Finite Volumes Method on Evolving Surfaces

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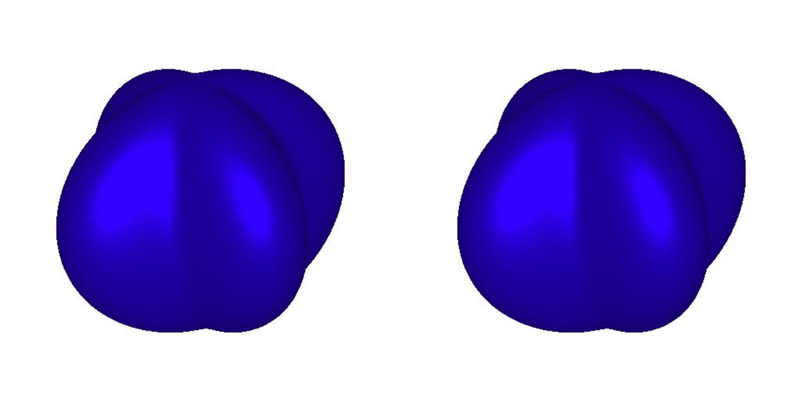
Martin Rumpf, INS, University of Bonn

Aussois, 2008-06-10

Outline

- Motivation
- Aim
- Problem setting
- Recall: Finite Volumes on flat and fixed domain
- Discretization
- Existence and uniqueness
- Stability
- Error estimate
- Numerical Results

Partial differential equation on surfaces



Aim: Finite volume approximation for advection dominated flow on evolving surfaces.

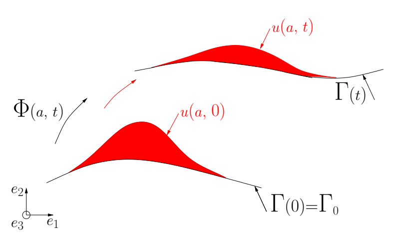
cf.:

G. Dziuk, C. M. Elliott "Finite elements on evolving surfaces" [2006]

Our approach is in addition based on:

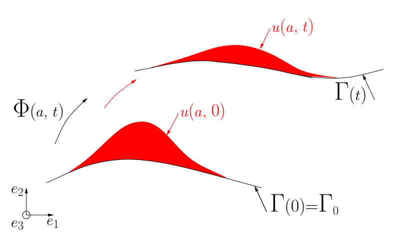
R. Eymard, T. Gallouët, R. Herbin "Finite Volume Method" [1997]

Problem Formulation



Problem Formulation

$\Gamma(t)$:	Family of 2-d smooth surfaces
$\Phi(\cdot, t)$:	Sufficiently smooth map, $\Phi(\Gamma(0), t) = \Gamma(t)$
$u(a,\, t)$:	density of a scalar quantity on $\Gamma(t)$



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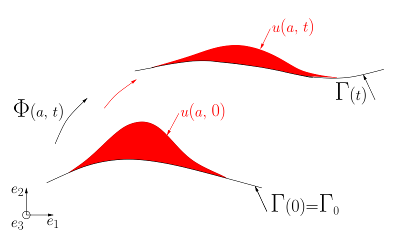

Find $u(\cdot,\, t)$ such that

$$\dot{u} + u\nabla_{\Gamma(t)} \cdot v - \nabla_{\Gamma(t)} \cdot \left(\mathcal{D}_0 \nabla_{\Gamma(t)} u\right) = g \quad \text{on } \Gamma(t)$$

$$u(\cdot,\, 0) = u_0(\cdot) \quad \text{on } \Gamma_0$$

+BC

$v(\cdot,\, t)$: Velocity field on $\Gamma(t)$, g : source term
$\mathcal{D}_0$: Symmetric elliptic diffusion tensor on the tangent plane

R. Eymard, T. Gallouët and R. Herbin, Finite Volume Methods [1997]

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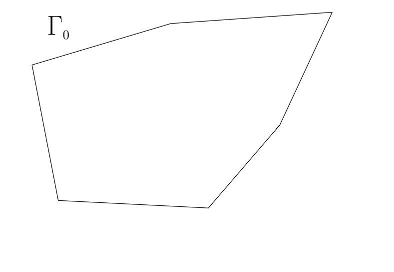

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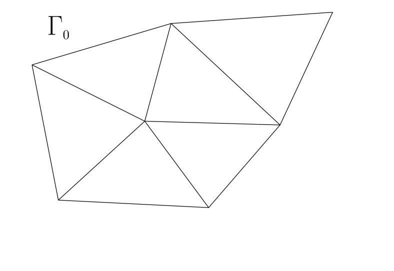

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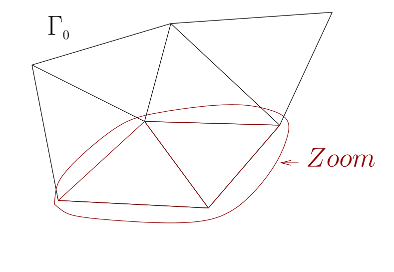

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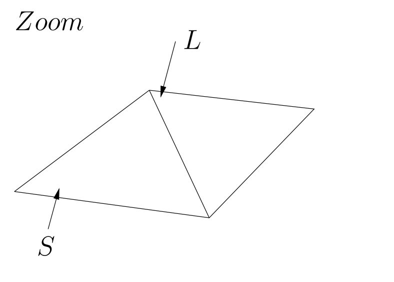

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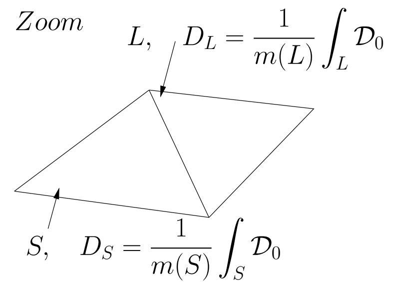

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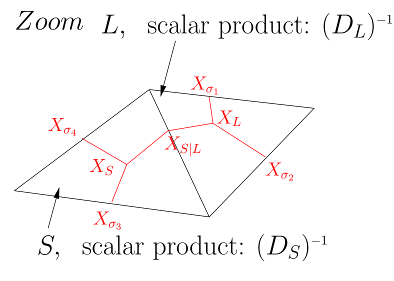

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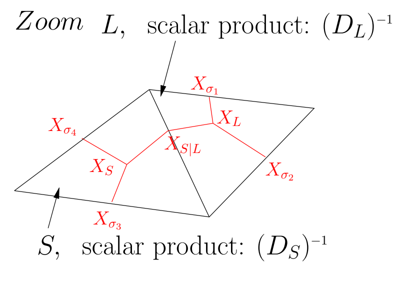

$$\int_S \int_{t_k}^{t_{k+1}} \dot{u} \quad - \quad \int_S \int_{t_k}^{t_{k+1}} \nabla \cdot (\mathcal{D}_0 \nabla u) \quad = \int_S \int_{t_k}^{t_{k+1}} g$$

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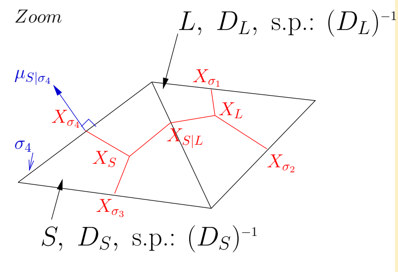

$$\int_S \int_{t_k}^{t_{k+1}} \dot{u} \ - \int_S \int_{t_k}^{t_{k+1}} \nabla \cdot (\mathcal{D}_0 \nabla u) = \int_S \int_{t_k}^{t_{k+1}} g$$

$$\int_S \int_{t_k}^{t_{k+1}} \dot{u} \approx m(S) \left(u_S^{k+1} - u_S^k\right)$$

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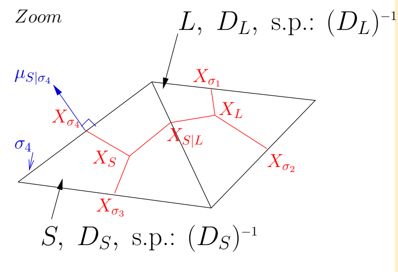

$$- \underbrace{\int_S \int_{t_k}^{t_{k+1}} \nabla \cdot (\mathcal{D}_0 \nabla u)}_{\approx}$$

$$- \tau \sum_{\sigma_i \subset \partial S} m(\sigma_i) \parallel D_S \mu_{S|\sigma_i} \parallel \frac{u_{\sigma_i}^{k+1} - u_S^{k+1}}{\parallel \overrightarrow{X_S X_{\sigma_i}} \parallel}$$

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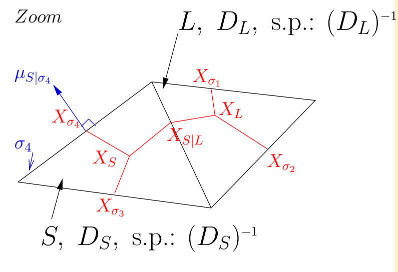

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$\Longrightarrow$ Using flux balance at interfaces

$$- \tau \sum_{\sigma_i \subset \partial S} m(\sigma_i) \frac{\parallel D_S \mu_{S|\sigma_i} \parallel \parallel D_{L_i} \mu_{L_i|\sigma_i} \parallel \cdot \left(u_{L_i}^{k+1} - u_S^{k+1}\right)}{\parallel D_{L_i} \mu_{L_i|\sigma_i} \parallel \parallel \overrightarrow{X_S\, X_{\sigma_i}} \parallel + \parallel D_S \mu_{S|\sigma_i} \parallel \parallel \overrightarrow{X_{L_i}\, X_{\sigma_i}} \parallel}$$

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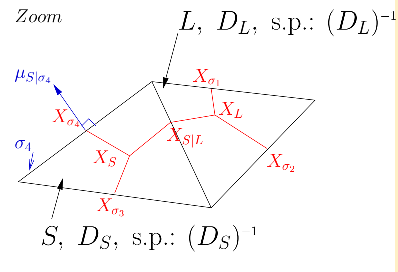

$$\int_S \int_{t_k}^{t_{k+1}} g \quad \approx \quad m(S)\tau g_S^{k+1}$$

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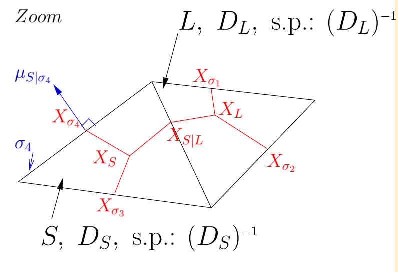


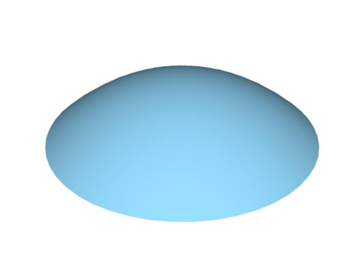
Find $u(\cdot, t)$ such that

$$m(S)\,(u_S^{k+1} - u_S^k)$$

$$+\tau \sum_{\sigma_i \subset \partial S} m(\sigma_i) \frac{\| D_S \mu_{S|\sigma_i} \| \| D_{L_i} \mu_{L_i|\sigma_i} \| \cdot (u_{L_i}^{k+1} - u_S^{k+1})}{\| D_{L_i} \mu_{L_i|\sigma_i} \| \| \overrightarrow{X_S X_{\sigma_i}} \| + \| D_S \mu_{S|\sigma_i} \| \| \overrightarrow{X_{L_i} X_{\sigma_i}} \|}$$

$$= m(S) \tau g_S^{k+1}$$

1



[1] cf. G. Dziuk ans C. M. Elliott, Finite Element on evolving surfaces [2006]

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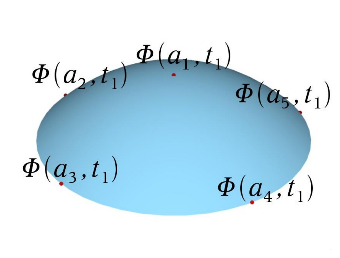

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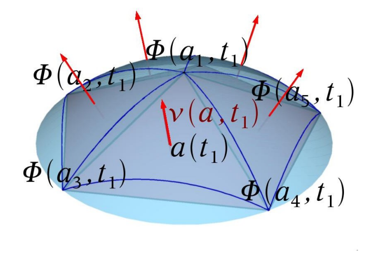

$$x(t) = a(t) - d(a(t))\nu(a(t))$$

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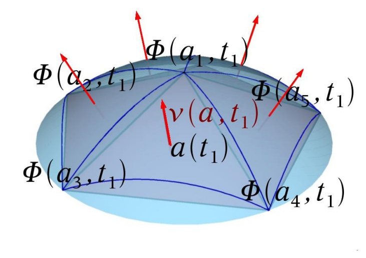

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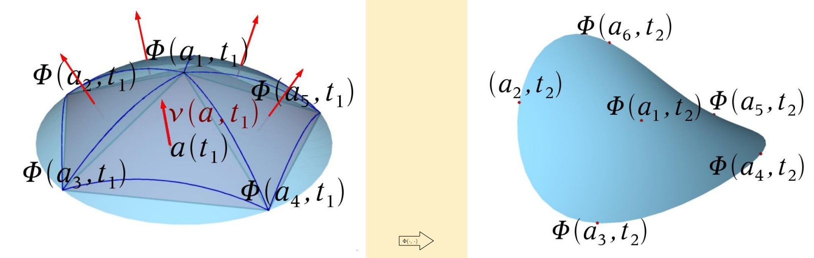

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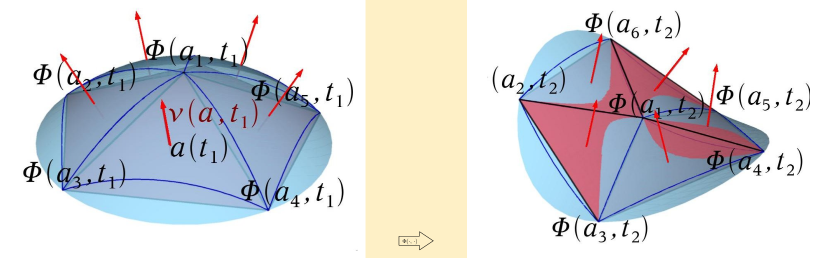

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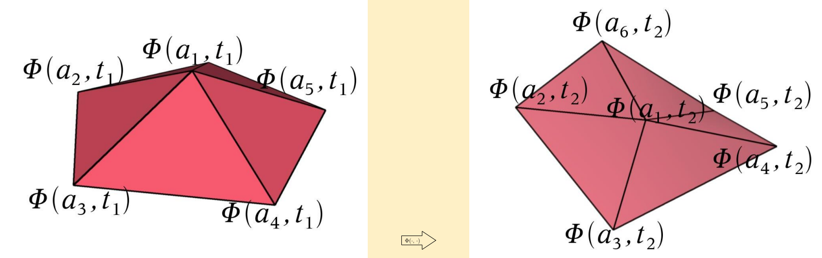
Finite Volumes Method on Evolving Surfaces (Discretization)

[1]



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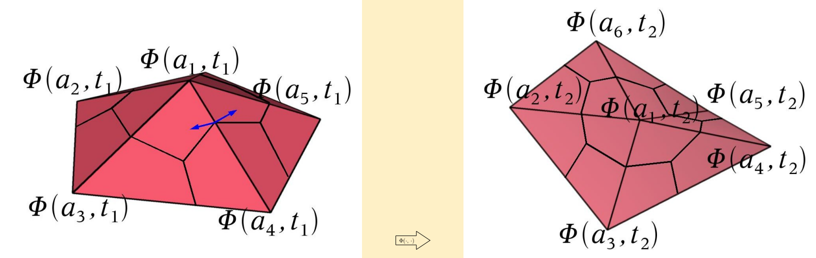

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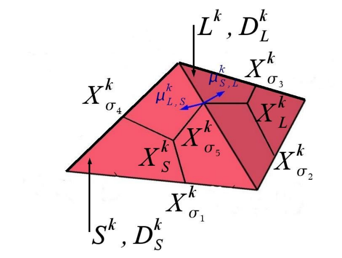
$Zoom$ of two cells at t_k



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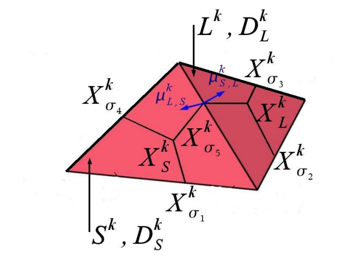

For the simplicity, we assume $\mathcal{D} = \mathcal{I}_d$

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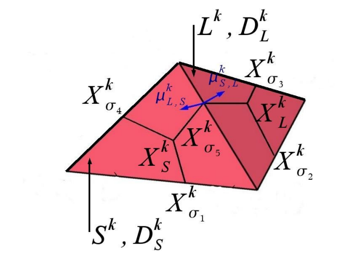

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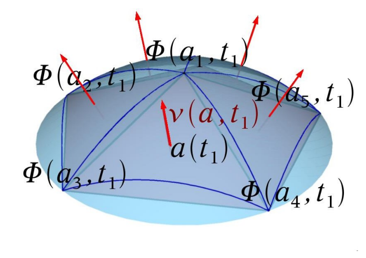
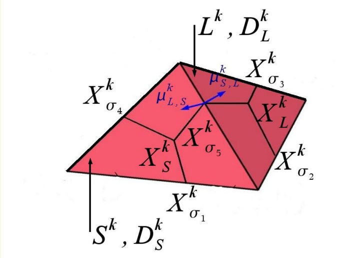
$$\int_{t_k}^{t_{k+1}} \int_{S^l(t)} \left(\dot{u} + u \nabla_{\Gamma(t)} \cdot v \right) - \int_{t_k}^{t_{k+1}} \int_{S^l(t)} \nabla \cdot (\mathcal{D}_0 \nabla u)$$
$$= \int_{t_k}^{t_{k+1}} \int_{S^l(t)} g$$





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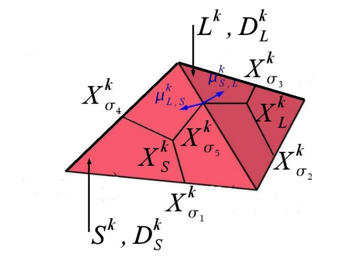

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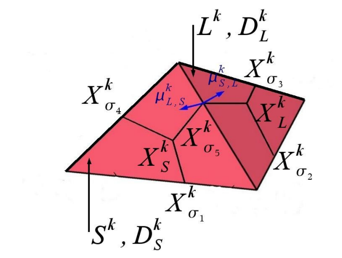

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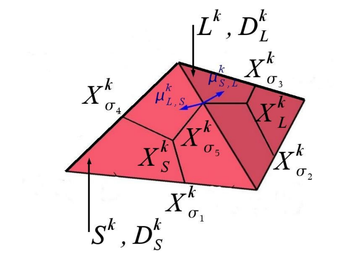
$$\approx m(S(t_{k+1}))\, u_S^{k+1} - m(S(t_k))\, u_S^k$$

$Zoom$ of two cells at t_k



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$$-\underbrace{\int_{t_k}^{t_{k+1}} \int_{S^l(t)} \nabla \cdot (\mathcal{D}_0 \nabla u)}_{\approx}$$

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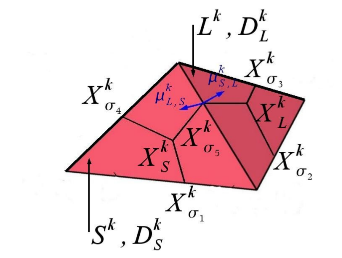

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$$\tau \sum_{\sigma_i \subset \partial S(t_{k+1})} m(\sigma_i) \frac{u_{\sigma_i}^{k+1} - u_S^{k+1}}{\| \overrightarrow{X_S^{k+1} X_{\sigma_i}^{k+1}} \|}$$

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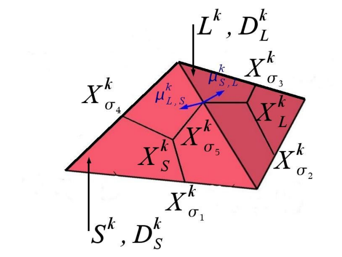

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$\Longrightarrow$ Using flux balance at interfaces

$$\tau \sum_{\sigma_i \subset \partial S} m(\sigma_i) \frac{1}{\| \overrightarrow{X_S^{k+1} X_{\sigma_i}^{k+1}} \| + \| \overrightarrow{X_{L_i}^{k+1} X_{\sigma_i}^{k+1}} \|} \cdot \left(u_{L_i}^{k+1} - u_S^{k+1}\right)$$

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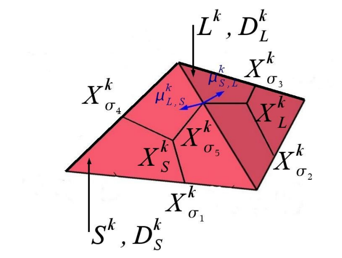
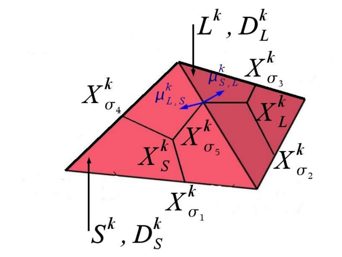


$$\int_{t_k}^{t_{k+1}} \int_{S^l(t)} g \quad \approx \quad \tau m(S(t_{k+1}))g_S^{k+1}$$

$Zoom$ of two cells at t_k



Find $\left\{ u_S^{k+1} \right\}_{S,\,k}$ such that:

$$m(S(t_{k+1}))\, u_S^{k+1} - m(S(t_k))\, u_S^k$$

$$- \tau \sum_{\sigma_i \subset \partial S} m(\sigma_i) \frac{1}{\| \overrightarrow{X_S^{k+1} X_{\sigma_i}^{k+1}} \| + \| \overrightarrow{X_{L_i}^{k+1} X_{\sigma_i}^{k+1}} \|} \cdot$$

$$\cdot \left(u_{L_i}^{k+1} - u_S^{k+1} \right) \qquad = \tau m(S(t_{k+1})) g_S^{k+1}$$

universitätbonn

Find $\left\{ u_S^{k+1} \right\}_{S,\, k}$ such that:

$$m(S(t_{k+1}))\, u_S^{k+1} - m(S(t_k))\, u_S^k$$

$$- \tau \sum_{\sigma_i \subset \partial S} m(\sigma_i) \frac{1}{\| \overrightarrow{X_S^{k+1}\ X_{\sigma_i}^{k+1}} \| + \| \overrightarrow{X_{L_i}^{k+1}\ X_{\sigma_i}^{k+1}} \|} \cdot$$

$$\cdot \left(u_{L_i}^{k+1} - u_S^{k+1} \right) \qquad = \tau m(S(t_{k+1})) g_S^{k+1}$$

Interpretation:

$$u^k = \{ u_S^k \} \equiv \sum_S u_S^k \chi_S$$

$$\left(u^l\right)^k = \left\{ \left(u^l\right)_S^k \right\} \equiv \sum_S \left(u^l\right)_S^k \chi_S$$

where $u^l\,(a,\, t_k) = \left(u^l\right)_S^k = u_S^k$ on $S^l(t_k)$

Theorem: The above system has a unique solution

Sketch of proof:

Recall scheme:

Find $\left\{ u_S^{k+1} \right\}_{S,\,k}$ such that:

$$m(S(t_{k+1}))\, u_S^{k+1} - m(S(t_k))\, u_S^k$$

$$- \tau \sum_{\sigma_i \subset \partial S} m(\sigma_i) \frac{1}{\| \overrightarrow{X_S^{k+1} X_{\sigma_i}^{k+1}} \| + \| \overrightarrow{X_{L_i}^{k+1} X_{\sigma_i}^{k+1}} \|} \cdot \left(u_{L_i}^{k+1} - u_S^{k+1} \right) \qquad = \tau m(S(t_{k+1})) g_S^{k+1}$$

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Theorem: The above system has a unique solution

Sketch of proof:

Induction

- Assume $g \equiv 0$
- Assume $u^k \equiv 0$
- For "$k+1$", multiply equations by corresponding u_S^{k+1}, $(\, S \subset \Gamma_h(t_{k+1})\,)$
- Sum the result over the Ss $\subset \Gamma_h(t_{k+1})$

Recall scheme:

Find $\left\{ u_S^{k+1} \right\}_{S,\,k}$ such that:

$$m(S(t_{k+1}))\, u_S^{k+1} - m(S(t_k))\, u_S^k - \tau \sum_{\sigma_i \subset \partial S} m(\sigma_i) \frac{1}{\| \overrightarrow{X_S^{k+1}\, X_{\sigma_i}^{k+1}} \| + \| \overrightarrow{X_{L_i}^{k+1}\, X_{\sigma_i}^{k+1}} \|} \cdot \left(u_{L_i}^{k+1} - u_S^{k+1} \right) = \tau m(S(t_{k+1})) g_S^{k+1}$$

Theorem: The above system has a unique solution

Sketch of proof:

Induction

- Sum the result over the Ss $\subset \Gamma_h(t_{k+1})$
- One obtains
  $\| u^{k+1} \|^2_{\mathbb{L}^2\left(\Gamma_h^{k+1}\right)} + \tau \| u^{k+1} \|^2_{1,\,\Gamma_h(t_{k+1})} = 0 \quad \Longrightarrow \quad u^{k+1} \equiv 0$

where

$$\| u^k \|^2_{1,\,\Gamma_h(t_k)} = \sum_{\sigma_i = S(t_k)|L(t_k)} m(\sigma_i) \frac{1}{\| \overrightarrow{X_S^k\, X_{\sigma_i}^k} \| + \| \overrightarrow{X_L^k\, X_{\sigma_i}^k} \|} \cdot (u_L^k - u_S^k)^2$$

Recall scheme:

Find $\{ u_S^{k+1} \}_{S,\,k}$ such that:

$$m(S(t_{k+1}))\, u_S^{k+1} - m(S(t_k))\, u_S^k - \tau \sum_{\sigma_i \subset \partial S} m(\sigma_i) \frac{1}{\| \overrightarrow{X_S^{k+1}\, X_{\sigma_i}^{k+1}} \| + \| \overrightarrow{X_{L_i}^{k+1}\, X_{\sigma_i}^{k+1}} \|} \cdot \left(u_{L_i}^{k+1} - u_S^{k+1}\right) = \tau m(S(t_{k+1}))g_S^{k+1}$$

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Proposition 1: (Stability, $\mathbb{L}^\infty(\mathbb{L})$ and $\mathbb{L}^2(\mathbb{H}^1_0)$)

$\{u^k\}$, solution of the above system with the dirichlet condition $u = 0$ on $\partial\Gamma(t)$, $\exists\ C \in \mathbb{R}$ such that

$$\sup_k \| u^{k+1} \|^2_{\mathbb{L}^2(\Gamma_h^{k+1})} + \sum_k \tau \left(\| u^{k+1} \|_{1,\Gamma_h^{k+1}} \right)^2 \leq C \left(\| u^0 \|^2_{\mathbb{L}^2(\Gamma_h^0)} + \tau \sum_k \| g^{k+1} \|^2_{\mathbb{L}^2(\Gamma_h^{k+1})} \right)$$

Sketch of proof:

Recall scheme:

Find $\left\{ u_S^{k+1} \right\}_{S,k}$ such that:

$$m(S(t_{k+1}))\, u_S^{k+1} - m(S(t_k))\, u_S^k$$

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Sketch of proof:

- For "$k+1$", multiply equations by corresponding u_S^{k+1}, ($S \subset \Gamma_h(t_{k+1})$)
- Sum the result over the $S^{k+1} \subset \Gamma_h(t_{k+1})$
- Use Cauchy-Schwarz inequality

Recall scheme:

Find $\{ u_S^{k+1} \}_{S,k}$ such that:

$$m(S(t_{k+1}))\, u_S^{k+1} - m(S(t_k))\, u_S^k - \tau \sum_{\sigma_i \subset \partial S} m(\sigma_i) \frac{1}{\| \overrightarrow{X_S^{k+1} X_{\sigma_i}^{k+1}} \| + \| \overrightarrow{X_{L_i}^{k+1} X_{\sigma_i}^{k+1}} \|} \cdot \left(u_{L_i}^{k+1} - u_S^{k+1} \right) = \tau m(S(t_{k+1})) g_S^{k+1}$$

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Sketch of proof:

One obtains

$$\begin{aligned} & \tfrac{1}{2} \|u^j\|^2_{\mathbb{L}^2(\Gamma_h^j)} + \tau \sum_{k=1\cdots j} \|u^k\|^2_{1,\ \Gamma_h^k} \\ \leq \quad & \tfrac{1}{2} \left(\|u^0\|^2_{\mathbb{L}^2(\Gamma_h^0)} + \tau \sum_{k=1\cdots k_{max}} \|g^k\|^2_{\mathbb{L}^2(\Gamma_h^k)} \right) + \tfrac{1}{2} \left(\delta(h,\tau) + 1\right) \tau \sum_{k=1\cdots j} \|u^k\|^2_{\mathbb{L}^2(\Gamma_h^k)} \end{aligned}$$

Recall scheme:

Find $\{ u_S^{k+1} \}_{S,\,k}$ such that:

$$m(S(t_{k+1}))\, u_S^{k+1} - m(S(t_k))\, u_S^k$$
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Sketch of proof:

One obtains

$$\frac{1}{2} \|u^j\|^2_{\mathbb{L}^2(\Gamma_h^j)} + \tau \sum_{k=1\cdots j} \|u^k\|^2_{1,\,\Gamma_h^k}$$

$$\leq \quad \frac{1}{2} \left(\|u^0\|^2_{\mathbb{L}^2(\Gamma_h^0)} + \tau \sum_{k=1\cdots k_{max}} \|g^k\|^2_{\mathbb{L}^2(\Gamma_h^k)} \right) + \frac{1}{2}\left(\delta(h,\tau)+1\right)\tau \sum_{k=1\cdots j} \|u^k\|^2_{\mathbb{L}^2(\Gamma_h^k)}$$

- $\delta(h,\tau) = \left(\frac{\frac{m(S(t_k))}{m(S(t_{k+1}))} - 1}{\tau} \right)$ is bounded
- Discrete version of Gronwall leads to the result.

Recall scheme:

Find $\{ u_S^{k+1} \}_{S,\,k}$ such that:

$$m(S(t_{k+1}))\, u_S^{k+1} - m(S(t_k))\, u_S^k$$

$$- \tau \sum_{\sigma_i \subset \partial S} m(\sigma_i) \frac{1}{\| \overrightarrow{X_S^{k+1} X_{\sigma_i}^{k+1}} \| + \| \overrightarrow{X_{L_i}^{k+1} X_{\sigma_i}^{k+1}} \|} \cdot \left(u_{L_i}^{k+1} - u_S^{k+1} \right) \qquad = \tau m(S(t_{k+1})) g_S^{k+1}$$

Proposition 2: (Stablity, $\mathbb{L}^2(\mathbb{L}^2)$ for $\partial_t u$, $\mathbb{L}^\infty(\mathbb{H}^1_0)$ for u)

$\{u^k\}$, solution of the above system with the dirichlet condition $u = 0$ on $\partial\Gamma(t)$, $\exists\ C \in \mathbb{R}$ such that

$$\sum_k \tau \parallel \frac{u^{k+1} - u^k}{\tau} \parallel^2_{\mathbb{L}^2(\Gamma_h^{k+1})} + \sup_k \left(\parallel u^{k+1} \parallel_{1,\Gamma_h^{k+1}} \right)^2 \leq C \left(\parallel u^0 \parallel^2_{\mathbb{L}^2(\Gamma_h^0)} + \parallel u^0 \parallel^2_{1,\Gamma_h^0} + \tau \sum_k \parallel g^{k+1} \parallel^2_{\mathbb{L}^2(\Gamma_h^{k+1})} \right)$$

Assumption: $\tau \leq \tau \quad \forall \ k$

Sketch of proof:

Recall scheme:

Find $\{ u_S^{k+1} \}_{S,k}$ such that:

$$m(S(t_{k+1}))\, u_S^{k+1} - m(S(t_k))\, u_S^k$$

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Assumption: $\tau \leq \tau \quad \forall \; k$

Sketch of proof:

- For "$k+1$", multiply equations by corresponding $\left(\frac{u^{k+1} - u^k}{\tau} \right)\Big|_{S^{k+1}}$, $\quad (S^{k+1} \subset \Gamma_h(t_{k+1}))$
- Sum the result over the $S^{k+1} \subset \Gamma_h(t_{k+1})$

Recall scheme:

Find $\{ u_S^{k+1} \}_{S,k}$ such that:

$$m(S(t_{k+1}))\, u_S^{k+1} - m(S(t_k))\, u_S^k$$

$$- \tau \sum_{\sigma_i \subset \partial S} m(\sigma_i) \frac{1}{\parallel \overrightarrow{X_S^{k+1} X_{\sigma_i}^{k+1}} \parallel + \parallel \overrightarrow{X_{L_i}^{k+1} X_{\sigma_i}^{k+1}} \parallel} \cdot \left(u_{L_i}^{k+1} - u_S^{k+1} \right) \qquad = \tau m(S(t_{k+1})) g_S^{k+1}$$

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Assumption: $\tau \le \tau \quad \forall \ k$

Sketch of proof:

- Use Cauchy-Schwarz inequality
- Use the previous result to obtain the solution

Recall scheme:

Find $\{u_S^{k+1}\}_{S,k}$ such that:

$$m(S(t_{k+1}))\, u_S^{k+1} - m(S(t_k))\, u_S^k$$

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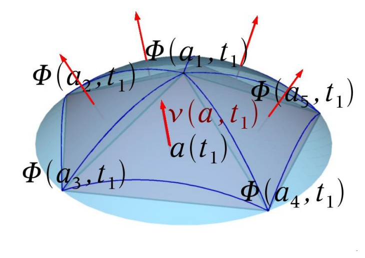
Theorem:

u: Real solution on $\Gamma(t)$,
u^k: Discrete solution on $\Gamma_h(t_k)$

$$E^k = \sum_S \left(u^{-l}\left(X_S^k, t_k\right) - u_S^k\right)\chi_S$$

$$\| E^k \|_{\mathbb{L}^2(\Gamma_h(t_k))} \leq C\,(h + \tau)$$

$$\tau = \tau$$



$$x(t) = a(t) - d(a(t))\nu(a(t))$$

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Theorem:
u: Real solution on $\Gamma(t)$,
u^k: Discrete solution on $\Gamma_h(t_k)$

$$E^k = \sum_S \left(u^{-l}\left(X_S^k, t_k\right) - u_S^k\right)\chi_S$$

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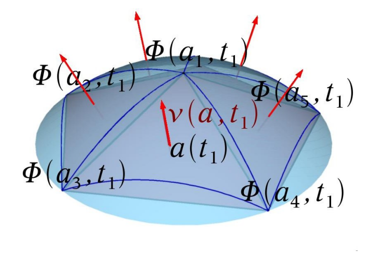

$$x(t) = a(t) - d(a(t))\nu(a(t))$$

Sketch of proof

- Consistency of the geometry approximation.
- Consistency of the approximation of differential opperators and integrals.
- Obtaintion of discrete equations with E^{k+1} as unknown.
- multiplication of each equation by E_S^{k+1}, and partial summation in S^{k+1}

Theorem:

u: Real solution on $\Gamma(t)$,

u^k: Discrete solution on $\Gamma_h(t_k)$

$$E^k = \sum_S \left(u^{-l}(X_S^k, t_k) - u_S^k\right)\chi_S$$

$$\| E^k \|_{\mathbb{L}^2(\Gamma_h(t_k))} \leq C\,(h + \tau)$$

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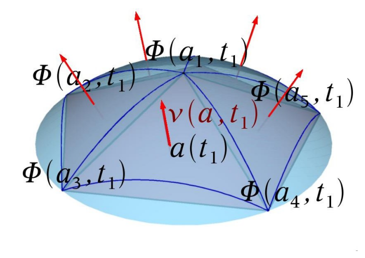


$$x(t) = a(t) - d(a(t))\nu(a(t))$$

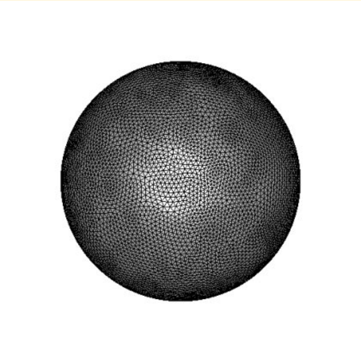
Sketch of proof

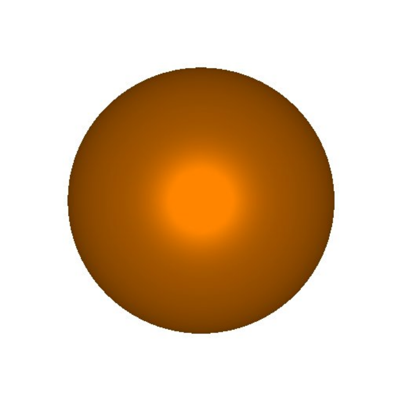
- Cauchy-Schwarz inequality separates terms; One obtains

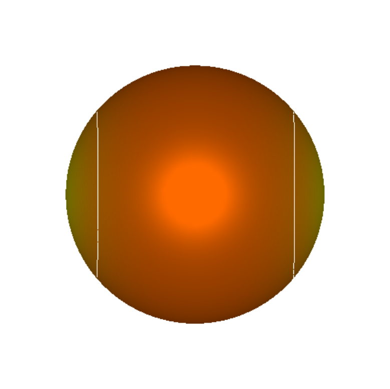
$$\begin{aligned} \frac{1}{2} \| E^{k+1} \|^2_{L^2(\Gamma_h^{k+1})} + \tau \frac{1}{2} \| E^{k+1} \|^2_{1,\Gamma_h^{k+1}} &\leq \frac{1}{2} \| E^k \|^2_{L^2(\Gamma_h^k)} + \frac{1}{2}\delta(h,\tau)\tau \| E^{k+1} \|^2_{L^2(\Gamma_h^{k+1})} \\ &+ \sqrt{m\,(\Gamma_h^{k+1})}C\tau(\tau + h) \| E^{k+1} \|_{L^2(\Gamma_h^{k+1})} + C\tau h^2 \end{aligned}$$

- Bound the sequence appearing on the right side to obtain the result.

Numerical Results: Heat equation on evolving sphere

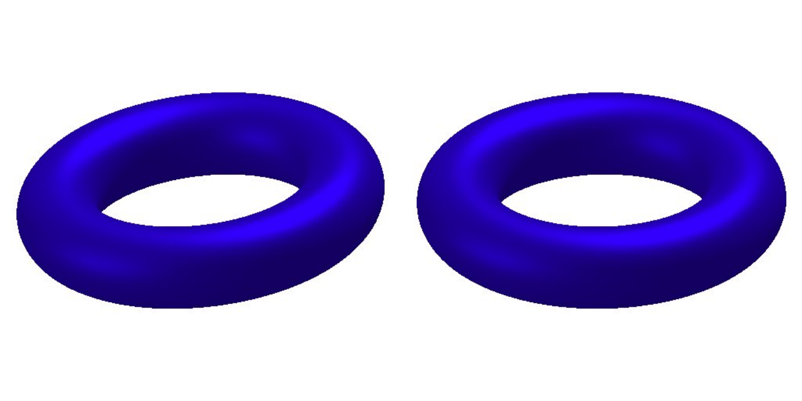






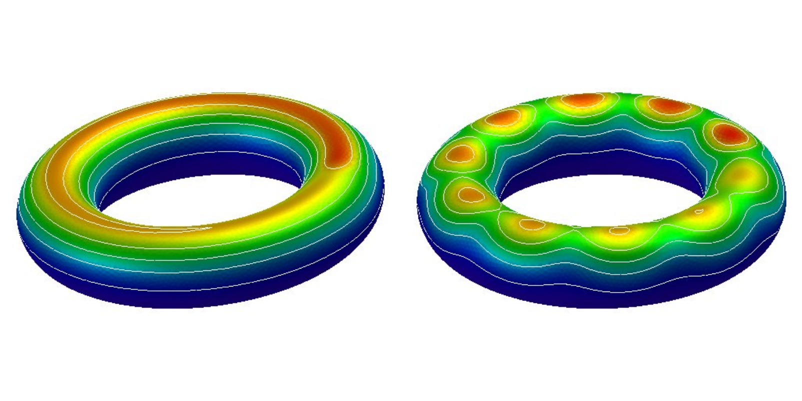
$$\mathcal{D}_0 = Id$$

Numerical Results: Heat equation on rotative torus



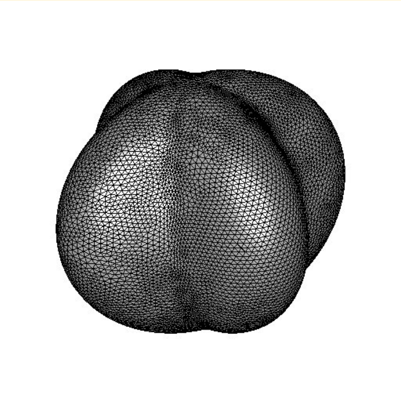
Time indepentent source term

Periodic source term

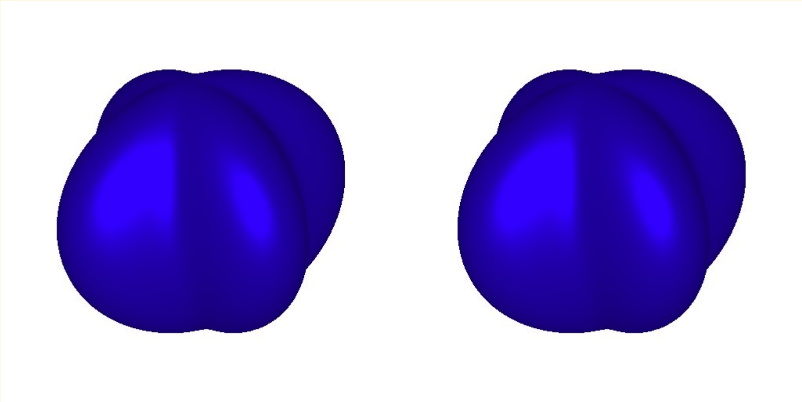


Time indepentent source term

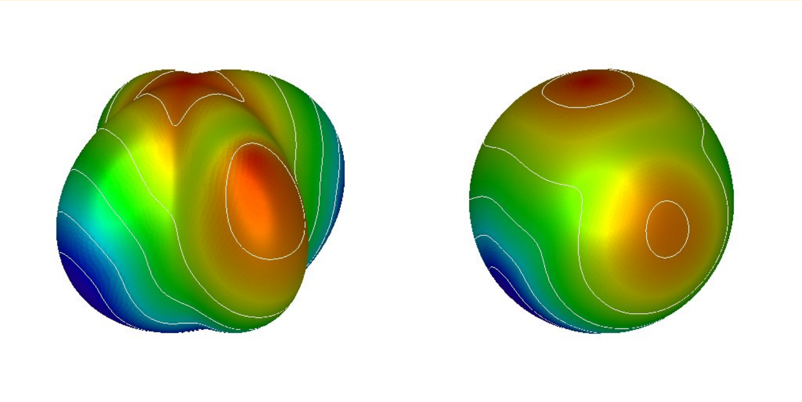
Periodic source term



Numerical Results: Advection-diffusion equation



Numerical Results: Advection-diffusion equation



THANK YOU FOR YOUR ATTENTION

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http://numod.ins.uni-bonn.de

Remark:

1) dA_h: Discrete measure on $\Gamma_h(t)$
dA: Discrete measure on $\Gamma(t)$
$h = \sup_S diam(S), \quad \delta_h, \quad \delta_h \, dA_h = dA \implies \sup_t \sup_{\Gamma(t)} |1 - \delta_h| \leq c\, h^2$

2) $u \in \chi(\Gamma_h(t))$, and u^l its lift

$$\| u \|^2_{\mathbb{L}^2(\Gamma_h(t))} \leq \frac{1}{\inf_t \inf_{\Gamma(t)} \delta_h} \| u^l \|^2_{\mathbb{L}^2(\Gamma(t))}$$

$$\| u^l \|^2_{\mathbb{L}^2(\Gamma(t))} \leq \sup_t \sup_{\Gamma(t)} \delta_h \| u \|^2_{\mathbb{L}^2(\Gamma_h(t))}$$