i^{(n+1)}}{\partial n_i} + \lambda u_i^{(n+1)} = -\dfrac{\partial u_j^{(n)}}{\partial n_j} + \lambda u_j^{(n)} \;\text{ on }\; \Gamma_{i,j} = \partial\Omega_i \cap \partial\Omega_j \end{cases}$$

where $\lambda > 0$.

Interest

- Reduce the size of systems to solve
- Use this algorithm as preconditioner

Problem

- The problem

$$\begin{cases} -\mathrm{div}(A(x)\cdot\nabla u(x)) = f(x), & x\in\Omega, \\ u = h, & \text{on } \partial\Omega. \end{cases} \qquad (1)$$

  - Ω is an open bounded polygonal domain of $\mathbb{R}^2$.
  - $A : \Omega \to \mathcal{M}_{2,2}(\mathbb{R})$, A uniformly elliptic and bounded.
  - $f \in L^2(\Omega)$, $h \in H^{\frac{1}{2}}(\partial\Omega)$.

The problem (1) is approximated by DDFV schemes

- standard DDFV scheme :
  if A lipschitz on $\Omega \rightsquigarrow A_{\mathcal{D}} = A(x_{\mathcal{D}})$.

  Domelevo-Omnes 05

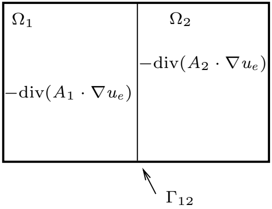
  Andreianov-Boyer-Hubert 07

- modified DDFV scheme (m-DDFV) :
  if A is discontinuous accross Γ
  $\rightsquigarrow A_{\mathcal{D}}$ appropriate function of A_1, A_2.

  Hermeline 03

  Boyer-Hubert 08

$$-\mathrm{div}^{\mathcal{T}}(A^{\mathfrak{D}}\cdot\nabla^{\mathfrak{D}}u^{\mathcal{T}}) = f^{\mathcal{T}}$$



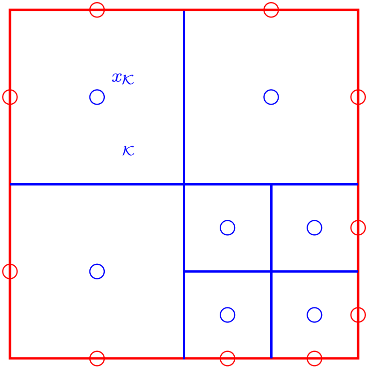
Schwarz algorithm and DDFV schemes

Outline of the study :

- Adaptation of the DDFV schemes to mixed Dirichlet/Fourier boundary conditions.
- Construction of a Schwarz algorithm called S-DDFV.
- Rewriting the DDFV schemes as possible limit problem of the S-DDFV algorithm.
- Finally, we prove the convergence of S-DDFV algorithm.

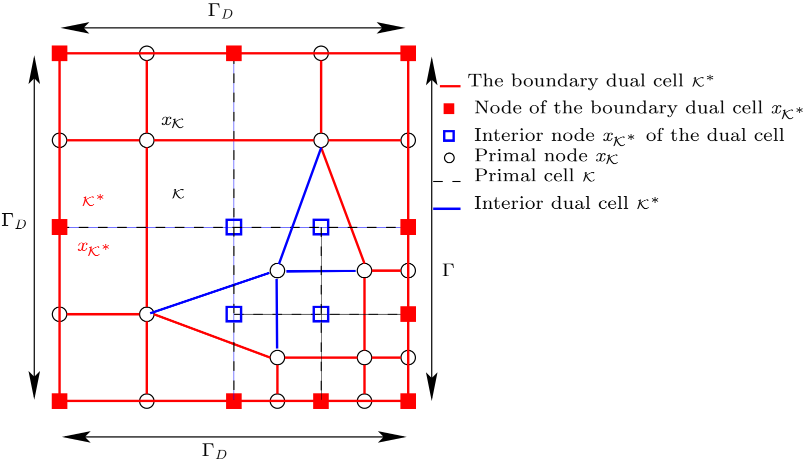
DDFV Meshes

Domelevo-Omnes 06, Andreianov-Boyer-Hubert 07

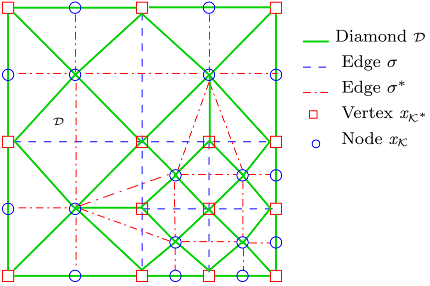


- Interior node $x_{\mathcal{K}}$ of the primal cell
- Interior primal cell $\mathcal{K}$
- Node of the boundary primal $x_{\mathcal{K}}$
- Boundary primal cell $\mathcal{K}$

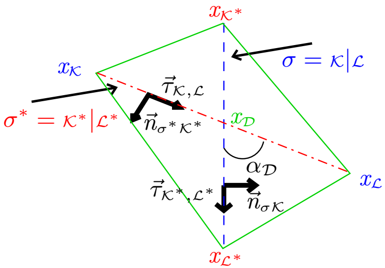
DDFV Meshes


Γ_D
$x_{\mathcal{K}}$
$\mathcal{K}^*$
$\mathcal{K}$
$x_{\mathcal{K}^*}$
Γ
The boundary dual cell $\mathcal{K}^*$
Node of the boundary dual cell $x_{\mathcal{K}^*}$
Interior node $x_{\mathcal{K}^*}$ of the dual cell
Primal node $x_{\mathcal{K}}$
Primal cell $\mathcal{K}$
Interior dual cell $\mathcal{K}^*$

DDFV Meshes



Diamond mesh $\mathfrak{D}$



$$\nabla^{\mathcal{D}} u^{\mathcal{T}} = \frac{1}{\sin\,\alpha_{\mathcal{D}}}\left(\frac{u_{\mathcal{L}} - u_{\mathcal{K}}}{m_{\sigma^*}}\vec{n}_{\sigma\mathcal{K}} + \frac{u_{\mathcal{L}^*} - u_{\mathcal{K}^*}}{m_{\sigma}}\vec{n}_{\sigma^*\mathcal{K}^*}\right)$$

Outline

First idea

- Adaptation the m-DDFV scheme to mixed Dirichlet/Fourier boundary conditions.

$$-\mathrm{div}(A \cdot \nabla u) = f, \quad \text{in } \Omega, \tag{2a}$$

$$u = h, \quad \text{on } \partial\Omega \setminus \Gamma, \tag{2b}$$

$$-(A \cdot \nabla u, \vec{n}) = \lambda u - g, \;\text{ on } \Gamma. \tag{2c}$$

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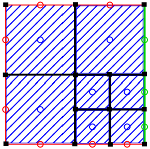
  - On the primal mesh :
    - Integrate the equation (2a) on interior primal cell $\kappa \in \mathfrak{M}$,
    - Impose the Fourier boundary condition (2c) on $\kappa \in \partial\mathfrak{M}_\Gamma$,
    - Impose the Dirichlet boundary condition (2b) on $\kappa \in \partial\mathfrak{M}_D$.



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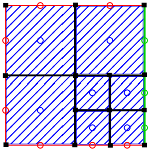
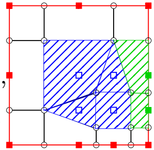


- On the dual mesh :
  - Integrate the equation (2a) on interior dual cell $\kappa^* \in \mathfrak{M}^*$,
  - Integrate the equation (2a) on boundary dual cell $\kappa^* \in \partial\mathfrak{M}^*_\Gamma$,
  - Impose the Dirichlet boundary condition (2b) on $\kappa^* \in \partial\mathfrak{M}^*_D$.



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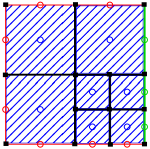

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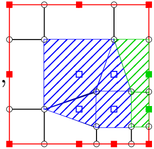
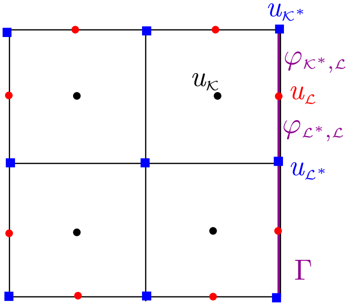


- The corresponding Schwarz algorithm does not converge to the m-DDFV scheme.

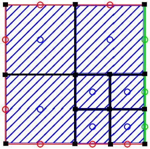
Second Idea : new unknowns

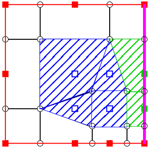
- New flux unknowns are used $\varphi_{\mathcal{K}^*,\mathcal{L}}$ on half edges in Γ.



m-DDFV with mixed boundary conditions

- m-DDFV scheme





$$\begin{cases} u_{\mathcal{K}} = h_{\mathcal{K}}, \quad \forall\, \mathcal{K} \in \partial\mathfrak{M}_D, \qquad u_{\mathcal{K}^*} = h_{\mathcal{K}^*}, \quad \forall\, \mathcal{K}^* \in \partial\mathfrak{M}^*_D, \\ -\mathrm{div}^{\mathcal{K}}(A^{\mathfrak{D}} \cdot \nabla^{\mathfrak{D}} u^{\mathcal{T}}) = f_{\mathcal{K}}, \quad \forall\, \mathcal{K} \in \mathfrak{M}, \\ -\mathrm{div}^{\mathcal{K}^*}(A^{\mathfrak{D}} \cdot \nabla^{\mathfrak{D}} u^{\mathcal{T}}) = f_{\mathcal{K}^*}, \quad \forall\, \mathcal{K}^* \in \mathfrak{M}^*, \\ -\sum_{\mathcal{D}\in\mathfrak{D}_{\mathcal{K}^*}} \frac{m_{\sigma^*}}{m_{\mathcal{K}^*}} (A_{\mathcal{D}} \cdot \nabla^{\mathcal{D}} u^{\mathcal{T}}, \vec{n}_{\mathcal{K}^*}) - \sum_{\substack{\mathcal{D}\in\mathfrak{D}_{\mathcal{K}^*} \\ \mathcal{D}\cap\Gamma\neq\emptyset}} \frac{m_{\mathcal{K}^*,\mathcal{L}}}{m_{\mathcal{K}^*}} \varphi_{\mathcal{K}^*,\mathcal{L}} = f_{\mathcal{K}^*}, \quad \forall\, \mathcal{K}^* \in \partial\mathfrak{M}^*_{\Gamma}, \\ \frac{m_{\mathcal{K}^*,\mathcal{L}}}{m_\sigma} \varphi_{\mathcal{K}^*,\mathcal{L}} + \frac{m_{\mathcal{L}^*,\mathcal{L}}}{m_\sigma} \varphi_{\mathcal{L}^*,\mathcal{L}} - (A_{\mathcal{D}} \cdot \nabla^{\mathcal{D}} u^{\mathcal{T}}, \vec{n}_{\sigma\mathcal{L}}) = 0, \quad \forall\, \mathcal{L} = [x_{\mathcal{K}^*} x_{\mathcal{L}^*}] \in \partial\mathfrak{M}_{\Gamma}, \\ \varphi_{\mathcal{K}^*,\mathcal{L}} + \lambda \frac{u_{\mathcal{K}^*} + u_{\mathcal{L}}}{2} = g_{\mathcal{K}^*,\mathcal{L}}, \quad \forall\, [x_{\mathcal{K}^*} x_{\mathcal{L}}] \in \partial\mathfrak{A}_{\Gamma}. \end{cases}$$

We shall write for short

$$\mathcal{L}^{\mathcal{T}}_{\Omega,\Gamma}(u^{\mathcal{T}}, \varphi^{\mathcal{T}}, f^{\mathcal{T}}, h^{\mathcal{T}}, g^{\mathcal{T}}) = 0. \tag{3}$$

m-DDFV

Boyer-Hubert 08

Theorem (Error estimate for m-DDFV, with Dirichlet boundary condition)

If $u_e \in H^2(\mathfrak{D})$ *solution of* (1), *then* $u^{\mathcal{T}}$ *and* $\nabla^{\mathfrak{D}} u^{\mathcal{T}}$ *are first order approximations of* u_e *and* ∇u_e*, respectively, in the* L^2 *norm.*

Boyer-Hubert-Krell Preprint

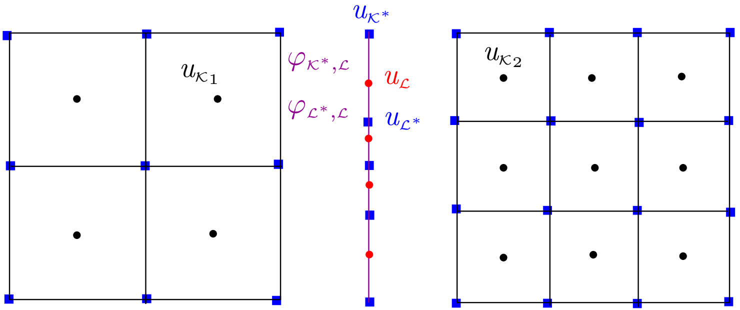
Theorem (Existence and uniqueness, with mixed boundary condition)

The scheme (3) *possesses a unique solution* $U^{\mathcal{T}} = (u^{\mathcal{T}}, \varphi^{\mathcal{T}}) \in \mathbb{R}^{\mathcal{T}} \times \Phi_{\Gamma}^{\mathcal{T}}$.

Outline

derivation of the S-DDFV algorithm

► The new flux unknows $\varphi_{\mathcal{K}^*,\mathcal{L}}$ are used on half edges in the interface Γ to discretize the condition (2c).



Iterative domain decomposition solver

- Choose $g_0^{\mathcal{T}_i} \in \Phi_\Gamma^{\mathcal{T}_i}$, $\forall\ i \in \{1,2\}$.
- $\forall\ n \geq 0,\ i \in \{1,2\}, j \neq i$:
  - Calculate $u_{n+1}^{\mathcal{T}_i}, \varphi_{n+1}^{\mathcal{T}_i}$ satisfying

$$\mathcal{L}_{\Omega_i,\Gamma}^{\mathcal{T}_i}(u_{n+1}^{\mathcal{T}_i}, \varphi_{n+1}^{\mathcal{T}_i}, f^{\mathcal{T}_i}, h^{\mathcal{T}_i}, g_n^{\mathcal{T}_j}) = 0. \tag{4}$$

  - Calculate $g_{n+1}^{\mathcal{T}_i}$ by

$$\forall\ [x_{\mathcal{K}^*} x_{\mathcal{L}}] \in \partial\mathfrak{A}_\Gamma, \quad g_{i,\mathcal{K}^*,\mathcal{L}}^{n+1} = -\varphi_{i,\mathcal{K}^*,\mathcal{L}}^{n+1} + \lambda \frac{u_{i,\mathcal{K}^*}^{n+1} + u_{i,\mathcal{L}}^{n+1}}{2} \tag{5}$$

Proposition

The initial data $g_0^{\mathcal{T}_i}$ being given, Algorithm (4)-(5) defines a unique sequence $(U_n^{\mathcal{T}_i})_n$ in $\mathbb{R}^{\mathcal{T}_i} \times \Phi_\Gamma^{\mathcal{T}_i}$, for $i = 1, 2$.

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- Express the m-DDFV scheme as limit problem of the S-DDFV algorithm.

m-DDFV scheme as limit of S-DDFV

Theorem

$\mathcal{T} = (\mathcal{T}_i)$ a mesh associated to the domain decomposition $\Omega = \cup_i \Omega_i$. $u^{\mathcal{T}}$ solution of the m-DDFV scheme on the mesh $\mathcal{T}$ with homogeneous Dirichlet condition.

$\forall\ i \in \{1,2\}$, $\exists\ (u^{\mathcal{T}_i}, \varphi^{\mathcal{T}_i}, g^{\mathcal{T}_i}) \in \mathbb{R}^{\mathcal{T}_i} \times \Phi_\Gamma^{\mathcal{T}_i} \times \Phi_\Gamma^{\mathcal{T}_i}$ *such that*

$$\begin{cases} \mathcal{L}_{\Omega_i,\Gamma}^{\mathcal{T}_i}(u^{\mathcal{T}_i}, \varphi^{\mathcal{T}_i}, f^{\mathcal{T}_i}, h^{\mathcal{T}_i}, g^{\mathcal{T}_i}) = 0, \\ u_{i,\mathcal{K}} = u_{\mathcal{K}}, \quad \textit{for } \mathcal{K} \in \mathfrak{M}_i \cup \partial\mathfrak{M}_{i,D}, \\ u_{i,\mathcal{K}^*} = u_{\mathcal{K}^*}, \quad \textit{for } \mathcal{K}^* \in \mathfrak{M}_i^* \cup \partial\mathfrak{M}_i^*, \\ \displaystyle\sum_{k=1}^{N} \left(\varphi_{i,\mathcal{K}_k^*,\mathcal{L}_k} - \varphi_{i,\mathcal{K}_{k+1}^*,\mathcal{L}_k} \right) = 0. \end{cases} \tag{6}$$

Convergence of the Schwarz algorithm

Boyer-Hubert-Krell Preprint

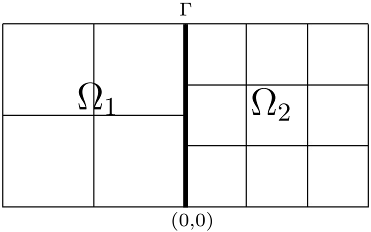
Theorem

- $\forall\ g_0^{\mathcal{T}_i} \in \Phi_\Gamma^{\mathcal{T}_i},\ i \in \{1,2\}$,
  - $(u_n^{\mathcal{T}_i})_{i=1,2}$ *solution of* (4)-(5) *converges towards* $u^{\mathcal{T}}$ *m-DDFV solution when* $n \longrightarrow \infty$.
  - $$\textit{If } \sum_{k=1}^{N} \left(g^0_{i,\mathcal{K}^*_k,\mathcal{L}_k} - g^0_{i,\mathcal{K}^*_{k+1},\mathcal{L}_k} \right) = \frac{\lambda}{2} \left(h_{\mathcal{K}^*_1} - h_{\mathcal{K}^*_{N+1}} \right), \quad i = \{1,2\},$$
    then, $\varphi^n_{i,\mathcal{K}^*,\mathcal{L}}$ *also converges towards* $\varphi^{\mathcal{T}}_{i,\mathcal{K}^*,\mathcal{L}}$ *when* $n \longrightarrow \infty$.
  - U^n *converges* $U^{\mathcal{T}}$ *when* $n \longrightarrow \infty$.

Outline

Case 1 - Two domains

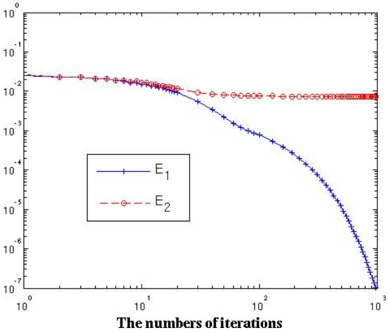
Mesh1



$$u_e(x,y) = \sin(\pi x)\sin(\pi y)\sin(\pi(x+y)),$$

$$A(x,y) = \begin{cases} \begin{pmatrix} 1.5 & 0.5 \\ 0.5 & 1.5 \end{pmatrix} & \text{if } x < 0 \\ \begin{pmatrix} 1.5 & 0.5 \\ 0.5 & 1 \end{pmatrix} & \text{if } x > 0. \end{cases}$$

Comparison between $E_1 = \dfrac{||u_n^{\mathcal{T}_i} - u^{\mathcal{T}_i}||_2}{||u^{\mathcal{T}_i}||_2}$ and $E_2 = \dfrac{||u_n^{\mathcal{T}_i} - u_e||_2}{||u_e||_2}$ as a function of the number of iterations. $\lambda = 160$



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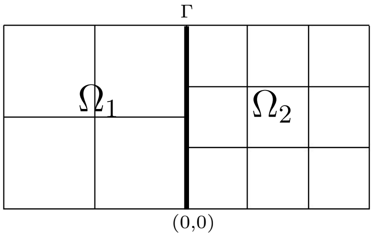

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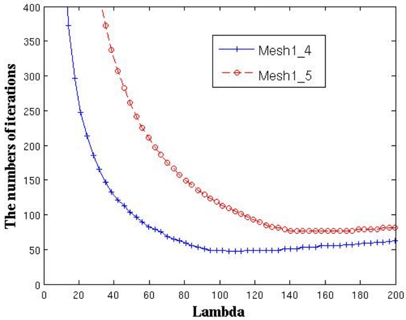
The number of necessary iterations as a function of λ.

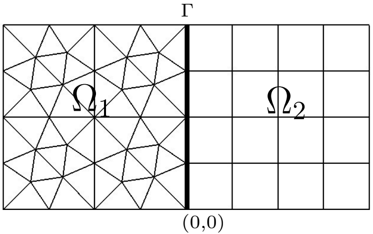


Stopping criterion

$$\frac{||u_n^{\mathcal{T}_i} - u^{\mathcal{T}_i}||_2}{||u^{\mathcal{T}_i}||_2} < 0.1 \frac{||u^{\mathcal{T}_i} - u_e||_2}{||u_e||_2}$$

Case 2 - two domains

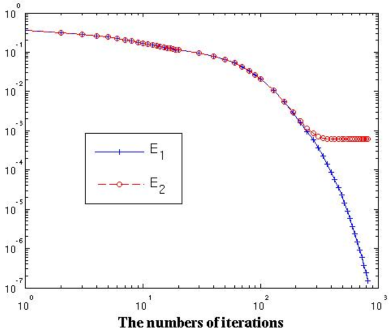
Mesh2



$$u_e(x,y) = \cos(2.5\pi x)\cos(2.5\pi y),$$

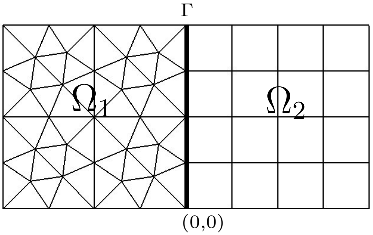
$$A(x,y) = \begin{cases} \begin{pmatrix} 1.5 & 0.5 \\ 0.5 & 1.5 \end{pmatrix} & \text{if } x < 0 \\ \begin{pmatrix} 1.5 & 0.5 \\ 0.5 & 1 \end{pmatrix} & \text{if } x > 0. \end{cases}$$

Comparison between $E_1 = \dfrac{||u_n^{\mathcal{T}_i} - u^{\mathcal{T}_i}||_2}{||u^{\mathcal{T}_i}||_2}$ and $E_2 = \dfrac{||u_n^{\mathcal{T}_i} - u_e||_2}{||u_e||_2}$ as a function of the number of iterations. $\lambda = 200$



Case 2 - two domains

Mesh2



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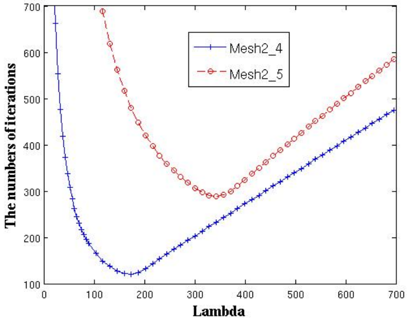
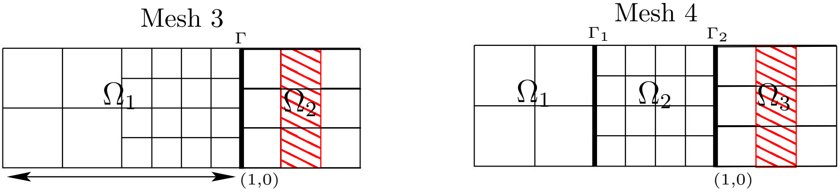


Stopping criterion

$$\frac{||u_n^{\mathcal{T}_i} - u^{\mathcal{T}_i}||_2}{||u^{\mathcal{T}_i}||_2} < 0.1 \frac{||u^{\mathcal{T}_i} - u_e||_2}{||u_e||_2}$$

Case 3 - Influence of the numbers of subdomains

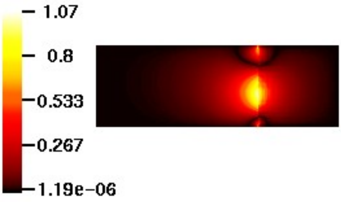
$$f(x,y) = \begin{cases} -1000\sin(2.5\pi(x-1.3)) & \text{if } 1.3 < x < 1.7, \\ 0 & \text{else.} \end{cases}$$



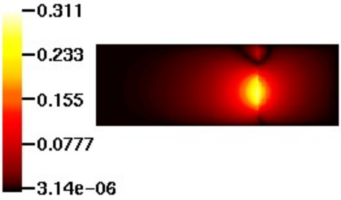
Primal meshes, $\lambda = 250$, Error $abs(u^n - u^{\mathcal{T}})$

Mesh3_5

n=1

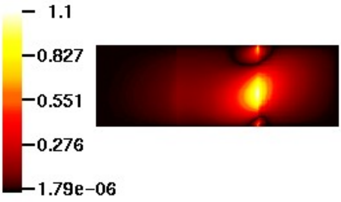


n=11

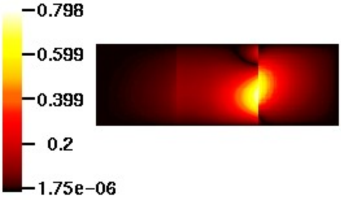


Mesh4_5

n=1



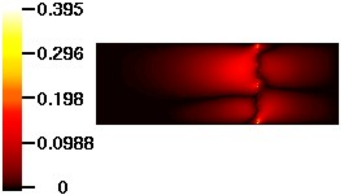
n=11



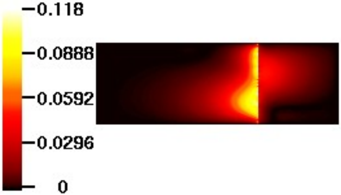
Dual meshes, $\lambda = 250$, Error $abs(u^n - u^{\mathcal{T}})$

Mesh3_5

n=1

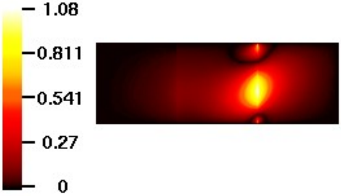


n=11

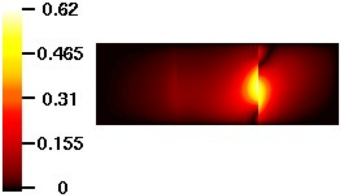


Mesh4_5

n=1

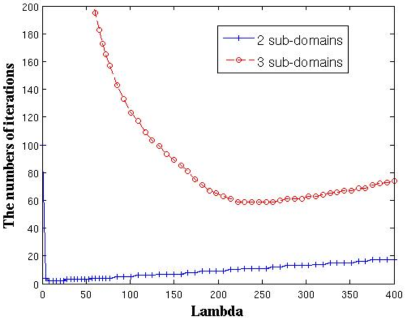


n=11



Influence of parameter λ

The number of necessary iterations as a function of λ.



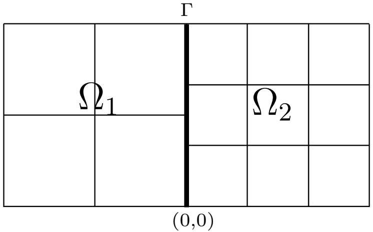
Stopping criterion

$$\frac{||u_n^{\mathcal{T}_i} - u^{\mathcal{T}_i}||_2}{||u^{\mathcal{T}_i}||_2} < 0.01 \frac{||u^{\mathcal{T}_i} - u_e||_2}{||u_e||_2}$$

Case 4 - Preconditioner

Comparison between the number of iterations required by solving the DDFV scheme with

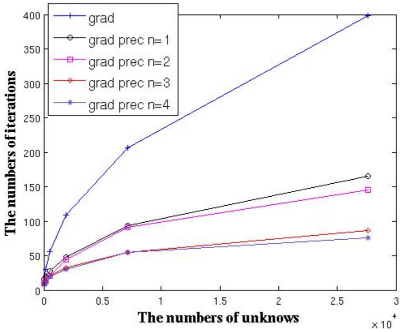
- the conjugate gradient
- the conjugate gradient preconditioned with n iterations of the S-DDFV algorithm.



$$u_e(x,y) = 16y(1-y)(1-x^2),$$

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\lambda = 205\ .$$



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