2D/3D DDFV scheme for anisotropic-heterogeneous elliptic equations, application to a bio-mathematics problem: electrocardiogram simulation.

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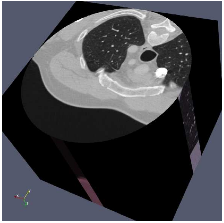
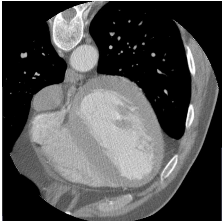
Outline :

1. Context, motivations.
2. 2D/3D DDFV sxheme definition,
   for anisotropic/heterogeneous elliptic equations.
3. Numerical analysis.
4. ECG simulation.

Medical data analysis

- Medical images (CT Scan, 3D)



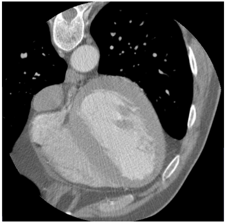


Heart Institute,
University of Ottawa.

Résolution : $1 \times 1 \times 2$ mm.

Pixels : 62 500 000

- Segmentation process (Level Set, Mumford-Shah functional)



$\longrightarrow$

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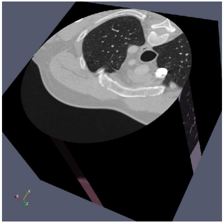

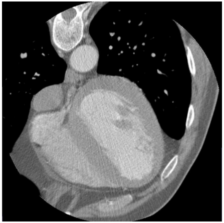

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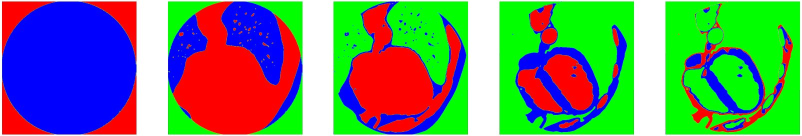

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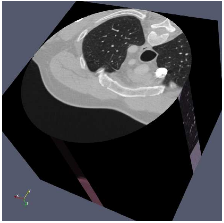

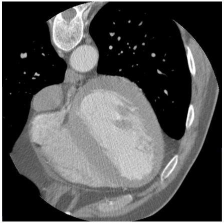

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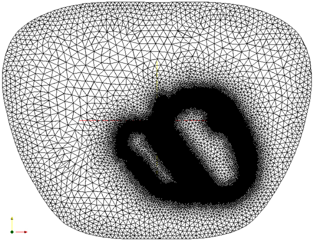
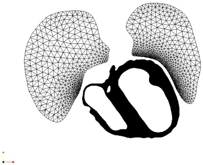
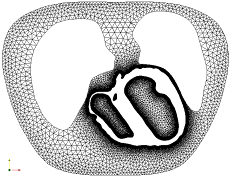
- Physiologically relevant mesh generation

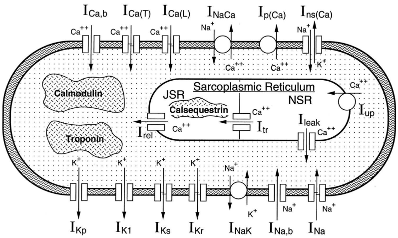






Available models

- Cardiac cells electrical activity
  $\longrightarrow$ EDO system



Available models

- Cardiac cells electrical activity

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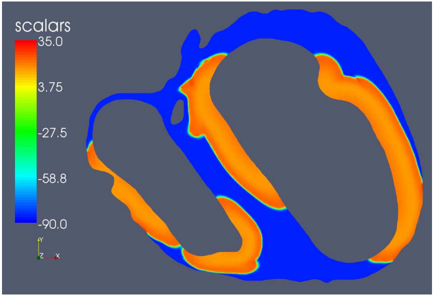
- Cardiac tissue electrical activity

  $\longrightarrow$ Electrical potential waves propagation

  $\longrightarrow$ Parabolic reaction diffusion equation

  $\longrightarrow$ Description of a *membrane potential* $V_m(x,t)$ in the heart



Available models

- Cardiac cells electrical activity
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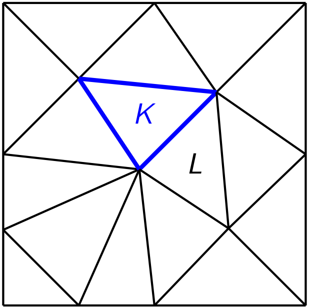
  - $\longrightarrow$ Electrical potential waves propagation
  - $\longrightarrow$ Parabolic reaction diffusion equation
  - $\longrightarrow$ Description of a *membrane potential* $V_m(x,t)$ in the heart
- Torso model
  - $\longrightarrow$ Elliptic anisotropic and heterogeneous equation
  - $\longrightarrow$ Description of a potential $\phi_T(x,t)$ in the whole torso
  - $\longrightarrow$ Body surface potential measurement : **ECG** simulation

$$\mathrm{div}(G\nabla\phi_T) = F(V_m)$$

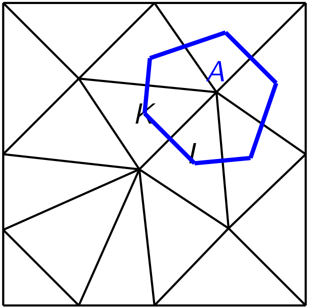
- $G = G(x)$ discontinuous (organs heterogeneity).
- G anisotropic (muscles fibres directions).

DDFV Meshes



- **Primal mesh** :
  one scalar unknown at every primal cell centre K, L, ...

DDFV Meshes



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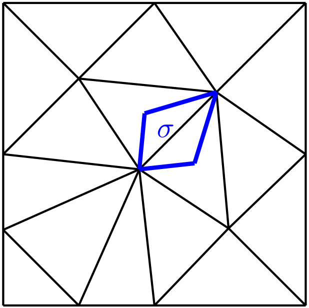
- **Dual mesh** :
  one scalar unknown at every mesh vertex A (*i.e.* every dual cell).

DDFV Meshes



- **Primal mesh** :
  one scalar unknown at every primal cell centre K, L, ...
- **Dual mesh** :
  one scalar unknown at every mesh vertex A (*i.e.* every dual cell).
- **Diamond mesh** :
  one flux vector associated to every interface σ.

DDFV Scheme principle

Construction of two discrete operators

- **Discrete gradient :** scalar data $\longrightarrow$ vectorial data
- **Discrete divergence :** vectorial data $\longrightarrow$ scalar data

satisfying a duality property, mimiking the *Green formula* in the continuous context.

Property (DDFV scheme 2D-3D)

- *By construction, the dicrete gradient and divergence operators fulfill a duality property : discrete Green Formula.*
- *Discretisation of elliptic equations* $\Longrightarrow$ **well posed** *symmetric, positive definite system.*
- *Holds for general anisotropic, heterogeneous tensors both in dimension 2 and 3.*

Discrete gradient

Scalar data $(\phi_C)_{C\in\mathcal{C}} \longrightarrow$ continuous piecewise affine function ϕ :

- $\mathcal{C}$ primal and dual cells set.
- $(\phi_C)_{C\in\mathcal{C}}$: one data per primal, dual cells

$\Longrightarrow$ function ϕ by P^1 interpolation on diamond cells halves.

- Discrete gradient : $(\nabla_h\phi_C)_\sigma$ on an interface σ.

$$(\nabla_h\phi_C)_\sigma = \frac{1}{|D(\sigma)|}\int_{D(\sigma)} \nabla\phi\, dx ,$$

$D(\sigma)$ diamond cell associated to σ.

- Discrete trace : $(\mathrm{Tr}_h\phi_C)_\sigma$ on boundary interfaces $\sigma \subset \partial\Omega$.

$$(\mathrm{Tr}_h\phi_C)_\sigma = \frac{1}{|\sigma|}\int_\sigma \phi\, dx ,$$

Discrete divergence

Vector data $(\mathbf{p}_\sigma) \longrightarrow$ piecewise constant vector field $\mathbf{p}$ on the diamond cells.

- Discrete divergence : $(\mathrm{div}_h \mathbf{p}_\sigma)_C$ on a cell $C \in \mathcal{C}$ (primal or dual).

$$(\mathrm{div}_h \mathbf{p})_C = \frac{1}{|C|} \int_{\partial C} \mathbf{p} \cdot \mathbf{n} \, dx$$

Theorem (discrete Green formula)

Being given a vector data $(\mathbf{p}_\sigma)$ and a scalar data (ϕ_C) :

$$\int_\Omega \nabla_h \phi \cdot \mathbf{p} \; dx = - \langle \phi_C, \mathrm{div}_h \mathbf{p} \rangle \; + \; \int_{\partial\Omega} Tr_h \phi \; \mathbf{p} \cdot \mathbf{n} \; ds,$$

$$\langle \phi_C, \mathrm{div}_h \mathbf{p} \rangle = \frac{1}{2} \sum_{C \in \mathcal{C}} (\mathrm{div}_h \mathbf{p})_C \; \phi_C \; |C|.$$

with : $\mathcal{C}$ primal and dual cells set.

Heterogeneous case

$$\text{div}(G\nabla\phi) = f \ , \quad \text{with :}$$

G discontinuous when crossing some media-interface Γ.
The mesh is supposed to be adapted to Γ :

$$\Gamma \subset \bigcup_{\sigma \text{ mesh interface}} \sigma \ .$$

Adaptation. When $\sigma \subset \Gamma$:

- Two discrete tensors are associated to σ (one on each side).
- Two discrete gradient vectors are computed on σ (one on each side).
- One supplementary normal flux continuity condition is imposed on σ.

$\longrightarrow$ Achieved by adding one auxiliary scalar unknown at the interface center, locally eliminated.

3D case : diamond cells

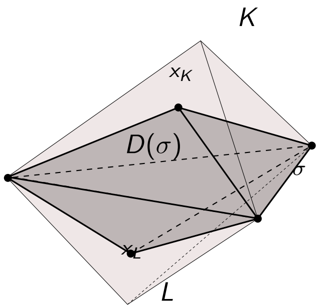


Fig.: 3D diamond cell

3D case : dual cells

$$P(A) = \bigcup_{\substack{\text{interfaces } \sigma, \\ A \in \sigma}} P_\sigma(A)$$

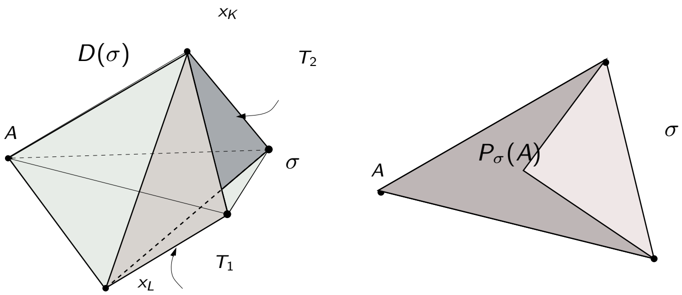
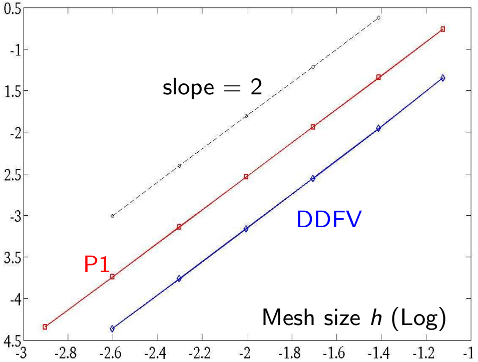


FIG.: 3D dual cells construction

2D convergence analysis

$$\Delta u = f \quad \text{on } \Omega$$
$$u_{|\partial\Omega} = 0 \quad \text{on } \partial\Omega$$

L^2 relative error (Log)



2D DDFV *vs* P1 : L^2-norm convergence

- $\Omega = (-1, 1)^2$
- f trigonometric
- DDFV *vs* P^1 finite element,
- 177 $\rightarrow$ 650 000 points triangular meshes.

2D convergence analysis

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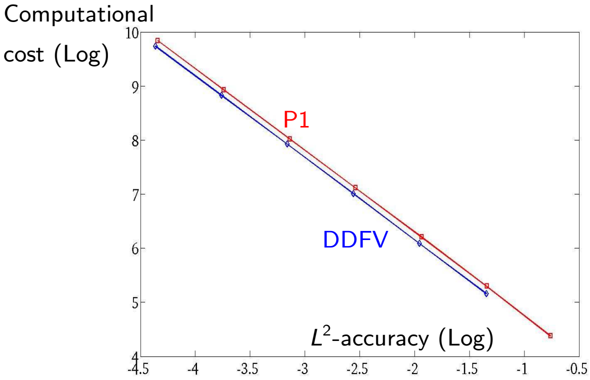

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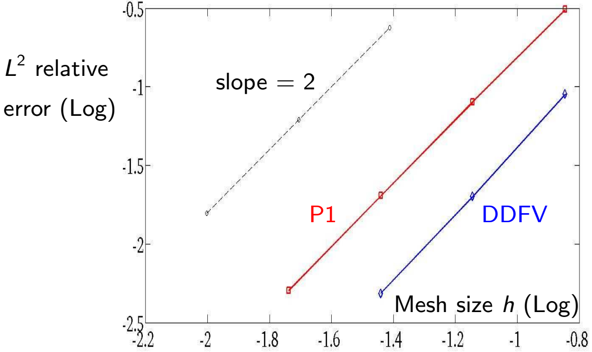
2D DDFV *vs* P1 : comparative computational costs

3D convergence analysis

$$\begin{aligned} \Delta u &= f \quad \text{on } \Omega \\ u_{|\partial\Omega} &= 0 \quad \text{on } \partial\Omega \end{aligned}$$



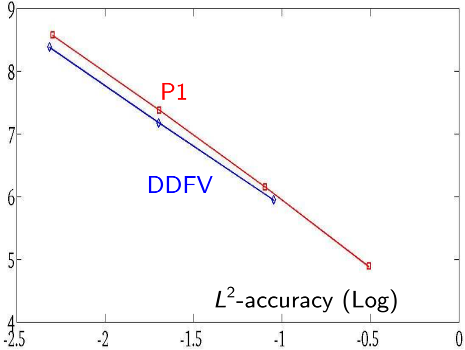
3D DDFV *vs* P1 : L^2-norm convergence

- $\Omega = (0,1)^3$
- f trigonometric
- DDFV *vs* P^1 finite element,
- 500 $\rightarrow$ 500 000 points tetrahadral meshes.

3D convergence analysis

$$\begin{aligned} \Delta u &= f \quad \text{on } \Omega \\ u_{|\partial\Omega} &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

Computational cost (Log)



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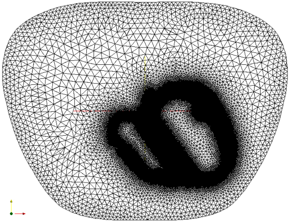
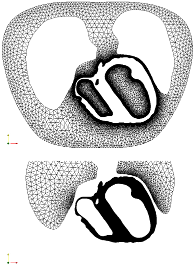
3D DDFV *vs* P1 : comparative computational costs

Numerical simulations : settings

- 2D mesh : from **DistMesh** code (P.-O. Persson, G. Strang).





→ Heart mesh

$\simeq$ 500 000 vertices

→ Entire mesh (heart+torso)

$\simeq$ 600 000 vertices

- Anisotropy :

$$\mathrm{div}(G\nabla\phi_T) = F(V_m) \,, \quad \text{at each time step.}$$

Ventricles :

$G = G(x)$: anisotropic, depending on muscular finres directions

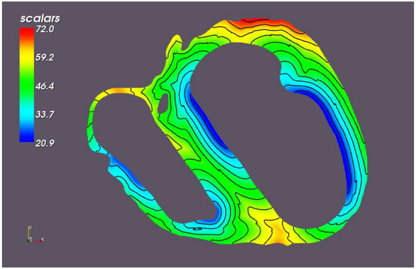
Lungs, ventricle cavity, remaining tissues :

$G = g_o \mathrm{Id}$: homogeneous, heterogeneous

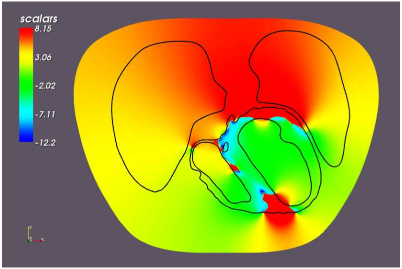
g_o discontinuous at organs boundaries.

ECG simulation : resultts

Simulation :

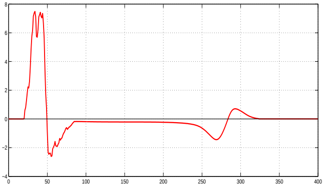


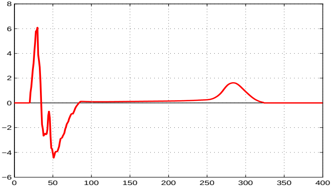
Membrane potemtial V_m Isochrones.



Torso potential ϕ_T.

Body surface potential recordings (ECG) :





Perspectives

1. Further numerical investigations of the DDFV scheme (behaviour in presence of strong anisotropy/heterogeneity).
2. Theoretical analysis (convergence analysis, discrete Poincaré inequality).
3. Behaviour on non conformal meshes, towards local mesh refinment.