An *A Posteriori* Error Bound for the Discrete Duality Finite Volume Discretization of the Laplace Equation

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Résumé

The main advantage of the Discrete Duality Finite Volume (DDFV) method applied to the Laplace equation, is that it may be used on any type of meshes, including highly non conforming meshes. In order to drive local refinement of the meshes, an a posteriori error estimator is required. This work presents such an estimator, which is derived thanks to the discrete variational formulation of the scheme. Tests on singular and stiff problems confirm the efficiency of the estimator.

1. The DDFV scheme

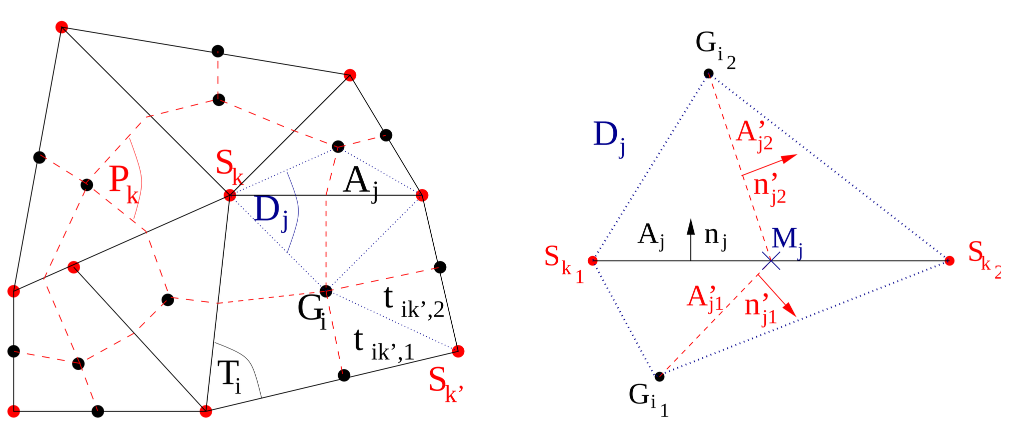

Fig. 1: *Notations related to the primal, dual and diamond cells*

Unknowns ϕ_i at G_i and unknowns ϕ_k at S_k.

Integration of $-\nabla\cdot\nabla\hat{\phi} = f$ over T_i and P_k.
Evaluation of the fluxes $\int \nabla\hat{\phi}\cdot\mathbf{n}$ over ∂T_i or ∂P_k in D_j using the discrete gradient $(\nabla_h\phi)_j$ defined over D_j :

$$(\nabla_h\phi)_j := \frac{1}{2\,|D_j|}\{\ [\phi_{k_2}-\phi_{k_1}]\,(|A'_{j1}|\mathbf{n}'_{j1}+|A'_{j2}|\mathbf{n}'_{j2}) + [\phi_{i_2}-\phi_{i_1}]\,|A_j|\mathbf{n}_j\}.$$

The scheme reads : Find (ϕ_i,ϕ_k) such that for all T_i and P_k

$$-\sum_{j\in\partial T_i}(\nabla_h\phi)_j\cdot|A_j|\mathbf{n}_j = |T_i|\bar{f}_i\,,$$

$$-\sum_{j\in\partial P_k}(\nabla_h\phi)_j\cdot(|A'_{j1}|\mathbf{n}'_{j1}+|A'_{j2}|\mathbf{n}'_{j2}) = |P_k|\bar{f}^P_k\,.$$

Equivalent discrete symmetric positive definite variational formulation, where ϕ_h is P^1 in each D_j.

$$\sum_j\int_{D_j}\nabla_h\phi_h\cdot\nabla_h\psi_h(\mathbf{x})\,d\mathbf{x} = \int_\Omega f\,\psi^*_h(\mathbf{x})\,d\mathbf{x} \qquad (1)$$

for any discrete ψ vanishing on Γ and with

$$\psi^*_h(\mathbf{x}) := \frac{1}{2}\left(\sum_{i\in[1,I]}\psi^T_i\mathbf{1}_{T_i}(\mathbf{x}) + \sum_{k\in[1,K]}\psi^P_k\mathbf{1}_{P_k}(\mathbf{x})\right)$$

2. Representation of the error

We seek to measure the broken H^1 semi-norm of the error

$$e^2 = \sum_j\int_{D_j}\left|\nabla\hat{\phi}-\nabla_h\phi_h\right|^2(\mathbf{x})\,d\mathbf{x}.$$

Orthogonal decomposition of the error in $(L^2(\Omega))^2$

$$\nabla\hat{\phi}-\nabla_h\phi_h = \nabla\hat{\Phi}+\nabla\times\hat{\Psi} \quad\text{and}\quad e^2 = \left\|\nabla\hat{\Phi}\right\|^2_{0,\Omega}+\left\|\nabla\times\hat{\Psi}\right\|^2_{0,\Omega}$$

with $\hat{\Phi}\in H^1_0(\Omega)$ and $\hat{\Psi}\in H^1(\Omega)$

$$e^2 = \sum_j\int_{D_j}(\nabla\hat{\phi}-\nabla_h\phi_h)\cdot\nabla\hat{\Phi}\,(\mathbf{x})\,d\mathbf{x} + \sum_j\int_{D_j}(\nabla\hat{\phi}-\nabla_h\phi_h)\cdot\nabla\times\hat{\Psi}\,(\mathbf{x})\,d\mathbf{x}.$$

Let i_1 (resp. i_2) be the first (resp. the second) of these two integrals. Applying the continuous variational formulation to $(\hat{\phi},\hat{\Phi})$ and considering an arbitrary $\Phi = (\Phi^T_i,\Phi^P_k)$ vanishing on Γ, for which (1) holds, we may write

$$i_1 = \int_\Omega f\,\left(\hat{\Phi}-\Phi^*_h\right)\,d\mathbf{x} - \sum_j\int_{D_j}\nabla_h\phi_h\cdot\left(\nabla\hat{\Phi}-\nabla_h\Phi_h\right)\,d\mathbf{x}$$

Using Green's formula over each diamond-cell, denoting by s any edge $[G_iS_k]$ when j runs over the whole set of diamond-cells, by $\mathbf{n}_s$ one of the two unit normal vectors to s, and by $[\nabla_h\phi_h\cdot\mathbf{n}_s]_s$ the jump of the normal component of $\nabla_h\phi_h$ through s in the direction $\mathbf{n}_s$, and finally applying the boundary conditions, we obtain

$$\begin{aligned}2i_1 =& \sum_{i\in[1,I]}\int_{T_i}f\,\left(\hat{\Phi}-\Phi^T_i\right)\,d\mathbf{x} + \sum_{k\in[1,K]}\int_{P_k}f\,\left(\hat{\Phi}-\Phi^P_k\right)\,d\mathbf{x}\\ &-\sum_{i\in[1,I]}\sum_{s\subset\overset{\circ}{T_i}}\int_s[\nabla_h\phi_h\cdot\mathbf{n}_s]_s\,\left(\hat{\Phi}-\Phi^T_i\right)(\sigma)\,d\sigma \qquad (2)\\ &-\sum_{k\in[1,K]}\sum_{s\subset\overset{\circ}{P_k}}\int_s[\nabla_h\phi_h\cdot\mathbf{n}_s]_s\,\left(\hat{\Phi}-\Phi^P_k\right)(\sigma)\,d\sigma\,.\end{aligned}$$

Applying the continuous Green formula to the $L^2(\Omega)$ dot product of $\nabla\hat{\phi}$ and $\nabla\times\hat{\Psi}$ and calculating the $L^2(\Omega)$ dot product of $\nabla_h\phi_h$ and $\nabla_h\times\Psi_h$, where the function Ψ_h is associated with an arbitrary $\Psi = (\Psi^T_i,\Psi^P_k)$ we may write

$$i_2 = -\sum_j\int_{D_j}\nabla_h\phi_h\cdot\left(\nabla\times\hat{\Psi}-\nabla_h\times\Psi_h\right)d\mathbf{x}.$$

Proceeding now as in the calculation of i_1, taking into account boundary conditions and denoting by $\boldsymbol{\tau}_s$ one of the two unit tangential vectors to s, we obtain

$$\begin{aligned}2i_2 =& -\sum_{i\in[1,I]}\sum_{s\subset\overset{\circ}{T_i}}\int_s[\nabla_h\phi_h\cdot\boldsymbol{\tau}_s]_s\,\left(\hat{\Psi}-\Psi^T_i\right)(\sigma)\,d\sigma \qquad (3)\\ &-\sum_{k\in[1,K]}\sum_{s\subset\overset{\circ}{P_k}}\int_s[\nabla_h\phi_h\cdot\boldsymbol{\tau}_s]_s\,\left(\hat{\Psi}-\Psi^P_k\right)(\sigma)\,d\sigma\,.\end{aligned}$$

3. A calculable error bound

In (2) we choose (Φ^T_i,Φ^P_k) as the L^2 projections of $\hat{\Phi}$ on the primal and dual cells.

$$\Phi^T_i = \frac{1}{|T_i|}\int_{T_i}\hat{\Phi}(\mathbf{x})\,d\mathbf{x}\ \forall i\ ,\ \Phi^P_k = \frac{1}{|P_k|}\int_{P_k}\hat{\Phi}(\mathbf{x})\,d\mathbf{x}\ \forall k\,.$$

Then the first term of (2) may be bounded by :

$$\begin{aligned}\left|\sum_{i\in[1,I]}\int_{T_i}f\left(\hat{\Phi}-\Phi^T_i\right)d\mathbf{x}\right| &= \left|\sum_{i\in[1,I]}\int_{T_i}\left(f-\bar{f}^T_i\right)\left(\hat{\Phi}-\Phi^T_i\right)d\mathbf{x}\right|\\ &\le \sum_{i\in[1,I]}\left\|f-\bar{f}^T_i\right\|_{0,T_i}\left\|\hat{\Phi}-\Phi^T_i\right\|_{0,T_i}\\ &\le \left(\sum_{i\in[1,N]}\left(C^T_i\,h^T_i\right)^2\left\|f-\bar{f}^T_i\right\|^2_{0,T_i}\right)^{1/2}\left\|\nabla\hat{\Phi}\right\|_{0,\Omega}.\end{aligned}$$

thanks to the Poincaré-type inequality

$$\exists C^T_i,\ \text{s.t.}\ \left\|\hat{\Phi}-\Phi^T_i\right\|_{0,T_i}\le C^T_i\,h^T_i\,\left\|\nabla\hat{\Phi}\right\|_{0,T_i}. \qquad (4)$$

Bounds for C^T_i depend on the shape of T_i and may be found, e.g. in works by Carstensen and Funken and by Verfürth. As soon as f is more regular than $L^2(\Omega)$, this is a Higher Order Term (H.O.T.) : asymptotically negligible as compared to the global convergence order of the method (which is at most $O(h)$). Other H.O.T. are provided by the second term in (2).
Let us now consider the main terms in the expression of i_1. Let T_i be a primal cell ; through Cauchy-Schwarz inequalities, we obtain

$$\left|\sum_{s\in\overset{\circ}{T_i}}\int_s[\nabla_h\phi_h\cdot\mathbf{n}_s]_s\left(\hat{\Phi}-\Phi^T_i\right)(\sigma)d\sigma\right|^2 \le C\left(\sum_{s\in\overset{\circ}{T_i}}|s|^{-1}\,\|[\nabla_h\phi_h\cdot\mathbf{n}_s]_s\|^2_{0,s}\right)\ \sum_{s\in\overset{\circ}{T_i}}|s|\left\|\hat{\Phi}-\Phi^T_i\right\|^2_{0,s}$$

For each segment s, we apply a trace inequality on each of the two triangles $t_{ik,1}$ and $t_{ik,2}$ defined on Figure 1.

$$\left\|\hat{\Phi}-\Phi^T_i\right\|^2_{0,s}\le C|s|^{-1}\left(\left\|\hat{\Phi}-\Phi^T_i\right\|^2_{0,t_{ik,\alpha}}+(h^T_i)^2\left\|\nabla\hat{\Phi}\right\|^2_{0,t_{ik,\alpha}}\right).$$

Summing the norms over the various $t_{ik;\alpha}$ into norms on the whole T_i and applying (4), there holds

$$\left|\sum_{i\in[1,I]}\sum_{s\in\overset{\circ}{T_i}}\int_s[\nabla_h\phi_h\cdot\mathbf{n}_s]_s\,\left(\hat{\Phi}-\Phi^T_i\right)(\sigma)\,d\sigma\right|\le\left\|\nabla\hat{\Phi}\right\|_{0,\Omega}\eta^T.$$

with $(\eta^T)^2 = \sum_i(\eta^T_i)^2$ and

$$(\eta^T_i)^2 = C(h^T_i)^2\left(\sum_{s\in\overset{\circ}{T_i}}|s|^{-1}\,\|[\nabla_h\phi_h\cdot\mathbf{n}_s]_s\|^2_{0,s}\right)$$

We obtain a similar bound for the dual terms in (2), and for both terms in (3). Thus, finally,

$$e\le\frac{1}{2}\left[\left(\eta^T+\eta^P\right)^2+\left(\eta'^T+\eta'^P\right)^2\right]^{1/2} + \text{H.O.T.}.$$

All the terms (including the H.O.T.) in the above bound are explicitly calculable. Checking the local efficiency of the quantities $\eta^T_i,\cdots$ is straightforward and can be performed in the usual way using bubble functions. We shall use the local quantities η in order to refine the mesh in an adaptive way.

4. Numerical results

4.1 Singular solution

We choose $\Omega =]-0.5;0.5[^2\setminus[0;0.5]\times\{0\}$. The data f is zero and the exact solution of the Laplace equation is $\hat{\phi} = r^{1/2}\sin(\theta/2)$ in cylindrical coordinates (r,θ) centered on $(0,0)$. The exact solution thus belongs to $H^{1+s}(\Omega)$, with $s<1/2$. An adaptive strategy based on the aggregation of the above estimators into one single local estimator η_i is employed. We refine T_i if $\eta_i\ge\frac{1}{2}\max_q(\eta_q)$.

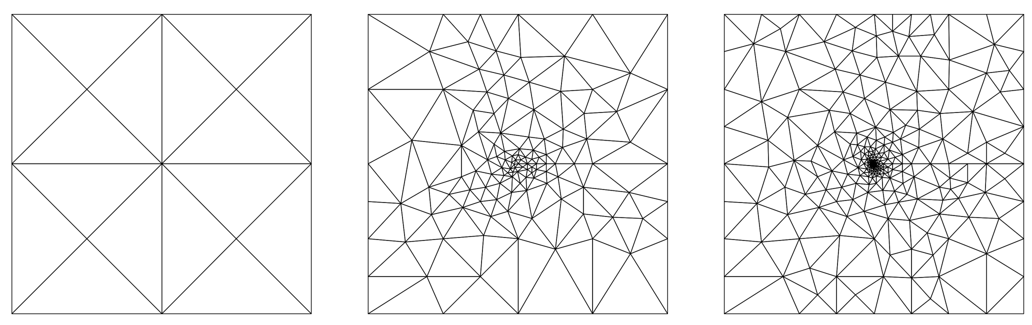

Fig. 2: *initial, 5th and 10th adaptive meshes*

The error behaves like $N^{-1/2}$, where N is the number of triangles on adaptive meshes and like $N^{-1/4}$ on uniformly refined meshes. Thus, the adaptive process enables us to recover the optimal rate of convergence. The efficiency η/e of the estimator tends to about 7 for adaptive meshes and to 12 for uniformly refined meshes.

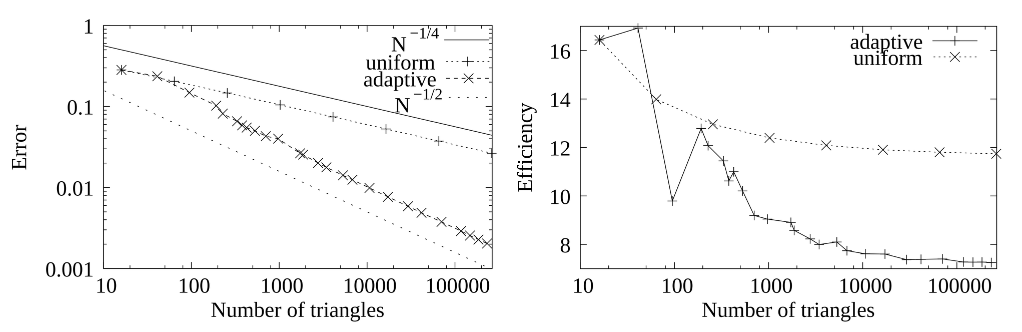

Fig. 3: *Error and efficiency curves for adaptive and uniform meshes*

4.2 Stiff solution

Let $\Omega =]-1,1[^2$. Let us choose f so that

$$\hat{\phi} = \cos(k\pi x)\cos(k\pi y)+\alpha\chi(r)\,\exp(1/\varepsilon^2)\,\exp[-1/(\varepsilon^2-r^2)]\,.$$

with $r=\sqrt{x^2+y^2}$ and $\chi(r)=1$ if $r\le\varepsilon$, while $\chi(r)=0$ if $r>\varepsilon$. We set $k=1/2$, $\alpha=10$ and $\varepsilon=1/4$.
We consider $\omega = [-1/4,1/4]^2$ and $\Omega\setminus\omega$ is uniformly meshed with squares of length h, while ω is uniformly meshed with squares of length $h_0 = h/2^p$. The mesh corresponding to $h=1/4$ and $h/h_0=4$ is presented on Fig. 4.

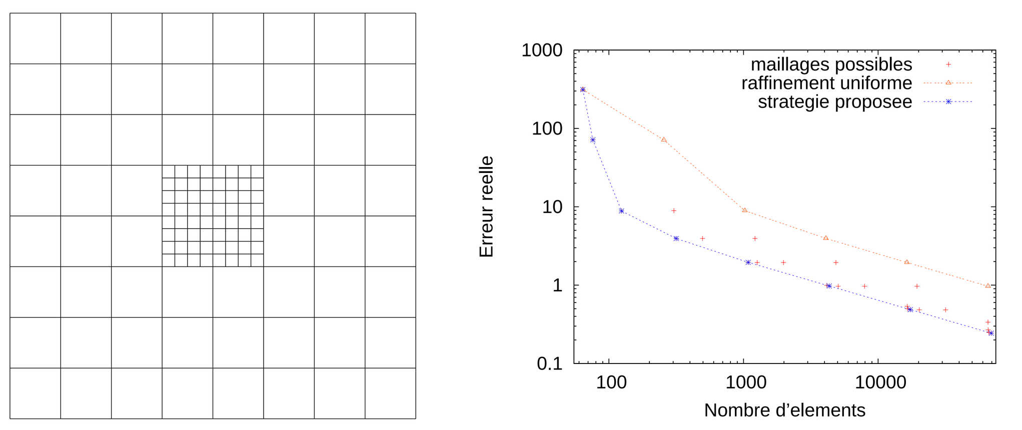

Fig. 4: *Left : Example of a non conforming mesh with $h=1/4$ and $p=2$. Right : Actual errors on meshes with various choices of h and p.*

The refinement is driven by the comparison of the average estimator inside and outside ω. When one of them is more than twice greater than the other, the corresponding part of the mesh is refined. If not both parts of the mesh are refined. The initial mesh is uniform with $h=1/4$. The right part of Fig. 4 presents a cloud of points with the actual errors corresponding to calculations performed on meshes obtained with an arbitrary choice of the couple (h,h_0). It appears that refining uniformly is the worst strategy, while the error curve for the proposed strategy is optimal. It leads to a refinement of the initial mesh in the central region up to $h/h_0=16$, with $h=1/4$, and then to a uniform refinement.

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