Numerical validation of a Bilayer Viscous model for Shallow Water equations.

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Introduction

In this work we present a new bi-layer one-dimensional Shallow-Water model including viscosity and friction effect on the bottom and the interface. It is obtained following [1] from an asymptotic analysis of non-dimensional and incompressible Navier-Stokes equations with hydrostatic approximation. In order to obtain the viscosity effects in the model we must take into account a second order approximation. To evaluate this model we perform a numerical test consisting of an internal dam-break problem. We make a comparison between the model obtained and the Navier-Stokes simulation.

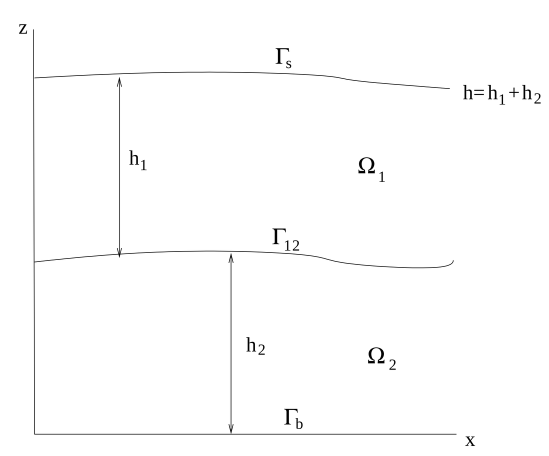
1. Problem Statement

We consider a periodic domain Ω in $\mathbb{R}^2$ with two layers of immiscible fluids like in the figure on right. If we denote h_i the height of each layer, we can define the domain as $\Omega = \Omega_1 \cup \Omega_2$ given by:

$$\Omega_1 = \{(x,z) \in \mathbb{R}^2 / x \in [a,b],\ z \in (h_2, h)\}$$

$$\Omega_2 = \{(x,z) \in \mathbb{R}^2 / x \in [a,b],\ z \in (0, h_2)\}$$

We also define $u_i = (v_i, w_i)$ the velocity of each layer, ρ_i the density, ν_i the kinematic viscosity and p_i the pressure.



So we consider the Navier-Stokes equations:

$$\begin{cases} \rho_i \partial_t u_i + (\rho_i u_i \nabla) u_i - \text{div}\ \sigma_i = -\rho_i g e_z; \\ \text{div}\ u_i = 0, \end{cases} \qquad \sigma_i = 2\rho_i \nu_i D(u_i) - p_i \text{Id},\ D(v) = \frac{\nabla v + \nabla^\perp v}{2}.$$

Boundary conditions:

- On the free surface $z = h(t,x)$: $\begin{cases} p_1 = \text{constant}, \\ \partial_t h + v_1 \cdot \nabla h = w_1; \\ \sigma_1 \cdot n_s = 0 \quad (n_s \text{ the normal outward vector to the free surface}) \end{cases}$
- At the interface $z = h_2(t,x)$: $\begin{cases} \partial_t h_2 + v_j \cdot \nabla h_2 = w_j, \quad \text{for } j = 1,2; \\ (\sigma_i \cdot n_{2,1})_\tau = \text{fric}(v_1, v_2) \quad \text{for } i = 1,2. \end{cases}$

  $n_{2,1}$ being the normal vector pointing from layer 2 to layer 1, and $\text{fric}(v_1, v_2)$ the friction term.
- At the bottom $z = 0$: $\begin{cases} (\sigma_2 \cdot n_b)_\tau = \alpha (u_2)_\tau \\ w_2 = 0. \end{cases}$

  with α a friction coefficient and n_b the normal outward vector to the bottom, and a non-penetration condition:

2. Derivation of the models

The derivation comprises the following steps:

1. Dimensionless equations: We take $\epsilon = H/L$ small $\leftrightarrow$ Hydrostatic approximation.
2. Integration of the hydrostatic system in the vertical direction.
3. Asymptotic analysis: $\nu_1 = \epsilon \nu_{1_0}$, $\quad \alpha = \epsilon \alpha_0$, $\quad \text{fric}(v_1, v_2) = \epsilon\, \text{fric}_0(v_1, v_2)$.

First order approximation

$$(SW) \begin{cases} \partial_t h_1 + \partial_x (h_1 v_1) = 0; \\ \rho_1 \partial_t (h_1 v_1) + \rho_1 \partial_x (h_1 v_1^2) + \rho_1 g h_1 \partial_x h_1 + \rho_1 g h_1 \partial_x h_2 = \\ \quad = -\text{fric}(v_1, v_2); \\ \partial_t h_2 + \partial_x (h_2 v_2) = 0; \\ \rho_2 \partial_t (h_2 v_2) + \rho_2 \partial_x (h_2 v_2^2) + \rho_2 g h_2 \partial_x h_2 + \rho_1 g h_2 \partial_x h_1 = \\ \quad = \text{fric}(v_1, v_2) - \alpha v_2. \end{cases}$$

Second order approximation

$$(VSW) \begin{cases} \partial_t h_1 + \partial_x (h_1 v_1) = 0; \\ \rho_1 \partial_t (h_1 v_1) + \rho_1 \partial_x (h_1 v_1^2) + \rho_1 g h_1 \partial_x h_1 + \rho_1 g h_1 \partial_x h_2 = \\ \quad = -\text{fric}(v_1, v_2) + 4\rho_1 \nu_1 \partial_x (h_1 \partial_x v_1); \\ \partial_t h_2 + \partial_x (h_2 v_2) = 0; \\ \rho_2 \partial_t (h_2 v_2) + \rho_2 \partial_x (h_2 v_2^2) + \rho_2 g h_2 \partial_x h_2 + \rho_1 g h_2 \partial_x h_1 = \\ \quad = \left(1 + \frac{\alpha\beta}{6\nu_2} h_2\right) \text{fric}(v_1, v_2) - \alpha\beta v_2 + 4\rho_2 \nu_2 \partial_x (h_2 \partial_x v_2). \end{cases}$$

where $\beta = \left(1 + \frac{\alpha h_2}{3\nu_2}\right)^{-1} \Rightarrow 1 + \frac{\alpha\beta}{6\nu_2} h_2 = 1 + \frac{1}{2}\frac{\alpha h_2}{3\nu_2 + \alpha h_2} \backsim \frac{3}{2}$.

3. Numerical results

Test data

Domain: $\Omega = [0, L]$, $L = 10$.

Physic data: $\mu = \mu_0\, \delta$, $\quad \kappa = \kappa_0\, \delta$, $\delta = \frac{h_{2_L} - h_{2_R}}{L}$, $h_{2_L} = -0.2$ and $h_{2_R} = -1.8$; $r = \frac{\rho_1}{\rho_2} = 0.98$; $\mu_1 = \mu_2 = \mu$

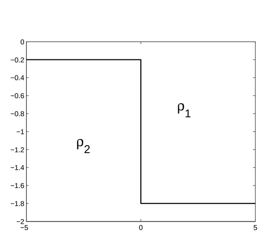
Friction term at the interface:
$\text{fric}(v_1, v_2) = -\kappa B(h_1, h_2)(v_2 - v_1)$, $\quad B(h_1, h_2) = \frac{h_1 h_2}{\frac{\rho_1}{\rho_2} h_1 + \frac{\rho_2}{\rho_1} h_2}$ and $\kappa > 0$.

Initial conditions: $h(t=0) = \begin{cases} h_{2_L} & x < 0; \\ h_{2_R} & x > 0, \end{cases} \qquad q(t=0) = 0.$

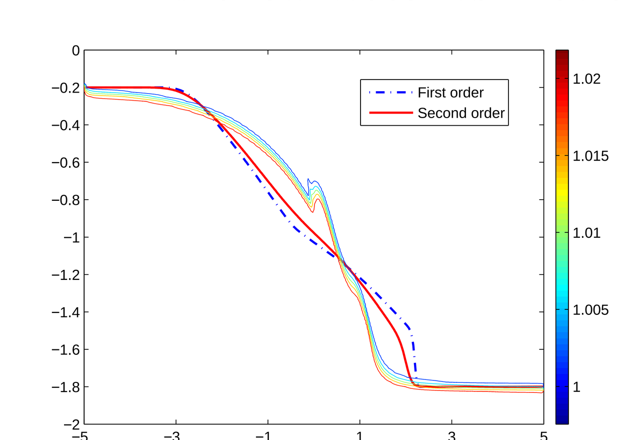
Navier-Stokes problem: $\begin{cases} \partial_t (\rho u) + \nabla(\rho u\, u) + \nabla p - \mu \Delta u = -\rho g; \\ \nabla \cdot u = 0; \\ \partial_t S + \nabla(uS) - \gamma \Delta S = 0. \end{cases}$

$\gamma = 10^{-5}$ being the molecular diffusion.

The density is updated by the state equation $\rho = \rho_0(1+S)$ with initial condition:



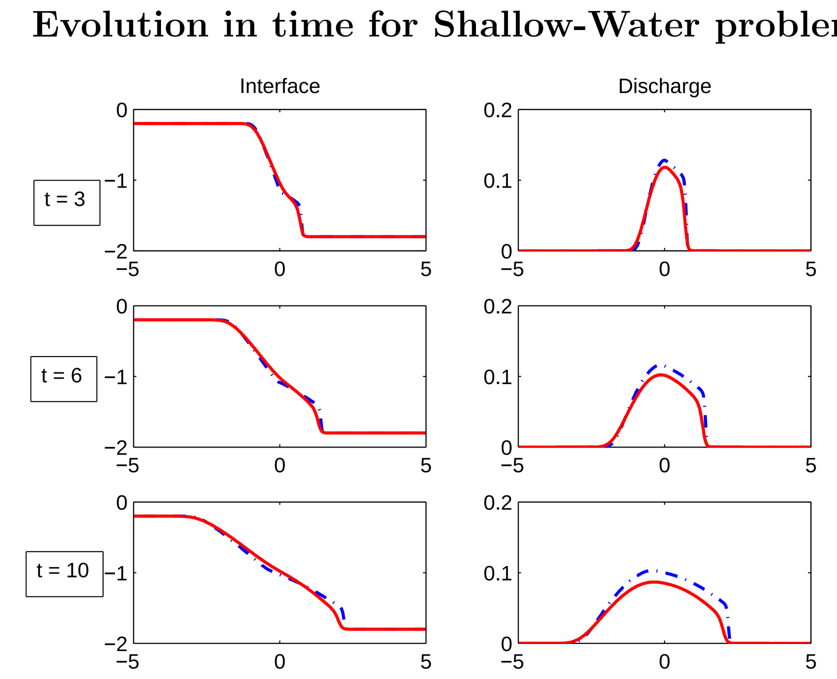
Shallow Water vs. Navier-Stokes solution



Interface.

x	u_{ns}	u_{sw}	u_{vsw}	$\|u_{ns} - u_{sw}\|$	$\|u_{ns} - u_{vsw}\|$
-1.75	0.035966	0.42879	0.036072	6.91×10^{-3}	1.06×10^{-4}
-1.425	0.044902	0.057095	0.047257	1.21×10^{-2}	2.35×10^{-3}
-0.825	0.068987	0.083247	0.066657	1.42×10^{-2}	2.33×10^{-3}
1.425	0.084920	0.109313	0.089799	2.43×10^{-2}	4.87×10^{-3}
1.925	0.038448	0.108732	0.074012	7.02×10^{-2}	3.55×10^{-2}

Evolution in time for Shallow-Water problems



The viscous effect is more quickly appreciable on the discharge profile that get further when the time increases. This result is similar with those obtained by Marche in [2] for one layer Shallow Water equations.

References

[1] J.-F. Gerbeau and B. Perthame. Derivation of viscous Saint-Venant system for laminar shallow water; numerical validation. *Discrete Contin. Dyn. Syst. Ser. B*, 1(1):89–102, 2001.

[2] F. Marche. Derivation of a new two-dimensional viscous shallow water model with varying topography, bottom friction and capillary effects. *Eur. J. Mech. B Fluids*, 26(1):49–63, 2007.