Finite Volume Scheme Satisfying Maximum and Minimum Principles for Anisotropic Diffusion Operators

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Outline

- Presentation
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Presentation

Let Ω be an open bounded convex subset of $\mathcal{R}^2$. We consider the following problem:

$$\begin{cases} \vec{q} = \overline{\overline{D}}\vec{\nabla}C \\ \omega\dfrac{\partial C}{\partial t} = div\vec{q} \text{ on } \Omega \ \forall \mathrm{t} > 0 \end{cases} \tag{1}$$

with

- ω, the porosity
- C, the radioactive element concentration
- $\overline{\overline{D}}$, a symmetric definite positive matrix
- Dirichlet boundary conditions

Bibliography

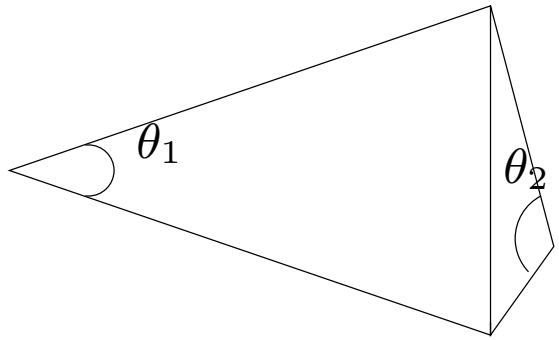
Finite volume scheme for diffusion operator

- I. Aavastsmak (98,04)
- F. Dubois (99), Coudière(99), F. Hermeline (00)
- R. Eymard, T. Gallouët, R. Herbin (00,04,07)
- Domelevo and P. Omnès (05), C.L.P(05)
- G. Manzini (06), C. Pierre (06), J. Droniou (06), S. Delcourte(06)
- B. Andreianov, F. Hubert and F. Boyer (07)
- L. Agelas, R. Masson, A. Di Pietro (08)
- ...

Bibliography (monotony)

1) standard linear schemes

In isotropic case : geometry assumptions. For example, for triangular cells :



Angle condition : $\theta_1 + \theta_2 \leq \pi$ for F.E and for F.V. (FV4). The sum of opposite angles of a triangle edge must be less or equal to π

Bibliography (monotony)

2) Schemes satisfying (or improving) monotony or maximum principles

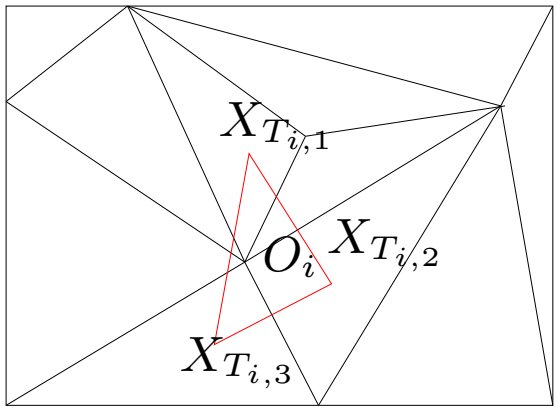
- Burman and Ern (04) non linear correction of Finite Elements
- C.L.P (05) (VFMON) Monotone but not DMMP
- G.Manzini and M. Putti (06, non linear scheme for FV "Diamond" scheme)
- K. Lipnikov, M. Shaskhov, D. Svyatskiy Y. Vassilevski(07)
- I. Kapyrin (07)
- J.M. Nordbotten, I Avvastamark, G.T. Eigestad (07)
- R. Eymard, T. Gallouët, R. Herbin (08) : SUSHIP scheme
- G. Yuan, Z. Sheng, JCP (June 08)

Isotropic homogenous case

We consider a grid $\mathcal{T}$ made up of N_{ma} triangular cells and N_f boundary edges.

- The unknowns are calculated at the intersection of the angle bisectors of each triangular cell
- Linear monotone reconstruction at the nodes of the grid.
- Non-linear calculation of the flux

Isotropic homogenous case



Grid composed of triangular cells

- So, there exists three positive or zero coefficients $\lambda_{i,j;j=1,2,3}$ with $\sum_{\{1\leq j\leq 3\}}\lambda_{i,j}=1$, such that $\vec{OO}_i=\sum_{\{1\leq j\leq 3\}}\lambda_{i,j}\vec{OX}_{T_{i,j}}$ (Figure 8).
- We approximate the value of the concentration C at the point O_i with the expression $C_{O_i}=\sum_{\{1\leq j\leq 3\}}\lambda_{i,j}C_{T_{i,j}}$
- We can construct several triangular cells which satisfy the desired property. So, we can introduce several points $X_{T_{i,j},j=1,..,N}$ and $\lambda_{i,j;j=1,..,N}$ (with $N>3$) such that $\vec{OO}_i=\sum_{\{1\leq j\leq N\}}\lambda_{i,j}\vec{OX}_{T_{i,j}}$.

Isotropic homogenous case

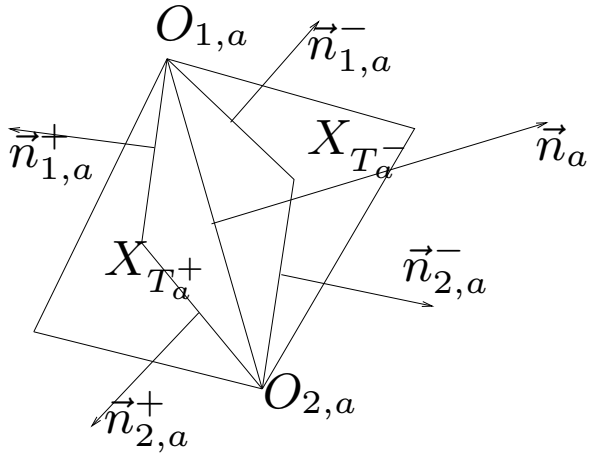
- Least square method. The coefficients $\lambda_{i,j}$ can be negative
- Other solution:
  One solves $\Delta C = 0$. Integrating on a disk around the node O_i,
  we obtain:

$$C_{O_i} = \sum_{j \in V_{O_i}} \lambda_j C_{A_j}$$

  with $\lambda_j \geq 0$. We obtain a linear exact formulation because the points A_j are the intersections of angle bisectors. Unfortunately, we can not generalize this method to heteregeneous cases.

Isotropic homogenous case

Calculation of the flux $\vec{q}.\vec{n_a}$



For an edge a belonging to $\mathcal{A}_{int}$, Green's formula :

$$\int_{(O_{1,a},X_{T_a^+},X_{T_a^-})} \vec{q} d\Omega = \int_{(O_{1,a},X_{T_a^+},X_{T_a^-})} \vec{\nabla} C d\Omega = \int_{\partial(O_{1,a},X_{T_a^+},X_{T_a^-})} C\vec{n} d\Gamma \tag{2}$$

Second order in space formula:

$$\vec{q}_{1,a} = \frac{1}{2SF_{1,a}} C_{O_{1,a}} (\vec{n}^+_{1,a} + \vec{n}^-_{1,a}) + \frac{1}{2SF_{1,a}} (-C_{T_a^-} \vec{n}^+_{1,a} - C_{T_a^+} \vec{n}^-_{1,a}) \tag{3}$$

Isotropic homogenous case

Integration on $(O_{2,a}, X_{T_a^+}, X_{T_a^-})$ leads to :

$$\vec{q}_{2,a} = \frac{1}{2SF_{2,a}} C_{O_{2,a}} (\vec{n}_{2,a}^+ + \vec{n}_{2,a}^-) + \frac{1}{2SF_{2,a}} (-C_{T_a^-} \vec{n}_{2,a}^+ - C_{T_a^+} \vec{n}_{2,a}^-) \quad (4)$$

The terms $\vec{q}_{1,a}.\vec{n}_a$ and $\vec{q}_{2,a}.\vec{n}_a$ can be written:

$$\vec{q}_{1,a}.\vec{n}_a = \gamma_1 (C_{O_{1,a}} - C_{T_a^+}) + \beta_1 (C_{T_a^-} - C_{T_a^+}) \quad (5)$$

and

$$\vec{q}_{2,a}.\vec{n}_a = \gamma_2 (C_{O_{2,a}} - C_{T_a^-}) + \beta_2 (C_{T_a^-} - C_{T_a^+}) \quad (6)$$

with β_1, β_2 positive coefficients, $\gamma_1 \gamma_2 \leq 0$.
Any combination $(\mu_{1,a}, \mu_{2,a})$ with $\mu_{1,a} + \mu_{2,a} = 1$, and
$\vec{q}.\vec{n}_a = \mu_{1,a} \vec{q}_{1,a}.\vec{n}_a + \mu_{2,a} \vec{q}_{2,a}.\vec{n}_a$ gives a consistent formulation.

Isotropic homogenous case

Let us assume $\gamma_2 \leq 0$, $\gamma_1 \geq 0$

- First solution : $\mu_{1,a} = \mu_{2,a} = 0.5$. It leads to the so-called "diamond" scheme (Y. Coudière).
- Second solution : $\mu_{1,a} = \dfrac{-\gamma_2 C_{O_{2,a}}}{\gamma_1 C_{O_{1,a}} - \gamma_2 C_{O_{2,a}}}$ and

  $\mu_{2,a} = \dfrac{\gamma_1 C_{O_{1,a}}}{\gamma_1 C_{O_{1,a}} - \gamma_2 C_{O_{2,a}}}$. It leads to the $VFMON$ scheme (C.L.P)
- Third solution :

  $\mu_{1,a} = \dfrac{|\vec{q}_{2,a}.\vec{n}_a|}{|\vec{q}_{2,a}.\vec{n}_a| + |\vec{q}_{1,a}.\vec{n}_a|}$, $\mu_{2,a} = \dfrac{|\vec{q}_{1,a}.\vec{n}_a|}{|\vec{q}_{2,a}.\vec{n}_a| + |\vec{q}_{1,a}.\vec{n}_a|}$
- Fourth solution :

  $\mu_{1,a} = \dfrac{-\gamma_2 |C_{O_{2,a}} - C_{T_a^-}|}{-\gamma_2 |C_{O_{2,a}} - C_{T_a^-}| + \gamma_1 |C_{O_{1,a}} - C_{T_a^+}|}$ and

  $\mu_{2,a} = \dfrac{\gamma_1 |C_{O_{1,a}} - C_{T_a^+}|}{-\gamma_2 |C_{O_{2,a}} - C_{T_a^-}| + \gamma_1 |C_{O_{1,a}} - C_{T_a^+}|}$

Isotropic homogenous case

If the edge a belongs to $\mathcal{A}_{ext}$

$$\vec{q}.\vec{n_a} = \{\frac{1}{2SF_a} C_{T_{a^+}} (\vec{n}^+_{1,a} + \vec{n}^+_{2,a}) + \frac{1}{2SF_a} (-C_{O_{1,a}} \vec{n}^+_{2,a} - C_{O_{2,a}} \vec{n}^+_{1,a})\}.\vec{n_a} \quad (7)$$

Calculation of the main unknown C_T

Integration of the mass conservation equation (the second equation of system (1)) over T.

$$\int_T \omega \frac{\partial C}{\partial t} d\Omega = S(T)\omega \frac{\partial C_T}{\partial t} = \int_T div\vec{q} d\Omega = \int_{\partial T} \vec{q}.\vec{n} d\Gamma = \sum_{j=1}^{j=3} \vec{q}.\vec{n}_{T,j} \quad (8)$$

Isotropic homogenous case

Properties of the algorithm

We choose an implicit time scheme. The discretization of the equation (8) leads to: $(SF\omega - \Delta t A(C^{n+1}))C^{n+1} = SF\omega C^n$ and

we denote $M = SF\omega - \Delta t A(C^{n+1})$.

Proposition *For any triangular mesh, the matrix M is an M-matrix*

For an edge $a \in \mathcal{A}_{int}$, we prove that the flux $\vec{q}.\vec{n}_a$ can be written:

$$\vec{q}.\vec{n}_a = f_1(C_{T_a^-} - C_{T_a^+}) + g_1(C_{O_{1,a}} - C_{T_a^+})$$

with $f_1(C_{O_{1,a}}, C_{O_{2,a}}, C_{T_a^-}, C_{T_a^+})$ and $g_1(C_{O_{2,a}}, C_{T_a^-})$ two positive functions. This is the flux coming from the cell T_a^+. Moreover, the flux $\vec{q}.(-\vec{n}_a)$ can be written:

$$\vec{q}.(-\vec{n}_a) = f_2(C_{T_a^+} - C_{T_a^-}) + g_2(C_{O_{2,a}} - C_{T_a^-})$$

with $f_2(C_{O_{1,a}}, C_{O_{2,a}}, C_{T_a^+}, C_{T_a^-})$ and $g_2(C_{O_{1,a}}, C_{T_a^+})$ two positive functions. This is the flux coming from the cell T_a^-.

Isotropic homogenous case

If $a \in \mathcal{A}_{ext}$, $\vec{q}.\vec{n}_a$ can be rewritten:

$$\vec{q}.\vec{n}_a = f_3(C_{O_{1,a}} - C_{T_a^+}) + f_4(C_{O_{2,a}} - C_{T_a^+})$$

with f_3 and f_4 which are also positive functions.
In any case :

$$\vec{q}.\vec{n}_a = \sum_{j \in V(T_a^+)} f_{5,j}(C_j - C_{T_a^+})$$

and

$$\vec{q}.(-\vec{n}_a) = \sum_{j \in V(T_a^-)} f_{6,j}(C_j - C_{T_a^-})$$

where $f_{5,j}$ and $f_{6,j}$ are positive.
We conclude that the matrix M is an M-matrix

Isotropic homogenous case

Remark 1 *The flux approximation is consistent because we use a second order in space formula.*

Remark 2 *Using properties of M-matrices, the previous scheme, denoted VFPMMD, satisfies discrete maximum and minimum principles.*

Remark 3 $\mu_{1,a}$ *and* $\mu_{2,a}$ *are not smooth. If we replace, for example, the expression* $|C_{O_{1,a}} - C_{T_a^+}|$ *by* $\{C_{O_{1,a}} - C_{T_a^+}\}^2$*, the resulting global matrix is no longer an M-matrix.*

Anisotropic heteregenous case

Let be $\overline{\overline{D}}_{T_a^+} = \frac{\int_{T_a^+} \overline{\overline{D}} d\Omega}{\int_{T_a^+} d\Omega}$. Let $\overline{\overline{D}}^{0.5}_{T_a^+} T_a^+$ be the image of T_a^+ with vertices $O_{1,a} O'_{2,a} O'_3$ ($O_{1,a}\vec{O'}_{2,a}{}^{\perp} = \overline{\overline{D}}^{0.5}_{T_a^+} O_{1,a}\vec{O}_{2,a}{}^{\perp}$ and $O_{1,a}\vec{O'}_3{}^{\perp} = \overline{\overline{D}}^{0.5}_{T_a^+} O_{1,a}\vec{O}_3{}^{\perp}$).

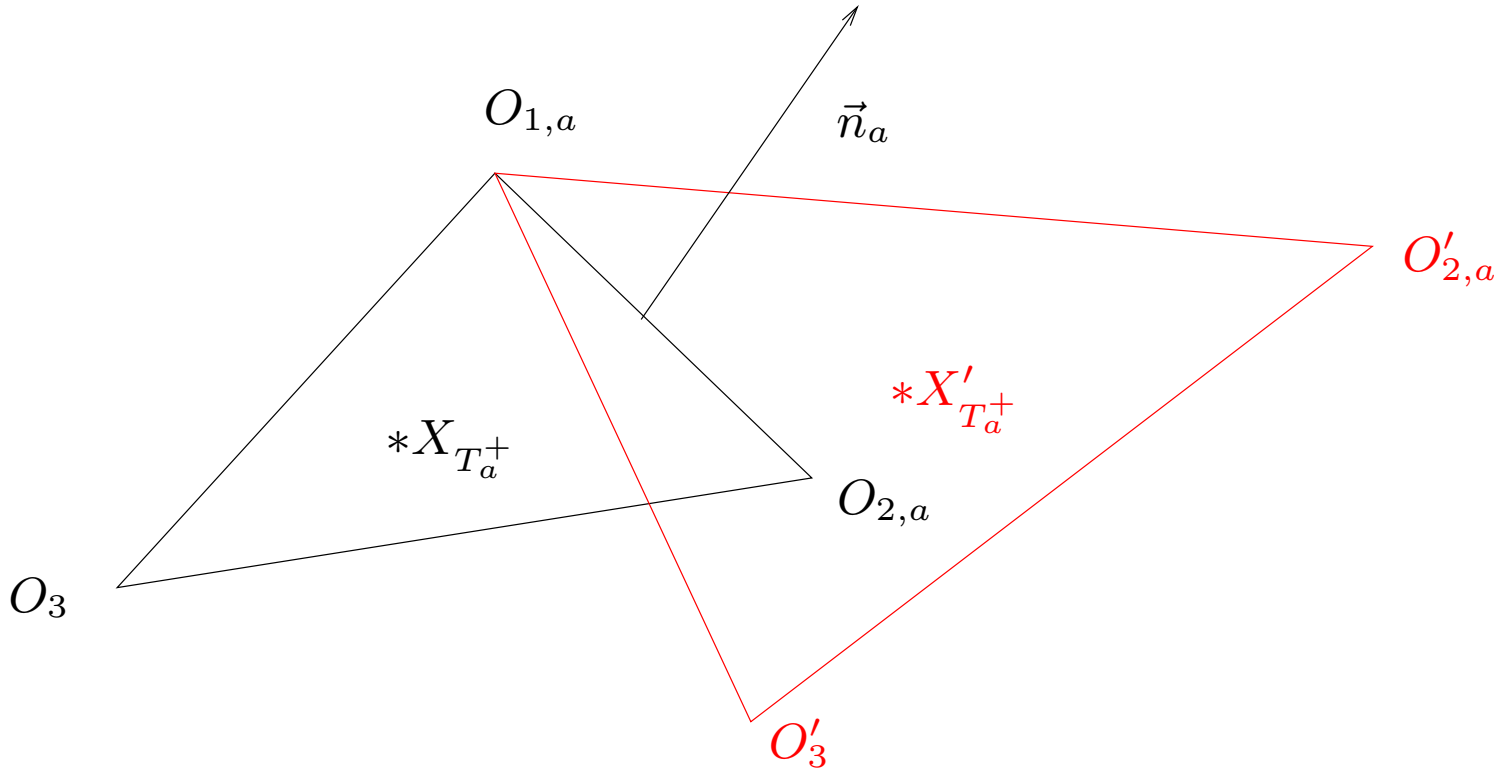


Image of a triangular cell by the application $\overline{\overline{D}}^{0.5}_{T_a^+}$

Anisotropic heteregenous case

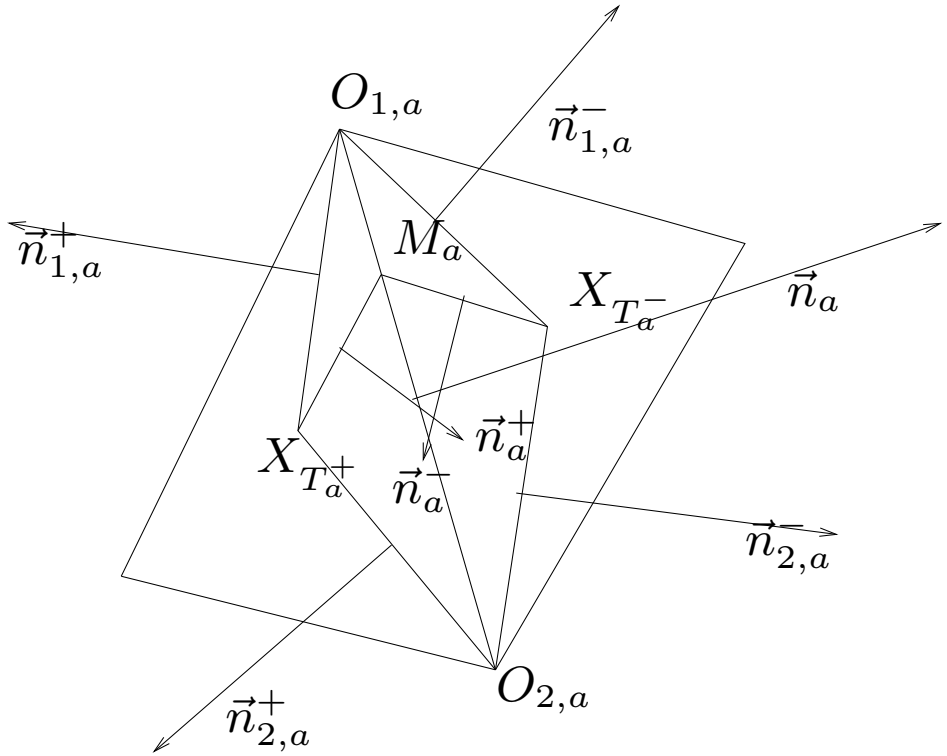
The point $X'_{T_a^+}$ is the center of the circle inscribed in the triangular cell $\overline{\overline{D}}^{0.5}_{T_a^+} T_a^+$. We define the point $X_{T_a^+}$ as the inverse image of the point $X'_{T_a^+}$. It satisfies the equality:

$$\vec{O_{1,a}X}_{T_a^+} = \frac{\vec{O_{1,a}O}_{2,a}|O_{1,a}O'_3| + \vec{O_{1,a}O}_3|O_{1,a}O'_{2,a}|}{|O'_{2,a}O'_3| + |O_{1,a}O'_3| + |O_{1,a}O'_{2,a}|}$$

Property of the point $X_{T_a^+}$

- the projection of the point $X_{T_a^+}$ in the direction $\overline{\overline{D}}_{T_{a^+}} \vec{n}_a$ on the edge $O_{1,a}, O_{2,a}$ is between the two nodes of the edge $O_{1,a}, O_{2,a}$.

Anisotropic heteregenous case



- The point M_a is the projection of the point $X_{T_a^+}$ in the direction $\overline{\overline{D}}_{T_a+}\vec{n}_a$ on the edge $O_{1,a}, O_{2,a}$ (using the previous proposition, this projection is inside the edge $O_{1,a}, O_{2,a}$).
- $C_{M_{i,a}}$ the value of the concentration at the point M_a associated with $O_{i,a}$

Anisotropic heteregenous case

As in DDFV scheme (F. Hermeline, P. Omnès, F. Hubert, F. Boyer), four gradients:

$$\begin{cases} \overline{\overline{D}}_{T_a^+}^{-1} \vec{q}_{1,a}^{+} = \dfrac{1}{2SF_{1,a}^+}(C_{T_a^+} - C_{M_{1,a}})\vec{n}_{1,a}^{+} + \dfrac{1}{2SF_{1,a}^+}(C_{T_a^+} - C_{O_{1,a}})\vec{n}_a^{+} \\ \overline{\overline{D}}_{T_a^-}^{-1} \vec{q}_{1,a}^{-} = \dfrac{1}{2SF_{1,a}^-}(C_{T_a^-} - C_{M_{1,a}})\vec{n}_{1,a}^{-} + \dfrac{1}{2SF_{1,a}^-}(C_{T_a^-} - C_{O_{1,a}})\vec{n}_a^{-} \end{cases} \tag{9}$$

and for $i = 2$:

$$\begin{cases} \overline{\overline{D}}_{T_a^+}^{-1} \vec{q}_{2,a}^{+} = \dfrac{1}{2SF_{2,a}^+}(C_{T_a^+} - C_{M_{2,a}})\vec{n}_{2,a}^{+} - \dfrac{1}{2SF_{2,a}^+}(C_{T_a^+} - C_{O_{2,a}})\vec{n}_a^{+} \\ \overline{\overline{D}}_{T_a^-}^{-1} \vec{q}_{2,a}^{-} = \dfrac{1}{2SF_{2,a}^-}(C_{T_a^-} - C_{M_{2,a}})\vec{n}_{2,a}^{-} - \dfrac{1}{2SF_{2,a}^-}(C_{T_a^-} - C_{O_{2,a}})\vec{n}_a^{-} \end{cases} \tag{10}$$

Anisotropic heteregenous case

Calculation of $C_{M_{i,a}}$

We delete the additional unknowns $C_{M_{i,a}}$ by imposing flux continuity.
We conclude that $C_{M_{1,a}}$ and $C_{M_{2,a}}$ can be simplified in the following way:

$$C_{M_{1,a}} = \alpha_1 C_{T_a^+} + \beta_1 C_{T_a^-} + \gamma_1 C_{O_{1,a}}$$

and

$$C_{M_{2,a}} = \alpha_2 C_{T_a^+} + \beta_2 C_{T_a^-} + \gamma_2 C_{O_{2,a}}$$

with α_1, α_2, β_1, β_2 positive coefficients, $\gamma_1\gamma_2 \leq 0$, $\alpha_1 + \beta_1 + \gamma_1 = 1$ and $\alpha_2 + \beta_2 + \gamma_2 = 1$.

Anisotropic heteregenous case

Calculation of the flux $\vec{q}.\vec{n}_a$

We obtain :

$$\vec{q}_{1,a}.\vec{n}_a = \beta_1(C_{T_a^-} - C_{T_a^+}) + \gamma_1(C_{O_{1,a}} - C_{T_a^+})$$

and

$$\vec{q}_{2,a}.\vec{n}_a = \beta_2(C_{T_a^-} - C_{T_a^+}) + \gamma_2(C_{O_{2,a}} - C_{T_a^-})$$

- The same equalities were described in the homogeneous case.
- β_1 and β_2 are positive coefficients and $\gamma_1\gamma_2 \leq 0$.

We can choose for $a \in \mathcal{A}_{int}$, $\vec{q}.\vec{n}_a$ in the form:

$$\vec{q}.\vec{n}_a = \frac{-\gamma_2|C_{O_{2,a}} - C_{T_a^-}|\vec{q}_{1,a}.\vec{n}_a + \gamma_1|C_{O_{1,a}} - C_{T_a^+}|\vec{q}_{2,a}.\vec{n}_a}{-\gamma_2|C_{O_{2,a}} - C_{T_a^-}| + \gamma_1|C_{O_{1,a}} - C_{T_a^+}|} \qquad (11)$$

- The resulting global matrix is also an M-matrix.

Discrete system

We solve equation (8) with a fixed point algorithm. We get:

$$(SF\omega - \Delta t A(C^i))C^{i+1} = SF\omega C^i$$

where i is a fixed point iteration.

Method 1

We choose to calculate the following fluxes:

$$\vec{q}^{i+1}.\vec{n}_a = \frac{-\lambda_2|C^i_{O_{2,a}} - C^i_{T_a^-}|\vec{q}^{i+1}_{1,a}.\vec{n}_a + \lambda_1|C^i_{O^i_{1,a}} - C^i_{T_a^+}|\vec{q}^{i+1}_{2,a}.\vec{n}_a}{-\lambda_2|C^i_{O_{2,a}} - C^i_{T_a^-}| + \lambda_1|C^i_{O_{1,a}} - C^i_{T_a^+}|} \qquad (12)$$

- Local conservation and convergence of the scheme at any iteration.
- The minimum and the maximum principles are satisfied once the convergence criterion is achieved.

Discrete system

Method 2

Another technique consists of inverting M-matrix at any iteration. We calculate two fluxes at each face.

- Local conservation and convergence of the scheme are satisfied once the convergence criterion is achieved.
- Minimum and maximum principles at any iteration.

Remark

Remark : Semi explicit scheme. We solve, with the method 2 :

$$(SF\omega - \Delta t A(C^n))C^{n+1} = SF\omega C^n$$

One linear system to solve. We obtain (DMMP) but constraint on Δt.

Generalisation to 3 dimensions

The scheme (VFMON) has been developed by I. Kapyrin in 3 dimensions. Using the same kind of method, it should be easy to generalize the VFPMMD scheme with tetrahedrons.

Numerical results

Stationary solution : analytical solution

$$\left\{ \begin{array}{l} div(\overline{\overline{D}}\vec{\nabla}C) = S \text{ on } \Omega =]0, 0.5[\times]0, 0.5[\\ C = sin(\pi x)sin(\pi y) \ for \ (x, y) \in \partial\Omega \end{array} \right. \tag{13}$$

$$\overline{\overline{D}} = \left(\begin{array}{cc} y_1^2 + \epsilon x_1^2 & -(1-\epsilon)x_1 y_1 \\ -(1-\epsilon)x_1 y_1 & x_1^2 + \epsilon y_1^2 \end{array} \right) \tag{14}$$

where $x_1 = x + 10^{-3}$ and $y_1 = y + 10^{-3}$. The parameter ϵ is equal to 10^{-3}. The anisotropy ratio : 10^3. (This case is close to the test 5 of the benchmark)

- Five grids, made up of about 30 triangular cells (the first) to 38000 triangular cells (the fifth)
- $VFPMMD$ Non-linearity. Fixed point algorithm. 30 iterations for the fifth grid
- $VFMON$
- $VFDIAMOND$ (linear)
- $VFPMMD_{1iteration}$ in the fixed point algorithm.

Numerical results

h	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$	$\frac{1}{256}$
Error L^2	1.4×10^{-2}	4.14×10^{-3}	1.13×10^{-3}	4×10^{-4}	1×10^{-4}

Error L^2 as a function of the discretization step for (VFPMMD)

h	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$	$\frac{1}{256}$
Error L^2	1.0×10^{-2}	2.7×10^{-3}	7.2×10^{-4}	1.6×10^{-4}	4.0×10^{-5}

Error L^2 as a function of the discretization step (VFMON)

h	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$	$\frac{1}{256}$
Error L^2	8.6×10^{-3}	1.8×10^{-3}	4.95×10^{-4}	7.7×10^{-5}	2×10^{-5}

Error L^2 as a function of the discretization step (VFDIAMOND)

h	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$	$\frac{1}{256}$
Error L^2	1.05×10^{-2}	2.55×10^{-3}	5.9×10^{-4}	$1. \times 10^{-4}$	$3. \times 10^{-5}$

Error L^2 as a function of the discretization step (VFPMDD 1 iteration)

Numerical results

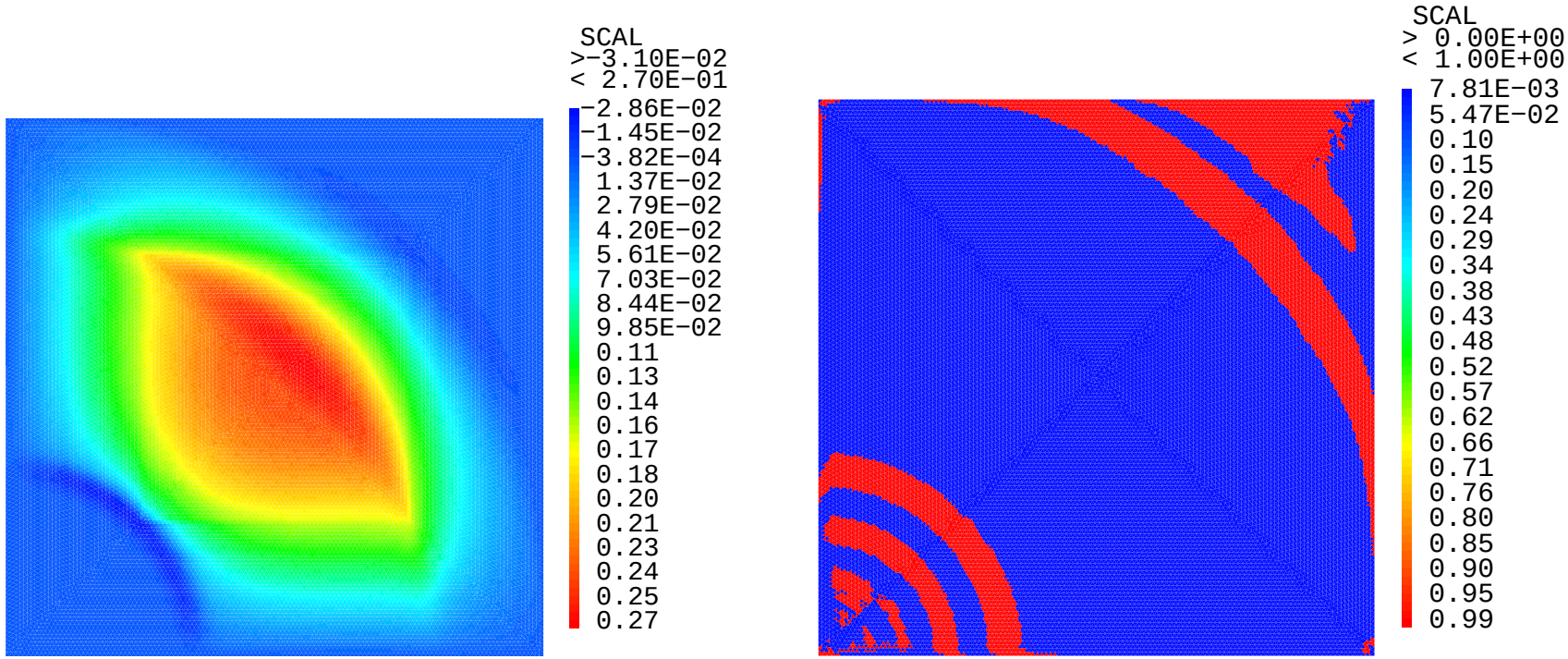
- Convergence of the 4 methods.
- $VFMON$ is a little more precise than $VFPMMD$
- $VFDIAMOND$ is the most precise scheme
- $VFMON$, $VFPMMD$ and $VFPMMD_{1iteration}$ satisfy DMP. (Not $VFDIAMOND$)

Numerical results

Stationary solution with discontinuous source

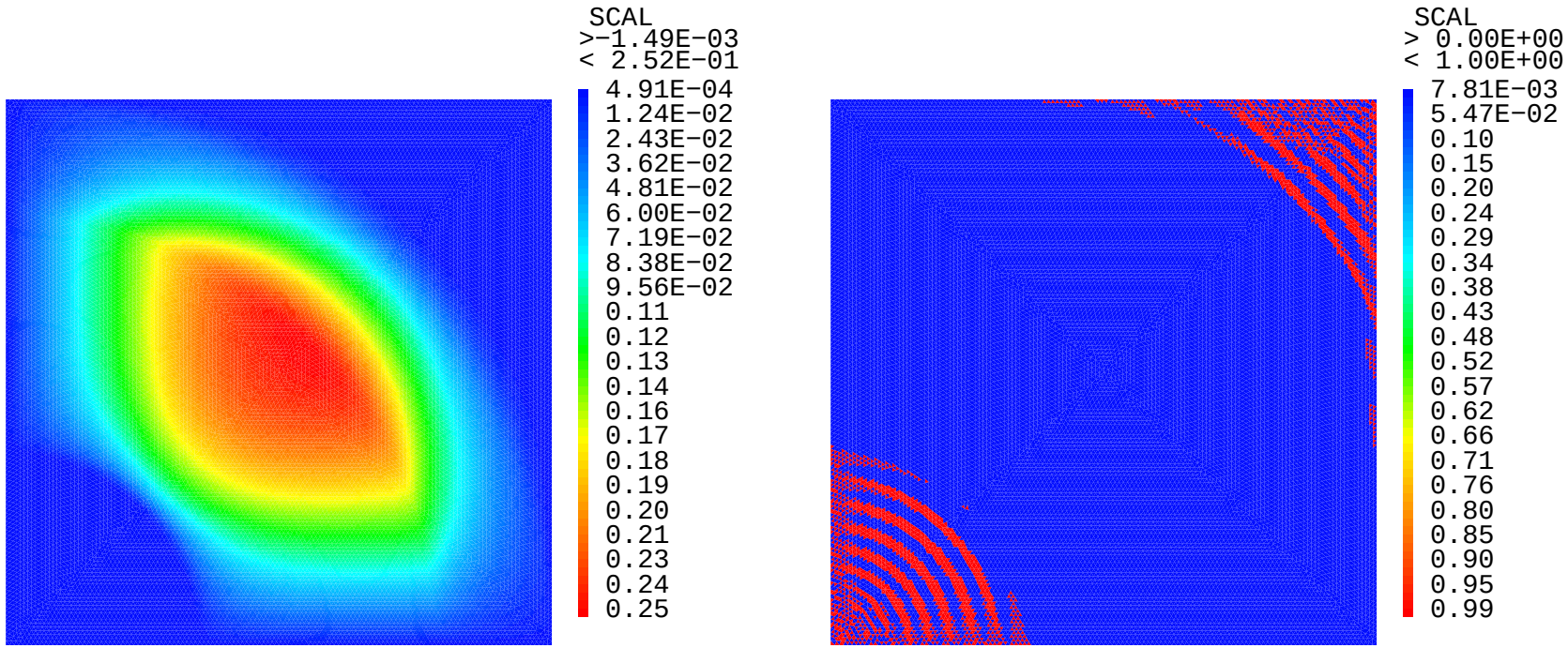
- $\Omega =]0, 0.5[\times]0, 0.5[$
- $S = 1$ on $]0.125, 0.375[\times]0.125, 0.375[$, and 0 elsewhere
- $C = 0$ on the boundary of Ω.
- Same diffusion tensor
- We show the concentrations and position of negative values obtained with M.H.F.E, VFSYM, F.E, VFMON and VFPMMD (F. Dabbène, G. Bernard-Michel, S. Gounand, C.L.P.).

Numerical results



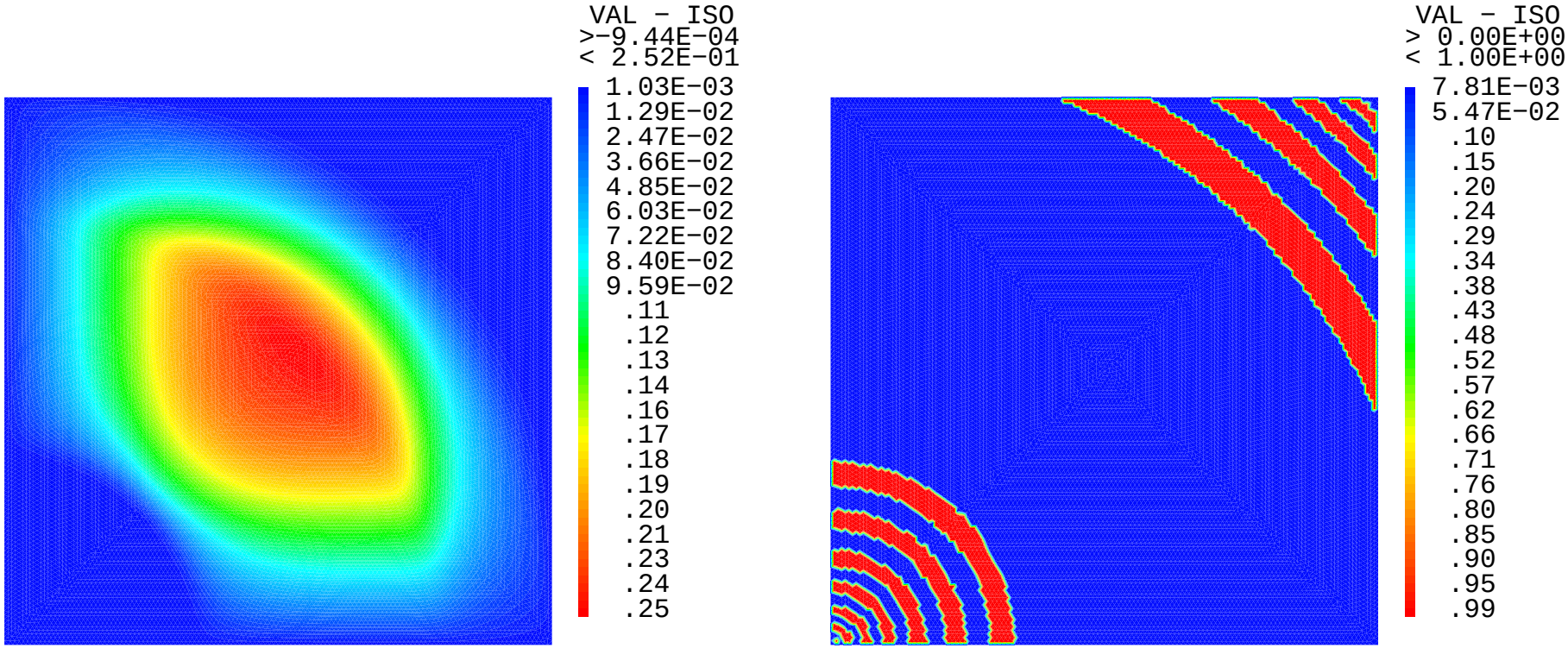
M.H.F.E (concentration and position of negative values) : 38000 meshes, minimum value -3.10×10^{-2}, 17 $\%$ oscillations

Numerical results



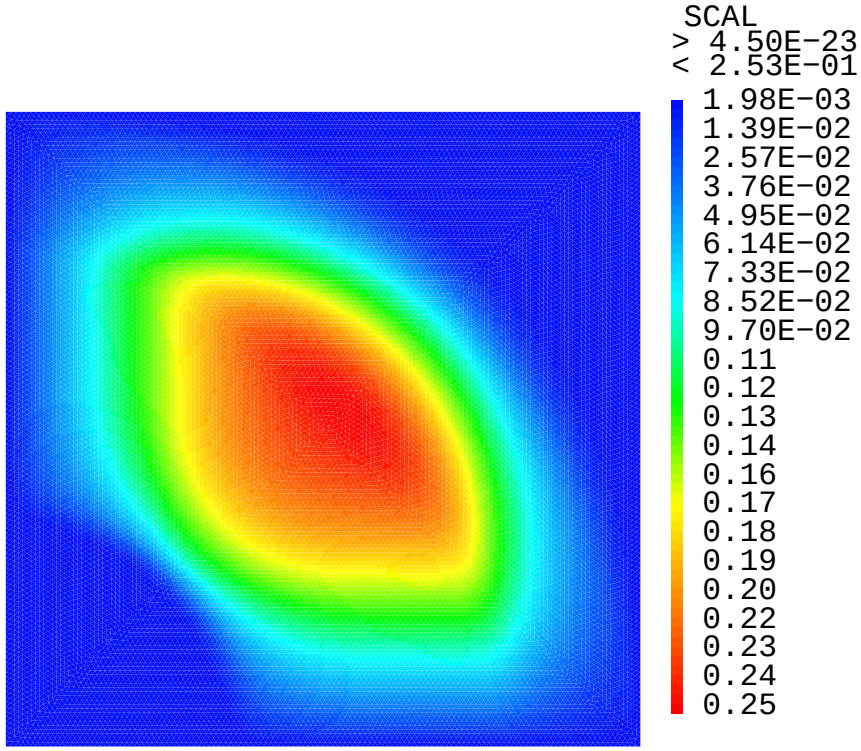
VFSYM (concentration and position of negative values) : 38000 meshes, minimum value -1.49×10^{-3},7 $\%$ oscillations

Numerical results



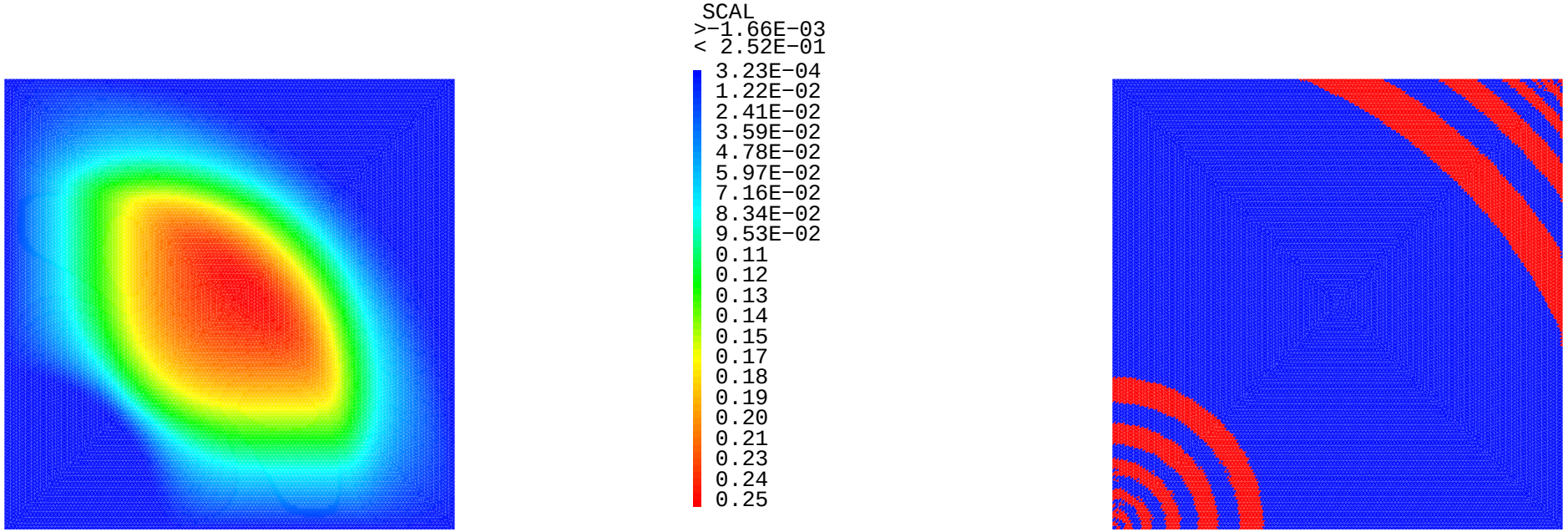
F.E (concentration and position of negative values) : 38000 meshes, minimum value -9.44×10^{-4}, 12 $\%$ oscillations

Numerical results



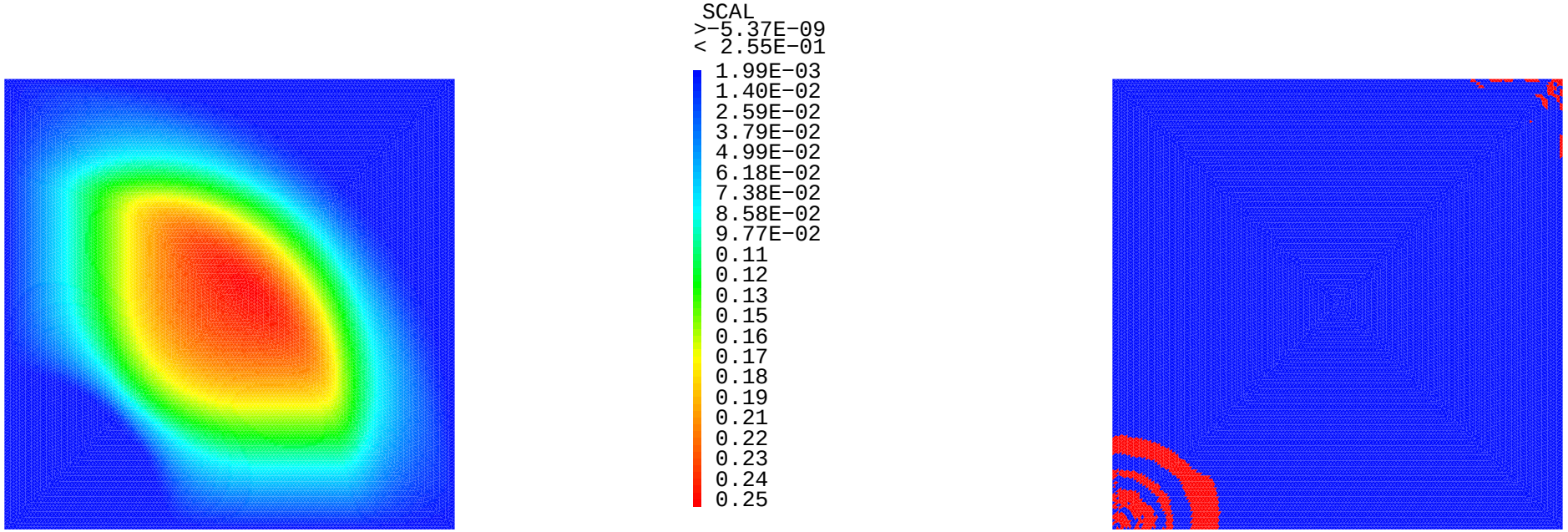
VFMON (concentration) : 38000 meshes

Numerical results



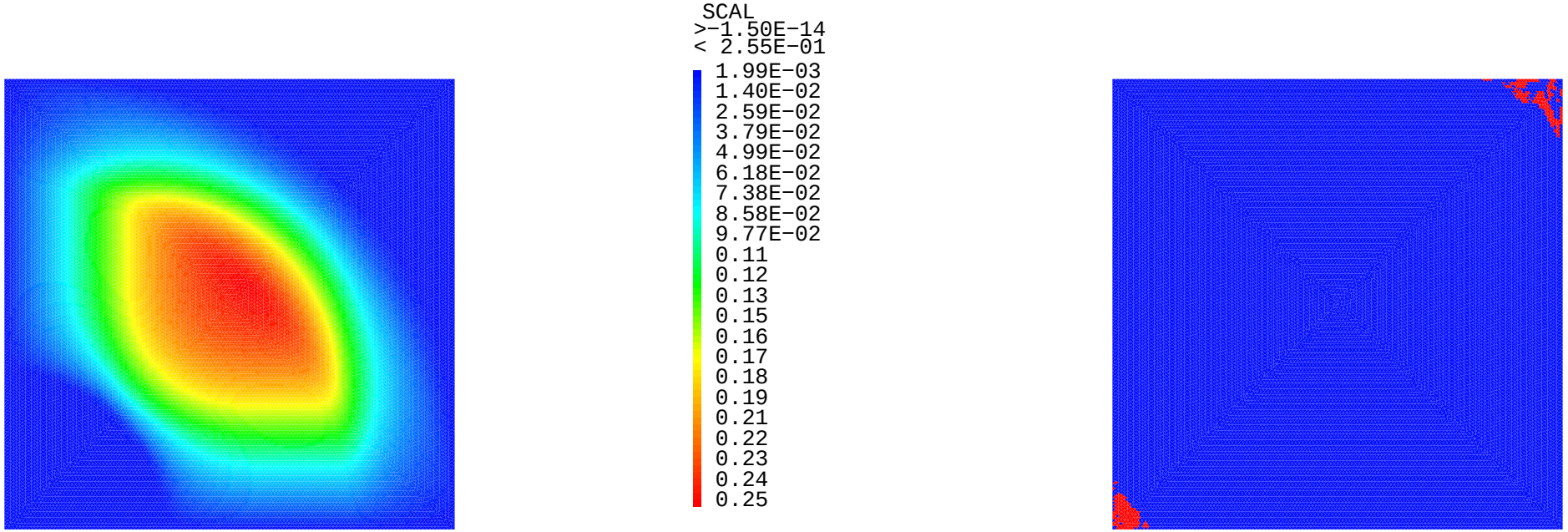
VFPMMD (1 iteration) (concentration and position of negative values) 38000 meshes, minimum value -1.7×10^{-3}, 13 $\%$ oscillations

Numerical results



VFPMMD (50 iterations) (concentration and position of negative values) 38000 meshes, minimum value -5.4×10^{-9}, 2.6 $\%$ oscillations

Numerical results



VFPMMD (200 iterations) (concentration and position of negative values) 38000 meshes, minimum value -1.5×10^{-14}, 0.8 $\%$ oscillations

Numerical results

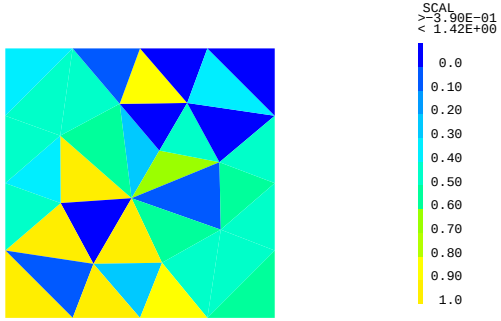
Stationary solution using boundary conditions with steep gradients (test 3 of the benchmark (R. Herbin, F. Hubert))

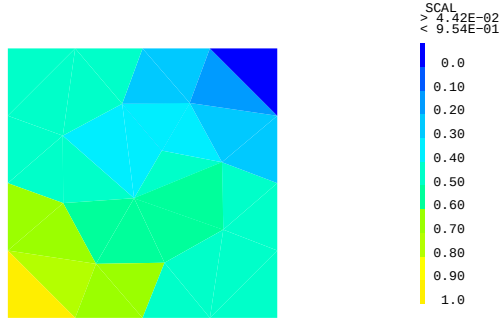
- $\Omega =]0, 1.0[\times]0, 1.0[$
- $\overline{\overline{D}} = R_\theta \begin{pmatrix} 1 & 0 \\ 0 & 10^{-4} \end{pmatrix} {R_\theta}^{-1}$, R_θ is the rotation of angle $\theta = 40$ degrees
- Dirichlet boundary conditions

$$C = \begin{cases} 1 \ on \ (0, .2) \times \{0.\} \cup \{0.\} \times ((0, .2) \\ 0 \ on \ (.8, 1.) \times \{1.\} \cup \{1.\} \times ((.8, .2) \\ \frac{1}{2} \ on \ (.3, 1.) \times \{0.\} \cup \{0.\} \times ((.3, 1.) \\ \frac{1}{2} \ on \ (0., 0.7) \times \{1.\} \cup \{1.\} \times ((0., 0.7) \end{cases}$$

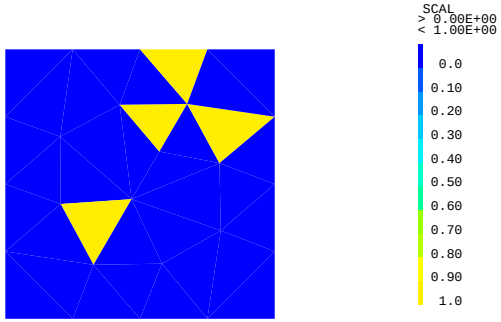
- Regular triangular cells (a coarse and a fine grid)

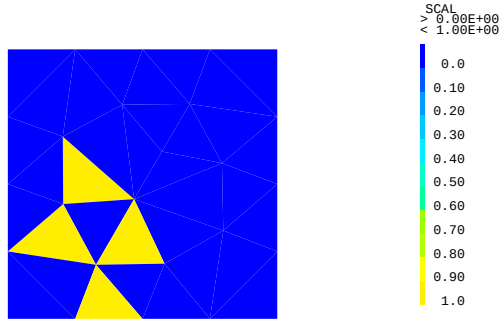
Numerical results





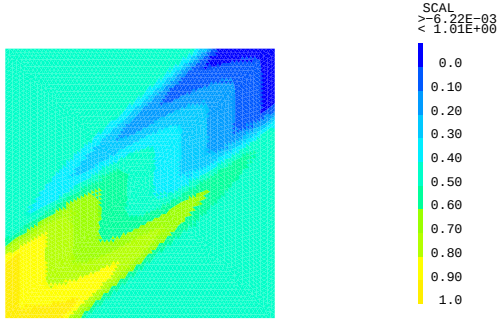
Concentration with VFSYM and VFPMMD schemes: 34 cells (for VFSYM scheme: maximum value 1.42 , minimum value -0.39, for VFPMMD scheme: maximum value 0.954 , minimum value 0.0442)

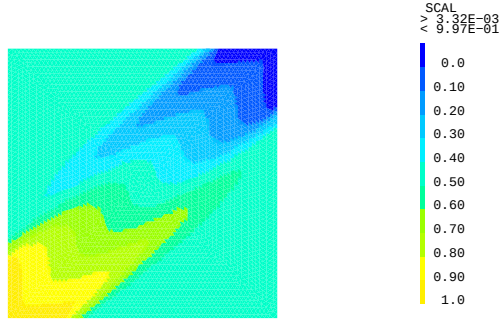




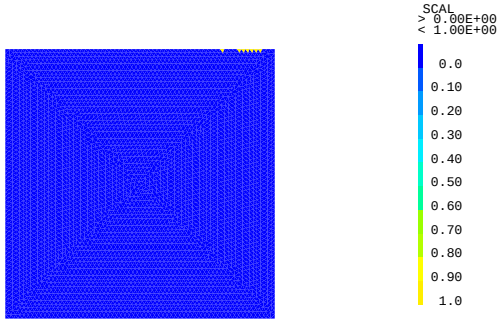
VFSYM scheme (position of negative values and values higher than 1)

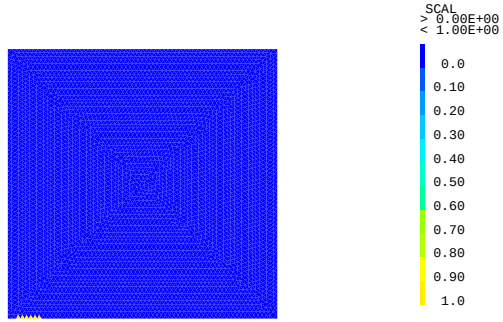
Numerical results





Concentration with VFSYM and VFPMMD schemes: 9500 cells, (for VFSYM scheme: maximum value 1.01 , minimum value -6.22×10^{-3}, for VFPMMD scheme: maximum value 0.997 , minimum value 3.32×10^{-3})





VFSYM scheme (position of negative values and values higher than 1)

Conclusion

- It seems to work on numerous cases!
- Test 8 and Test 9 of the Benchmark
- The computer cost can be cheap (Explicit scheme, or a few iterations in the fixed point Algorithm)
- Application to chemical transport or nonlinear models (Richards)
- In the future VFMPFA (DMP)
- Arbitrary grids