Bench of anisotropic problems

Numerical results with a cell-centered finite volume scheme (VFSYM) for diffusion operators

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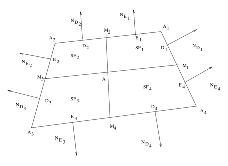
Description of the scheme

- Ω_k the quadrangular cell (A_1, A_2, A_3, A_4), SF_A the area of Ω_k, A the barycenter of Ω_k, M_{i+1} the middle of the edge A_iA_{i+1} (by convention $A_5 = A_1$), D_i and E_i the middles of the edges M_iA_i and A_iM_{i+1} (by convention $M_5 = M_1$).
- $\vec{n}_{D_i}$ and $\vec{n}_{E_i}$ the normal vectors to the edges M_iA_i and A_iM_{i+1} with the same length as these edges.
- Δ_{A_i} the quadrangular cell (A, M_i, A_i, M_{i+1}), SF_i its area and $\partial\Delta_{A_i}$ its boundary.
- F_{A_i} and N_{A_i}, the set and the number of edges around the point A_i in the grid of Ω.



Case of quadrangular cells

Assumptions of the finite volume discretization

- The concentration C is constant inside Ω_k, the vector $\vec{q}$ is constant on Δ_{A_i}.
- The matrix $\overline{\overline{K}}$ is constant on Ω_k, the concentration C is constant on the edges M_iA_i and A_iM_{i+1}.

We denote C_A (respectively C_{D_i} and C_{E_i}) the value of the concentration C at the point A (respectively on the edges M_iA_i and A_iM_{i+1}), f_A the value of the source term f at the point A, $\vec{q}_{A_i}$ the value of $\vec{q}$ on Δ_{A_i}, $\overline{\overline{K}}_A$ the value of the matrix $\overline{\overline{K}}$ at the point A.

Description of the scheme

Integration of the first equation of the system (1) over Δ_{A_i}, using Green's formula leads to:

$$\overline{\overline{K}}_A^{-1}\vec{q}_{A_i} = \frac{1}{SF_i}(C_{D_i} - C_A)\vec{n}_{D_i} + \frac{1}{SF_i}(C_{E_i} - C_A)\vec{n}_{E_i} \qquad (1)$$

So, we deduce the flux f_{D_i} and f_{E_i} through the interface M_iA_i and A_iM_{i+1}

$$\begin{cases} f_{D_i} = \vec{n}_{D_i}.\overline{\overline{K}}_A\vec{n}_{D_i}\dfrac{1}{SF_i}(C_{D_i} - C_A) + \vec{n}_{D_i}.\overline{\overline{K}}_A\vec{n}_{E_i}\dfrac{1}{SF_i}(C_{E_i} - C_A) \\ f_{E_i} = \vec{n}_{E_i}.\overline{\overline{K}}_A\vec{n}_{D_i}\dfrac{1}{SF_i}(C_{K_i} - C_A) + \vec{n}_{E_i}.\overline{\overline{K}}_A\vec{n}_{E_i}\dfrac{1}{SF_i}(C_{E_i} - C_A) \end{cases} \qquad (2)$$

Applying the flux continuity conditions on F_{A_i}, we deduce the interface values (C_{E_i}, C_{D_i}, etc.) as a function of the main unknowns, inversing a small matrix M_{A_i} of dimension N_{A_i}. Then we reconstruct all the fluxes around the point A_i. Let us integrate the mass conservation equation (the second equation of system (1)) over a cell Ω_k. We get:

$$\int_{\Omega_k} div\vec{q}d\Omega = \int_{\partial\Omega_k} \vec{q}.\vec{n}d\Gamma = \sum_{1\leq i\leq 4} f_{D_i} + f_{E_i} = \int_{\Omega_k} fd\Omega = SF_A f_A \qquad (3)$$

So, we are able to calculate all the values located at the barycenter of each cell of the grid.

Results for Test 1.1

umin= 0.0, **umax**=1.0.

- Triangular mesh **mesh1** $\rightsquigarrow$ **ocvl2**= 2.00, **ocvgradl2**= 1.00.

i	nunkw	nnmat	sumflux	erl2	ergrad	ratiol2	ratiograd
1	56	584	ϵ	5.46E-02	1.51E-01		
2	224	2656	ϵ	1.34E-02	7.53E-02	2.02E+00	1.00E+00
3	896	11312	ϵ	3.34E-03	3.74E-02	2.00E+00	1.00E+00
4	3584	46672	ϵ	8.36E-04	1.86E-02	2.00E+00	1.00E+00
5	14336	189584	ϵ	2.09E-04	9.31E-03	2.00E+00	1.00E+00
6	57344	764176	ϵ	5.22E-05	4.65E-03	2.00E+00	1.00E+00
7	229376	3068432	ϵ	1.31E-05	2.32E-03	2.00E+00	1.00E+00

i	erflx0	erflx1	erfly0	erfly1	erflm	umin	umax
1	2.00E-02	2.00E-02	2.00E-02	2.00E-02	1.76E-01	8.00E-02	9.51E-01
2	5.12E-03	5.12E-03	5.12E-03	5.12E-03	9.26E-02	1.93E-02	9.88E-01
3	1.28E-03	1.28E-03	1.28E-03	1.28E-03	4.97E-02	4.70E-03	9.97E-01
4	3.22E-04	3.22E-04	3.22E-04	3.22E-04	2.57E-02	1.16E-03	9.99E-01
5	8.06E-05	8.06E-05	8.06E-05	8.06E-05	1.31E-02	2.87E-04	9.99E-01
6	2.05E-05	2.05E-05	2.05E-05	2.05E-05	6.60E-03	7.15E-05	9.99E-01
7	5.03E-06	5.03E-06	5.03E-06	5.03E-06	3.31E-03	1.78E-05	9.99E-01

- Distorted quadrangular mesh **mesh4_j_i**

grid	nunkw	nnmat	sumflux	erl2	ergrad
C	289	2401	ϵ	2.55E-02	6.39E-02
F	1089	9409	ϵ	6.30E-03	1.74E-02

grid	erflx0	erflx1	erfly0	erfly1	erflm	umin	umax
C	8.00E-03	7.30E-03	6.57E-04	1.49E-02	9.00E-01	7.34E-03	9.59E-01
F	1.53E-03	1.71E-03	6.76E-04	2.60E-03	2.67E-01	2.33E-03	9.89E-01

- **Comments**

  For test 1.1, for triangular cells, as the mesh and the solution is regular, we observe no oscillations. For distorted quadrangular meshes, although the scheme is not linear exact, we observe that it is of order 2 in space for the solution. The error erflm seems to be large. We do not see any oscillations appearing. Moreover, the approximate solution remains within the bounds of the exact solution.

Results for Test 1.2

umin= 0.0, **umax**=$1 + \sin(1)$.

- Triangular mesh **mesh1** $\rightsquigarrow$ **ocvl2**= 2.00, **ocvgradl2**= 1.00.

i	nunkw	nnmat	sumflux	erl2	ergrad	ratiol2	ratiograd
1	56	584	ϵ	7.40E-03	1.03E-01		
2	224	2656	ϵ	1.81E-03	5.09E-02	2.00E+00	1.00E+00
3	896	11312	ϵ	4.50E-04	2.52E-02	2.00E+00	1.00E+00
4	3584	46672	ϵ	1.12E-04	1.26E-02	2.00E+00	1.00E+00
5	14336	189584	ϵ	2.82E-05	6.30E-03	2.00E+00	1.00E+00
6	57344	764176	ϵ	7.05E-06	3.15E-03	2.00E+00	1.00E+00
7	229376	3068432	ϵ	1.76E-06	1.57E-03	2.00E+00	1.00E+00

i	erflx0	erflx1	erfly0	erfly1	erflm	umin	umax
1	2.63E-03	3.63E-02	4.28E-02	6.55E-02	2.50E-01	4.81E-03	1.37E+00
2	2.55E-04	8.65E-03	1.22E-02	2.00E-02	1.23E-01	1.16E-03	1.60E+00
3	2.20E-05	2.11E-03	3.47E-03	5.96E-03	6.09E-02	2.84E-04	1.72E+00
4	2.53E-05	5.22E-04	9.74E-04	1.73E-03	3.03E-02	7.13E-05	1.78E+00
5	1.10E-05	1.29E-04	2.70E-04	4.96E-04	1.52E-02	1.78E-05	1.81E+00
6	3.96E-06	3.24E-05	7.43E-05	1.39E-04	7.57E-03	4.45E-06	1.82E+00
7	1.27E-06	8.08E-06	2.02E-05	3.88E-05	3.78E-03	1.11E-06	1.83E+00

Results for Test 2 Numerical locking

Triangular mesh **mesh1**. **umin**= -1, **umax**= 1.

- $\delta = 10^5 \rightsquigarrow$ **ocvl2**= 2.38, **ocvgradl2**= 1.47.

i	nunkw	nnmat	sumflux	erl2	ergrad	ratiol2	ratiograd
1	56	584	ϵ	2.66E-01	3.15E-01		
2	224	2656	ϵ	5.91E-01	1.77E+01	-1.15E+00	-1.15E+00
3	896	11312	ϵ	1.05E-01	6.18E-01	2.49E+00	1.52E+00
4	3584	46672	ϵ	1.84E-02	2.11E-01	2.50E+00	1.55E+00
5	14336	189584	ϵ	3.30E-03	7.80E-02	2.48E+00	1.52E+00
6	57344	764176	ϵ	6.02E-04	2.61E-02	2.45E+00	1.50E+00
7	229376	3068432	ϵ	1.15E-04	9.44E-03	2.38E+00	1.47E+00

i	erflx0	erflx1	erflm	umin	umax
1	0.00E+00	0.00E+00	5.68E+02	-1.06E+00	1.06E+00
2	0.00E+00	0.00E+00	2.27E+02	-1.76E+00	1.80E+00
3	0.00E+00	0.00E+00	1.15E+02	-1.10E+00	1.10E+00
4	0.00E+00	0.00E+00	5.79E+01	-1.01E+00	1.01E+00
5	0.00E+00	0.00E+00	2.90E+01	-1.00E+00	1.00E+00
6	0.00E+00	0.00E+00	1.45E+01	-9.99E-01	9.99E-01
7	0.00E+00	0.00E+00	7.29E+00	-9.99E-01	9.99E-01

- $\delta = 10^6 \rightsquigarrow$ **ocvl2**= 2.31, **ocvgradl2**= 1.50.

i	nunkw	nnmat	sumflux	erl2	ergrad	ratiol2	ratiograd
1	56	584	ϵ	2.66E-01	3.15E-01		
2	224	2656	ϵ	1.86E+00	5.59E+01	-2.80E+00	-4.14E+00
3	896	11312	ϵ	3.29E-01	1.95E+00	2.50E+00	1.52E+00
4	3584	46672	ϵ	5.75E-02	6.66E-01	2.52E+00	1.55E+00
5	14336	189584	ϵ	1.01E-02	2.30E-01	2.51E+00	1.53E+00
6	57344	764176	ϵ	1.79E-03	8.05E-02	2.49E+00	1.51E+00
7	229376	3068432	ϵ	3.60E-04	2.84E-02	2.31E+00	1.50E+00

i	erflx0	erflx1	erflm	umin	umax
1	0.00E+00	0.00E+00	1.80E+03	-1.07E+00	1.07E+00
2	0.00E+00	0.00E+00	7.19E+02	-3.62E+00	3.62E+00
3	0.00E+00	0.00E+00	3.65E+02	-1.43E+00	1.43E+00
4	0.00E+00	0.00E+00	1.83E+02	-1.05E+00	1.05E+00
5	0.00E+00	0.00E+00	9.18E+01	-1.00E+00	1.00E+00
6	0.00E+00	0.00E+00	4.60E+01	-1.00E+00	1.00E+00
7	0.00E+00	0.00E+00	2.30E+01	-1.00E+00	1.00E+00

- **Comments**

  For test 2, we observe that the scheme is of order higher than 2 in space for the solution, and of order 1.5 for the gradient. We observe oscillations for the coarsest meshes but no numerical locking effect.

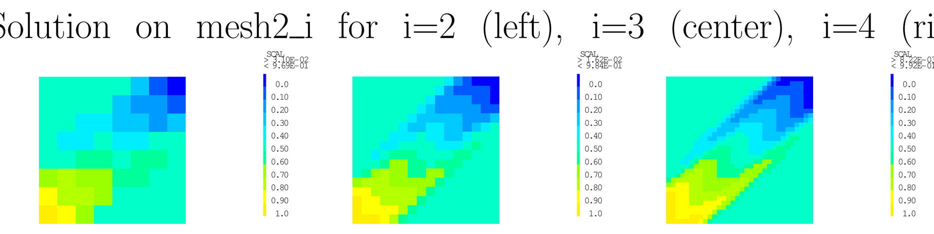
Results for Test 3 : Oblique flow

- Uniform rectangular mesh **mesh2**. **umin**= 0.0, **umax**= 1.0.

i	nunkw	nnmat	sumflux	umin	umax
1	16	100	ϵ	6.85E-02	9.32E-01
2	64	484	ϵ	3.10E-02	9.69E-01
3	256	2116	ϵ	1.62E-02	9.84E-01
4	1024	8836	ϵ	8.22E-03	9.92E-01
5	4096	36100	ϵ	4.08E-03	9.96E-01
ref	262144	2353156	ϵ	4.92E-04	9.99E-01

i	flux0	flux1	fluy0	fluy1	ener1	ener2	eren
ref							
1	-1.95E-01	1.95E-01	-1.18E-01	1.18E-01	2.20E-01	2.20E-01	ϵ
2	-1.94E-01	1.94E-01	-1.02E-01	1.02E-01	2.40E-01	2.40E-01	ϵ
3	-1.93E-01	1.93E-01	-1.01E-01	1.01E-01	2.38E-01	2.38E-01	ϵ
4	-1.93E-01	1.93E-01	-9.91E-02	9.91E-02	2.42E-01	2.42E-01	ϵ
5	-1.93E-01	1.93E-01	-9.88E-02	9.88E-02	2.42E-01	2.42E-01	ϵ
ref	-1.93E-01	1.93E-01	-9.87E-02	9.87E-02	2.42E-01	2.42E-01	ϵ

- Solution on mesh2_i for i=2 (left), i=3 (center), i=4 (right)



- **Comments**

  For test 3, we observe that the scheme respects the maximum principle.

Results for Test 5 : Heterogeneous rotating anisotropy

- Non conforming rectangular mesh **mesh5**. **umin**= 0.0, **umax**= 1.0. $\rightsquigarrow$ **ocvl2**= 1.99, **ocvgradl2**= 1.44.

i	nunkw	nnmat	sumflux	erl2	ergrad	ratiol2	ratiograd
1	16	100	ϵ	1.71E+00	1.13E+00		
2	64	484	ϵ	2.57E-01	3.22E-01	2.73E+00	1.81E+00
3	256	2116	ϵ	3.77E-02	9.43E-02	2.77E+00	1.77E+00
4	1024	8836	ϵ	6.98E-03	3.06E-02	2.43E+00	1.62E+00
5	4096	36100	ϵ	1.57E-03	1.06E-02	2.15E+00	1.53E+00
6	16384	145924	ϵ	3.88E-04	3.82E-03	2.02E+00	1.47E+00
7	65536	586756	ϵ	9.75E-05	1.41E-03	1.99E+00	1.44E+00

i	erflx0	erflx1	erfly0	erfly1	erflm	umin	umax
1	1.04E-01	2.40E-02	1.04E-01	2.40E-02	2.98E-01	-8.67E-01	2.57E+00
2	3.22E-02	1.16E-02	3.22E-02	1.16E-02	1.50E-01	-2.47E-01	1.05E+00
3	1.00E-02	4.09E-03	1.00E-02	4.09E-03	7.63E-02	-6.02E-02	1.01E+00
4	3.00E-03	1.41E-03	3.00E-03	1.41E-03	3.83E-02	-1.50E-02	1.00E+00
5	8.62E-04	4.85E-04	8.62E-04	4.85E-04	1.92E-02	-3.74E-03	1.00E+00
6	2.41E-04	1.64E-04	2.41E-04	1.64E-04	9.60E-03	-9.35E-04	1.00E+00
7	6.64E-05	5.34E-05	6.64E-05	5.34E-05	4.79E-03	-2.34E-04	1.00E+00

- **Comments**

  For test 5, the scheme is of order 2 in space for the solution and of order around 1.5 for the gradient. We observe large oscillations for the coarsest meshes.

Results for Test 6 and Test 7

- **Test 6** Oblique drain, min $= -1.2$, max $= 0$, coarse (C) and fine (F) oblique meshes, **mesh6** and **mesh7**

grid	nunkw	nnmat	sumflux	erl2	ergrad
C	210	1348	ϵ	ϵ	ϵ

grid	erflx0	erflx1	erfly0	erfly1	erflm	umin	umax
C	ϵ	ϵ	ϵ	ϵ	ϵ	-1.14E+00	-5.39E-02

- **Test 7** Oblique barrier, min $= -5.575$, max $= 0.575$, coarse oblique mesh **mesh6**

nunkw	nnmat	sumflux	erl2	ergrad
210	1348	ϵ	ϵ	ϵ

erflx0	erflx1	erfly0	erfly1	erflm	umin	umax
ϵ	ϵ	ϵ	ϵ	ϵ	-5.53E+00	5.37E-01

- **Comments**

  For tests 6 and 7, the new scheme (variant of VFSYM) is very accurate (about the machine precision).

Results for Test 8 and Test 9

- **Test 8** Perturbed parallelograms mesh **mesh8**, **umin**= 0.0, **umax**= 1.0.

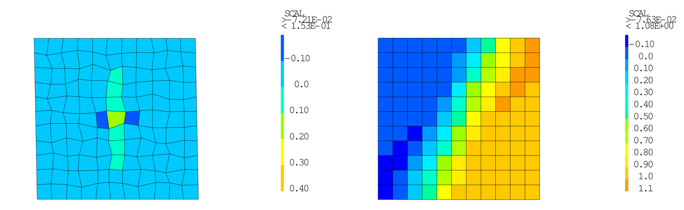
nunkw	nnmat	sumflux	umin	umax
121	961	ϵ	-7.21E-02	1.52E-01

flux0	flux1	fluy0	fluy1
1.37E-04	-1.15E-04	4.96E-01	5.04E-01

- **Test 9** Anisotropy with wells. Square uniform grid **mesh9**, **umin**= 0.0, **umax**= 1.0,

nunkw	nnmat	sumflux	umin	umax
121	929	ϵ	-7.63E-02	1.07E+00

- Solutions of Test 8 (left), Test 9 (right)



- **Comments**

  For tests 8 and 9, we observe large oscillations.