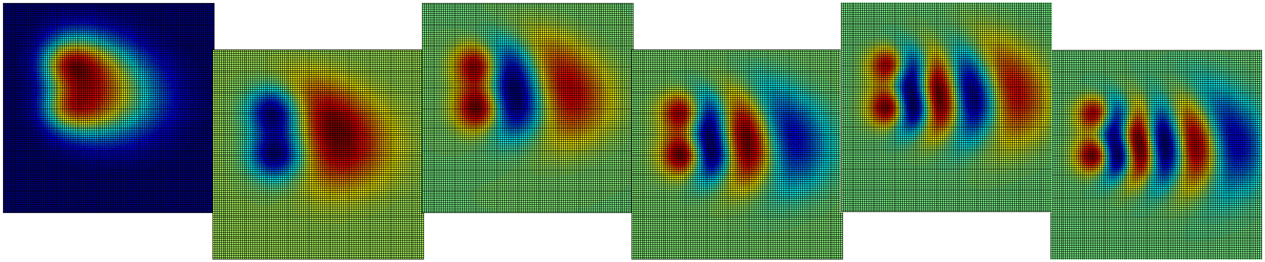
WESTFÄLISCHE WILHELMS-UNIVERSITÄT MÜNSTER

CeNoS Center for Nonlinear Science

Institut für Numerische und Angewandte Mathematik

Adaptive Basis Enrichment for the Reduced Basis Method applied to Finite Volume Schemes



Bernard Haasdonk and Mario Ohlberger

FVCA5, Aussois, June 08-13, 2008

mario.ohlberger@uni-muenster.de www.uni-muenster.de/math/u/ohlberger

Outline

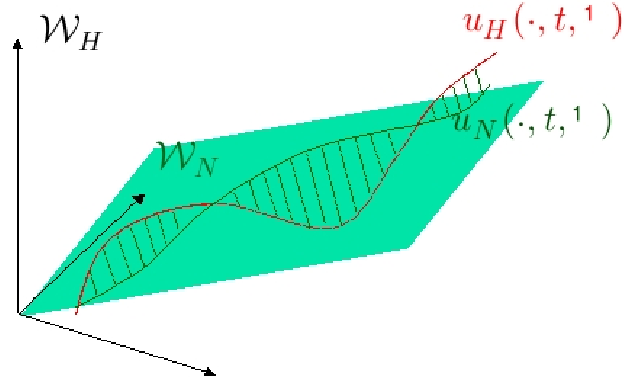
- Reduced Basis Method
- Adaptive Basis Enrichment
- Numerical Experiments
- Outlook

Reduced Basis Method for Parametrized Evolution Equations

[Haasdonk, Ohlberger '08], [Haasdonk, Ohlberger '08]
[Haasdonk, Ohlberger, Rozza '07]

Goal: Fast "Online"-Simulation of Complex Evolution Systems for

- Parameter Optimization
- Desig Optimization
- Optimal Control
- Integration into System Simulation



Ansatz:

- Reduced Basis Method (RB)

$$\dim(\mathrm{W_N}) \ll \dim(\mathrm{W_H})\ !$$

mario.ohlberger@uni-muenster.de www.uni-muenster.de/math/u/ohlberger

Example: Convection-Diffusion Problem

$$\partial_t c(\boldsymbol{\mu}) + \nabla \cdot (\mathbf{v}(\boldsymbol{\mu}) c(\boldsymbol{\mu}) - d(\boldsymbol{\mu}) \nabla u(\boldsymbol{\mu})) = 0 \;\; \text{in } \Omega \times [0, T_{\max}],$$

$$c(\cdot, 0; \boldsymbol{\mu}) = u_0(\boldsymbol{\mu}) \;\; \text{in } \Omega,$$

$$c(\boldsymbol{\mu}) = b_{\mathrm{dir}}(\boldsymbol{\mu}) \;\; \text{in } \Gamma_{\mathrm{dir}} \times [0, T_{\max}],$$

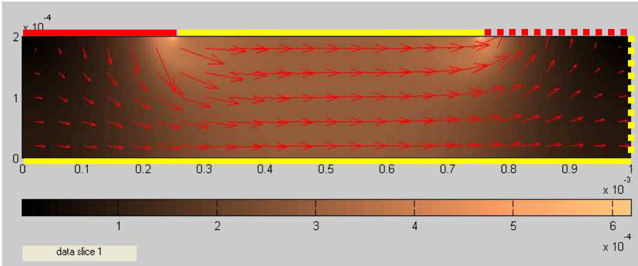
$$(\mathbf{v}(\boldsymbol{\mu}) c(\boldsymbol{\mu}) - d(\boldsymbol{\mu}) \nabla c(\boldsymbol{\mu})) \cdot \mathbf{n} = b_{\mathrm{neu}}(u; \boldsymbol{\mu}) \;\; \text{in } \Gamma_{\mathrm{neu}} \times [0, T_{\max}].$$

Discretization by Finite Volumes $\Longrightarrow$ $\mathbf{c_H}(\boldsymbol{\mu}) \in \mathbf{W_H}$.

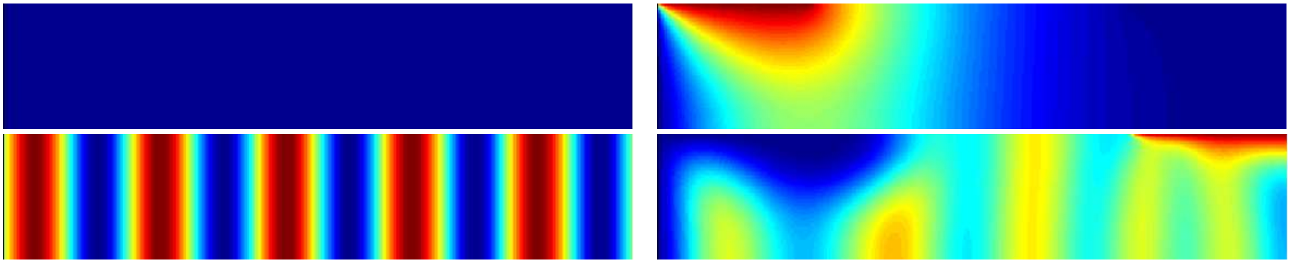
Example: Convection-Diffusion Problem

Parameter:

- Initial Data
- Boundary Values
- Diffusion Parameter



Possible Variations of the Solution:



Model Reduction: Reduced Basis Method

Goal: Find $\mathbf{c}(\cdot, t; \boldsymbol{\mu}) \in L^2(\Omega)$ for $t \in [0, T]$, $\boldsymbol{\mu} \in P \subset \mathbb{R}^p$ with

$$\partial_t \mathbf{c}(\boldsymbol{\mu}) + \mathbf{L}_{\boldsymbol{\mu}}(\mathbf{c}(\boldsymbol{\mu})) = 0 \qquad \text{in } \Omega \times [0, T],$$

plus suitable Initial and Boundary Conditions.

Assumption: FV/DG Approximation $c_H(\boldsymbol{\mu}) \in W_H$ for given Parameter $\boldsymbol{\mu}$

Ansatz (RB): Define low dimensional Subspace $W_N \subset W_H$
and project FV/DG Scheme onto the Subspace
$\Longrightarrow$ RB Approximation $c_N(\boldsymbol{\mu}) \in W_N$.

Requirement:
- Efficient Choice of W_N (Exponential Convergence in N)
- Offline–Online Decomposition for fast Calculation of $c_N(\boldsymbol{\mu})$
- Error Control for $||c_H(\boldsymbol{\mu}) - c_N(\boldsymbol{\mu})||$

Model Reduction: Reduced Basis Method

Assumption: **FV/DG Scheme for Evolution Equations of the Form**

$$c_H^0 = P[c_0(\boldsymbol{\mu})], \qquad L_I^k(\boldsymbol{\mu})[c_H^{k+1}(\boldsymbol{\mu})] = L_E^k(\boldsymbol{\mu})[c_H^k(\boldsymbol{\mu})] + b^k(\boldsymbol{\mu}).$$

with timestep counter k and $\mathbf{c_H^k}(\boldsymbol{\mu}) \in \mathbf{W_H}$.

RB Method: **Let $W_N \subset W_H$ be given, $\{\varphi_1, ..., \varphi_N\}$ a ONB of W_N.**
Sought: $\mathbf{c_N^k}(\boldsymbol{\mu}) = \sum_{n=1}^{N} \mathbf{a_n^k}(\boldsymbol{\mu})\varphi_n$ with

$$\mathbf{L}_I^k(\boldsymbol{\mu})\mathbf{a}^{k+1} \;=\; \mathbf{L}_E^k(\boldsymbol{\mu})\mathbf{a}^k + \mathbf{b}^k(\boldsymbol{\mu}).$$

We then have

$$(\mathbf{L}_I^k(\boldsymbol{\mu}))_{nm} := \int_\Omega \varphi_n L_I^k(\boldsymbol{\mu})[\varphi_m], \qquad (\mathbf{L}_E^k(\boldsymbol{\mu}))_{nm} := \int_\Omega \varphi_n L_E^k(\boldsymbol{\mu})[\varphi_m],$$

$$(\mathbf{a}^0(\boldsymbol{\mu}))_n = \int_\Omega P[c_0(\boldsymbol{\mu})]\varphi_n, \qquad (\mathbf{b}^k(\boldsymbol{\mu}))_n := \int_\Omega \varphi_n b^k(\boldsymbol{\mu}).$$

A Posteriori Error Estimate

Definition: **Residuum of the FV/DG Method at Time t^k**

$$R^{k+1}(\boldsymbol{\mu})[c_N] := \frac{1}{\Delta t}\left(L_I^k(\boldsymbol{\mu})[c_N^{k+1}(\boldsymbol{\mu})] - L_E^k(\boldsymbol{\mu})[c_N^k(\boldsymbol{\mu})] - b^k(\boldsymbol{\mu})\right)$$

Theorem: **A Posteriori Error Estimate**

$$\left\|c_N^k(\boldsymbol{\mu}) - c_H^k(\boldsymbol{\mu})\right\|_{L^2(\Omega)} \leq \sum_{l=0}^{k-1} \Delta t \, (C_E)^{k-1-l} \, \left\|R^{l+1}(\boldsymbol{\mu})[c_N(\boldsymbol{\mu})]\right\|_{L^2(\Omega)}$$

Offline–Online Decomposition

Goal: All Steps for the Calculation of $c_N(\boldsymbol{\mu})$ and for the Calculation of the Error Estimator are split into Two Parts:

- **Offline**–Step: Complexity **depending** on $\dim(W_H)$
- **Online**–Step: Complexity **independent** of $\dim(W_H)$

Constrained: Affine Parameter Dependency of the Evolution Scheme

$$L_I^k(\boldsymbol{\mu})[\cdot] \;=\; \sum_{q=1}^{Q} \underbrace{L_I^{k,q}[\cdot]}_{\text{depending on } x} \quad \underbrace{\sigma_{L_I}^q(\boldsymbol{\mu})}_{\text{depending on } \boldsymbol{\mu}}$$

$\Longrightarrow$ **Precompute offline:** $(L_I^{k,q})_{nm} := \int\limits_\Omega \varphi_n L_I^{k,q}[\varphi_m]$

$\Longrightarrow$ **Assemble online:** $(L_I^k(\boldsymbol{\mu}))_{nm} := \sum\limits_{q=1}^{Q} (L_I^{k,q})_{nm}\sigma_{L_I}^q(\boldsymbol{\mu})$

Numerical Results

CPU-Time Comparison for the Convetion-Diffsuion Problem:

Diskretization: 40×200 **Elements,** $K = 200$ **time steps**

	timedependent data			constant data		
	Reference	RB online	RB offline	Reference	RB online	RB offline
implizit	155.94s	16.67s	447.16s	45.67s	1.02s	2.41s
Factor		9.44			44.77	
explizit	105.97s	16.53s	437.20s	1.51s	0.79s	2.31s
Factor		6.41			1.91	

Discretization: 80×400 **Elements,** $K = 1000$ **time steps**

	time dependent data			constant data		
	Reference	RB online	RB offline	Reference	RB online	RB offline
implizit	4043.18s	143.57s	8693.90s	924.91s	6.18s	9.22s
Factor		28.27			149.66	
explizit	2758.20s	134.00s	8506.60s	17.41s	3.64s	8.83s
Factor		20.58			4.78	

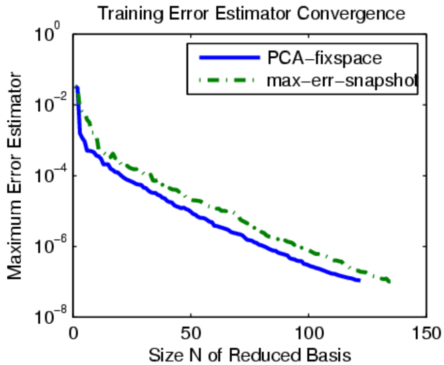
Efficient Choice of W_N

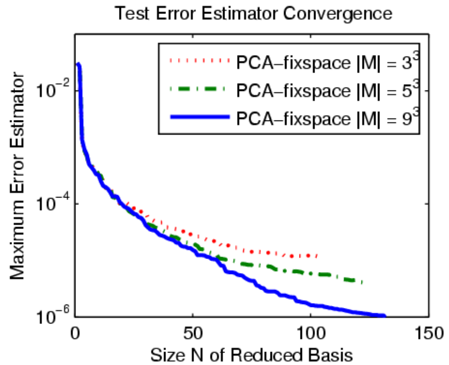
Idea:

- Construct $\tilde{W}_{\tilde{N}}$ as the span of Snapshots $c_H(\mu)$, $\mu \in D \subset P$.
- Use Error Estimator for a efficient Choice of the Snapshots with Guaranteed Error Control on a Training Set.
- Reduce $\tilde{W}_{\tilde{N}}$ zu W_N with Principal Component Analysis (PCA).

Goal: Exponential Convergence in N!

Preliminary Result: Convergence in N for Training and Test Sets





Improvement through Adaptive Basis Enrichement

Algorithms for Basis Enrichemnt: Fixed / Adaptive Training Sets

$\text{ESGREEDY}(\Phi_0, M_{train}, \varepsilon_{tol}, M_{val}, \rho_{tol})$

- $\Phi := \Phi_0$
- **repeat**
  - $\boldsymbol{\mu}^* := \arg\max_{\boldsymbol{\mu} \in M_{train}} \Delta(\boldsymbol{\mu}, \Phi)$
  - **if** $\Delta(\boldsymbol{\mu}^*) > \varepsilon_{tol}$
    - **then**
      - $\varphi := \text{ONBASISEXT}(u_H(\boldsymbol{\mu}^*), \Phi)$
      - $\Phi := \Phi \cup \{\varphi\}$
  - $\varepsilon := \max_{\boldsymbol{\mu} \in M_{train}} \Delta(\boldsymbol{\mu}, \Phi)$
  - $\rho := \max_{\boldsymbol{\mu} \in M_{val}} \Delta(\boldsymbol{\mu}, \Phi)/\varepsilon$
- **until** $\varepsilon \leq \varepsilon_{tol}$ **or** $\rho \geq \rho_{tol}$
- **return** Φ, ε

$\text{RBADAPTIVE}(\Phi_0, \mathcal{M}_0, \varepsilon_{tol}, M_{val}, \rho_{tol})$

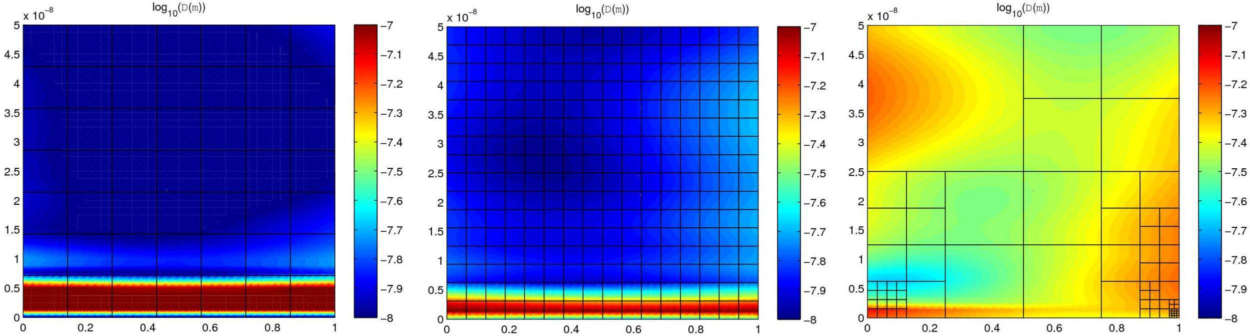
- $\Phi := \Phi_0, \mathcal{M} := \mathcal{M}_0$
- **repeat**
  - $M_{train} := V(\mathcal{M})$
  - $[\Phi, \varepsilon] := \text{ESGREEDY}(\Phi, M_{train}, \varepsilon_{tol}, M_{val}, \rho_{tol})$
  - **if** $\varepsilon > \varepsilon_{tol}$
    - **then**
      - $\boldsymbol{\eta} = \text{ELEMENTINDICATORS}(\mathcal{M}, \Phi, \varepsilon)$
      - $\mathcal{M} := \text{MARK}(\mathcal{M}, \boldsymbol{\eta})$
      - $\mathcal{M} := \text{REFINE}(\mathcal{M})$
- **until** $\varepsilon \leq \varepsilon_{tol}$
- **return** Φ

Here, $\mathcal{M}$ denotes a Partition of the Parameter Space,
and $V(\mathcal{M})$ are the Vertices of $\mathcal{M}$.

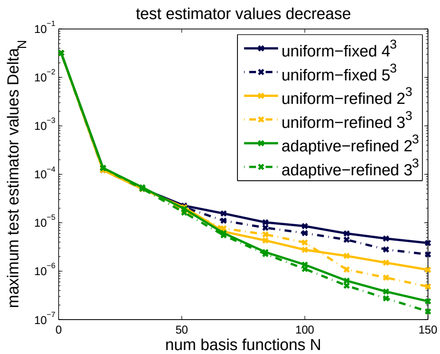
mario.ohlberger@uni-muenster.de www.uni-muenster.de/math/u/ohlberger

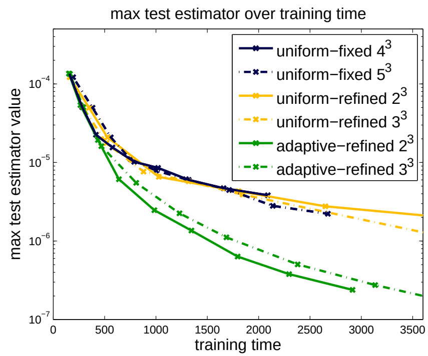
Improvement through Adaptive Basis Enrichement

Error Distribution for Uniform / Adaptive Trainig Sets



Exponential Convergence and CPU-Efficiency





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Outlook

- Generalize to Non-Affine Parameter Dependency
- **Generalize to Non-Linear Evolution Equations**
- **Error Estimates for more Complex Systems**

First Results based on Empirical Interpolation:

B. Haasdonk, M. Ohlberger, G. Rozza.
A reduced basis method for evolution schemes with parameter-dependent explicit operators.
Preprint 09/07-N, FB 10, Universität Münster (2007). Accepted for publication in ETNA.

Thank you for your attention!

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Software: DUNE ALUGrid GRAPE

`www.uni-muenster.de/math/num/Arbeitsgruppen/ag_ohlberger/software`