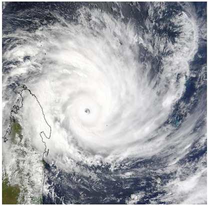
Influence of cell geometry on the accuracy of the Roe-scheme in the low Mach number regime

FVCA 2008



Felix Rieper

Institute for Applied Mathematics and Scientific Computing
BTU Cottbus / Germany

Mach number and Compressibility

$$\mathrm{M} = \frac{u}{a} > 1$$

$$\tilde{\rho} = \mathcal{O}(1)$$

fully compressible

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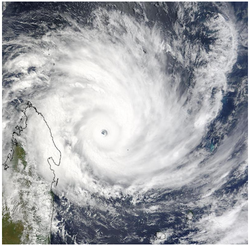
$$\tilde{\rho} = \mathcal{O}(\mathrm{M}^2) \approx 0$$

'incompressible'

Mach number and Compressibility

		
$\mathrm{M} = \frac{u}{a} > 1$	$0 < \mathrm{M} < 0.2$	$\mathrm{M} \approx 10^{-3}$
$\tilde{\rho} = \mathcal{O}(1)$	$\tilde{\rho} = f(T, h, \mathrm{M}^2)$	$\tilde{\rho} = \mathcal{O}(\mathrm{M}^2) \approx 0$
fully compressible	weakly compressible	'incompressible'

Compressible-flow solvers at low Mach numbers

- Stiffness problem

$$\frac{|\lambda_{\text{acoustic}}|}{|\lambda_{\text{flow}}|} \approx \frac{a}{u} = \frac{1}{\mathrm{M}}$$

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$$p = 1 + \mathcal{O}(\mathrm{M}^2)$$

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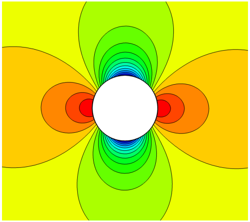
$$p = 1 + \mathcal{O}(\mathrm{M}^2)$$

- 'Accuracy problem'

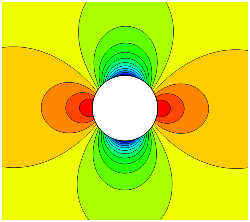
$$p = 1 + \mathcal{O}(\mathrm{M}\Delta x) + \mathcal{O}(\mathrm{M}^2)$$

The accuracy problem: wrong pressure

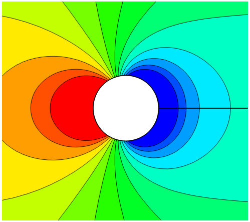


Incompressible flow

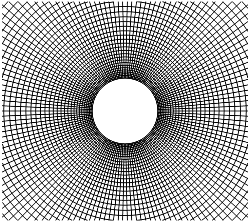
The accuracy problem: wrong pressure



Incompressible flow



Roe at $M = 10^{-3}$



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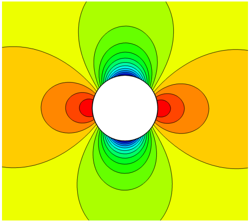

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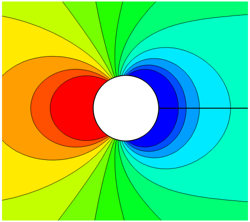
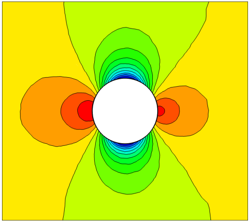


Roe at $\mathrm{M} = 10^{-3}$



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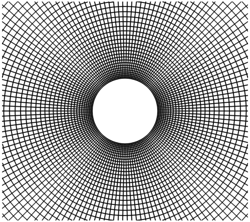
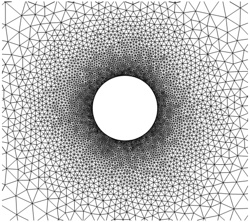
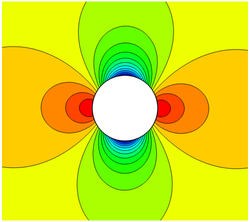




Contents

- Experiments: Accuracy and grid
- Review: asymptotics for Euler equations and Roe scheme
- New: triangular grid cells

Accuracy problem: grid structure or cell geometry?



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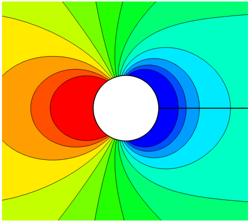

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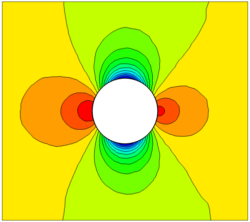

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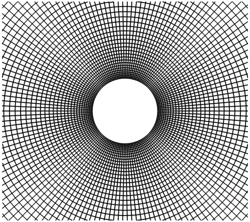
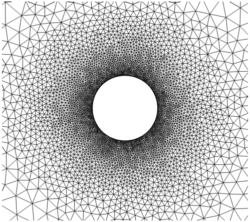
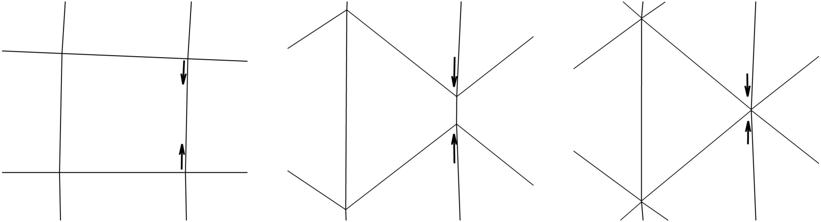




Numerical experiment no. 1



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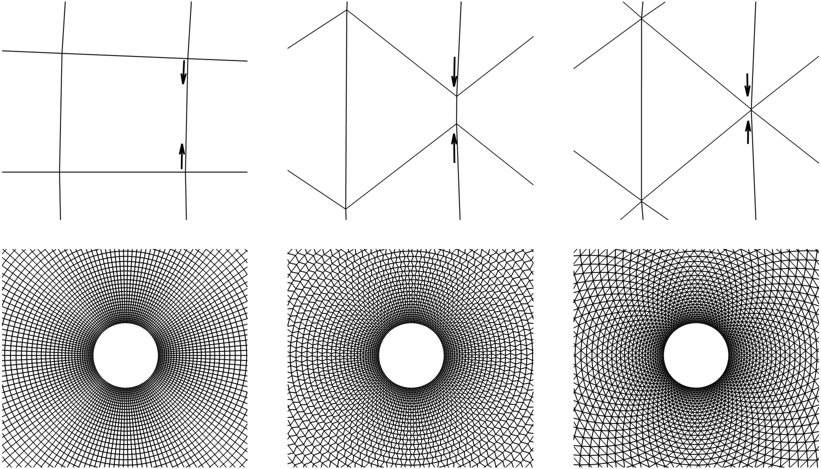

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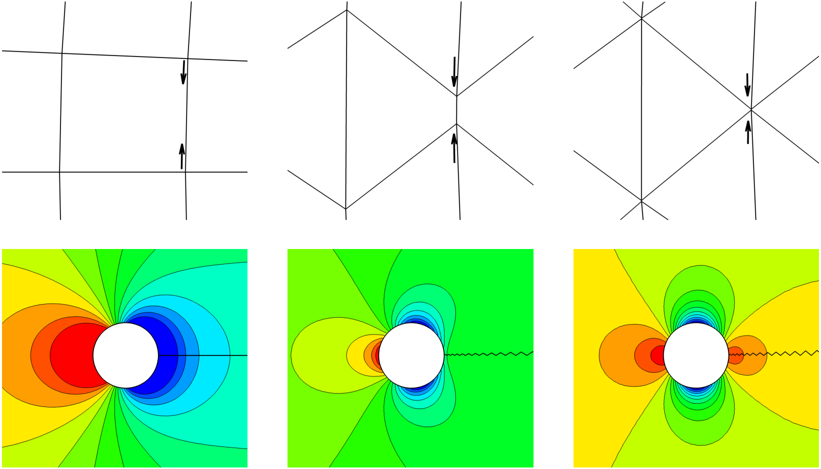
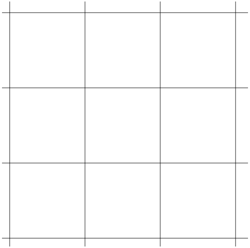
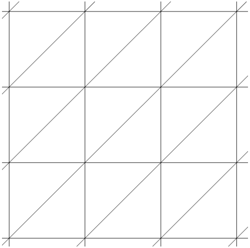


Numerical experiment no. 2





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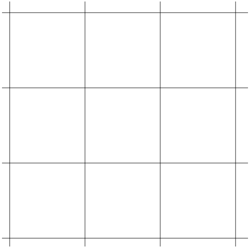

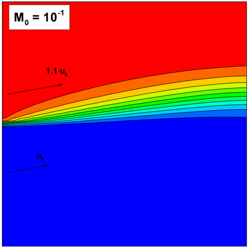

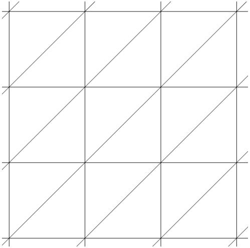

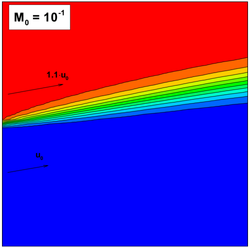

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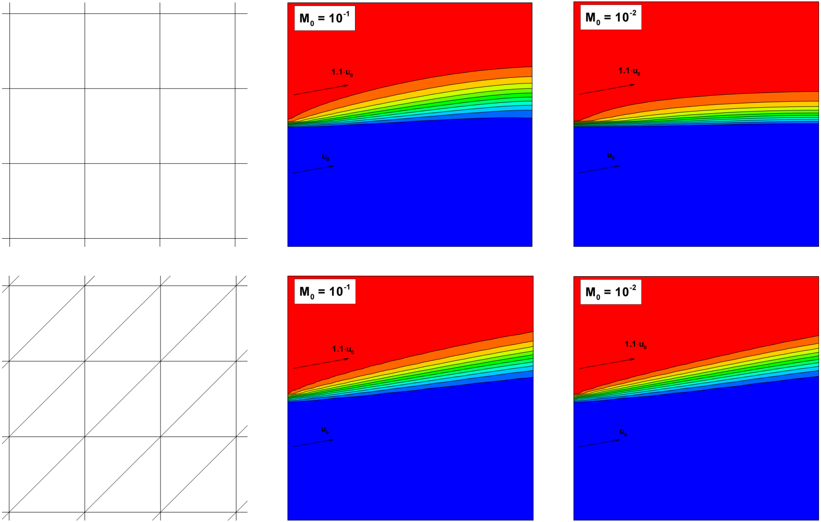


The Euler equations

Inviscid flow:

$$\begin{aligned} \rho_t + \nabla \cdot (\rho \mathbf{u}) &= 0 \\ \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} + \frac{1}{\mathrm{M}^2} \frac{1}{\rho} \nabla p &= 0 \\ p_t + \mathbf{u} \cdot \nabla p + \gamma p \nabla \cdot \mathbf{u} &= 0 \end{aligned}$$

Asymptotic 3-term expansion:

$$\rho = \rho^{(0)} + \mathrm{M}\rho^{(1)} + \mathrm{M}^2\rho^{(2)} + \mathcal{O}(\mathrm{M}^2)$$

$$\mathbf{u} = \mathbf{u}^{(0)} + \mathrm{M}\mathbf{u}^{(1)} + \mathrm{M}^2\mathbf{u}^{(2)} + \mathcal{O}(\mathrm{M}^2)$$

$$p = p^{(0)} + \mathrm{M}p^{(1)} + \mathrm{M}^2p^{(2)} + \mathcal{O}(\mathrm{M}^2)$$

Review: asymptotic Euler equations

$$\nabla p^{(0)} = 0$$

$$\nabla p^{(1)} = 0$$

$$\Rightarrow p = p^{(0)} + \qquad \mathrm{M}^2 p^{(2)} + \mathcal{O}(\mathrm{M}^2)$$

$$\mathbf{u}_t^{(0)} + \mathbf{u}^{(0)} \cdot \nabla \mathbf{u}^{(0)} + \frac{1}{\rho^{(0)}} \nabla p^{(2)} = 0$$

The Roe scheme

Update for conservative variables q_i:

$$\frac{\mathrm{d}}{\mathrm{d}t} q_i + \frac{1}{A} \sum_{l \in \nu(i)} \phi(q_i, q_l, \mathbf{n}_{il}) \delta_{il} = 0$$

with the Roe flux function:

$$\phi(q_i, q_l, \mathbf{n}_{il}) = \frac{\mathbf{f}(q_i) + \mathbf{f}(q_l)}{2} \cdot \mathbf{n}_{il} + \frac{1}{2} \sum_{k=1}^{4} \mathbf{r}_k(q_{il}) |\lambda_k(q_{il})| \Delta w_k(\Delta_{il} q) \ .$$

Review: Roe on Cartesian grids

Guillard & Viozat (1999):

$$\nabla p^{(1)} \sim u_{xx}^{(0)} \Delta x \neq 0$$

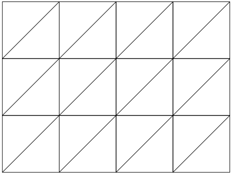
Modified equation:

$$u_t + u \cdot \nabla u + \frac{1}{\mathrm{M}^2} \frac{1}{\rho} \nabla p \sim \frac{1}{\mathrm{M}} u_{xx} \Delta x$$

Roe on triangular grid 1/2

Ingredients of the proof:

- steady problems: $\partial/\partial t = 0$
- $\rho^{(0)} = \text{const}$
- simple geometry
- asymptotic equations for momentum and energy



here occurs a miracle . . .

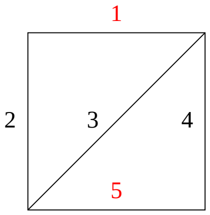
Roe on triangular grid 2/2

$$p^{(1)} \leadsto p, \quad U^{(0)} \leadsto U$$

$$\Delta_1 p + \Delta_1 U = 0$$
$$\Delta_5 p - \Delta_5 U = 0$$

$$\Delta_3 p = 0$$
$$\Delta_3 U = 0$$

$$\Delta_2 p - \Delta_2 U = 0$$
$$\Delta_4 p + \Delta_4 U = 0$$



Roe on triangular grid 2/2

$$p^{(1)} \rightsquigarrow p, \quad U^{(0)} \rightsquigarrow U$$

$$\Delta_1 p + \Delta_1 U = 0$$
$$\Delta_5 p - \Delta_5 U = 0$$
$$\Delta_3 p = 0$$
$$\Delta_3 U = 0$$
$$\Delta_2 p - \Delta_2 U = 0$$
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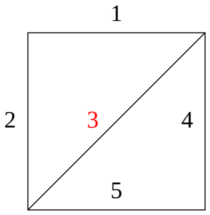

Roe on triangular grid 2/2

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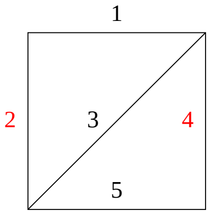


One solution – two implications

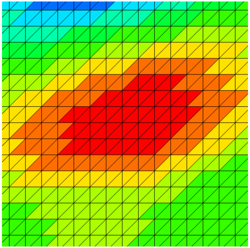
Unique solution:

$$\boxed{\Delta_i p^{(1)} = 0\,, \quad \Delta_i U^{(0)} = 0 \quad \forall\, i \in \Omega \setminus \partial\Omega}$$

Two implications:

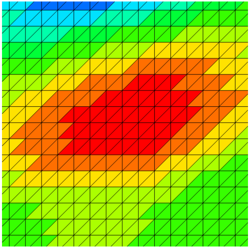
- $p^{(1)} = \text{const}$
- normal component of $\mathbf{u}$ does not jump at cell interfaces

Zero-jump constraint on $\mathbf{u}$



u-pairing

Zero-jump constraint on $\mathbf{u}$



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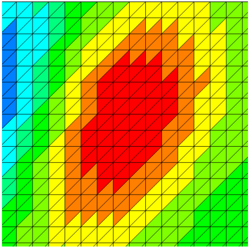
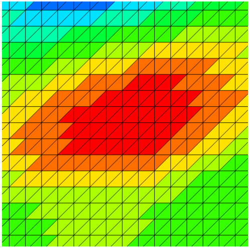


v-paring

Zero-jump constraint on $\mathbf{u}$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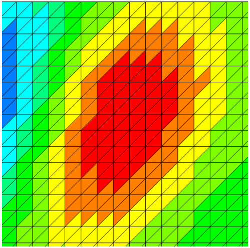
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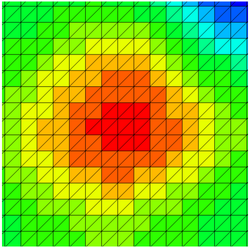
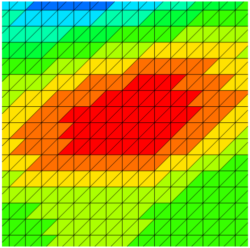
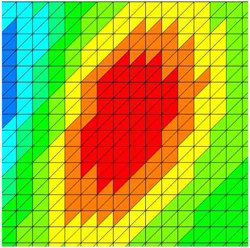

u_{diag}-pairing

Zero-jump constraint on $\mathbf{u}$



u-pairing



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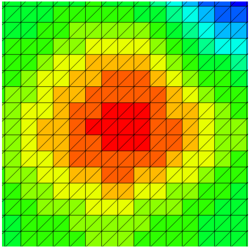
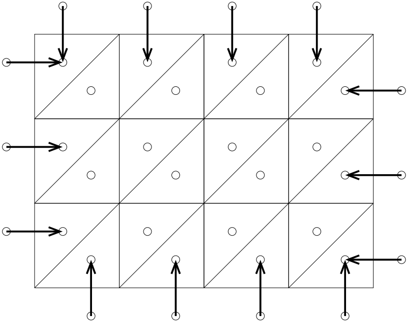


u_{diag}-pairing

Open question:

- How many degrees of freedom are left for $\mathbf{u}$?

Degrees of freedom for $\mathbf{u}$



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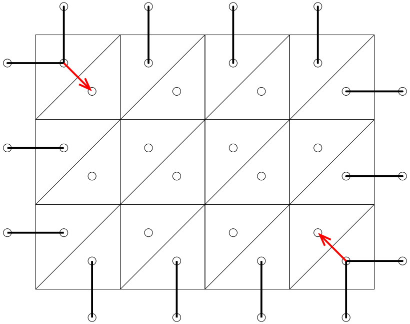

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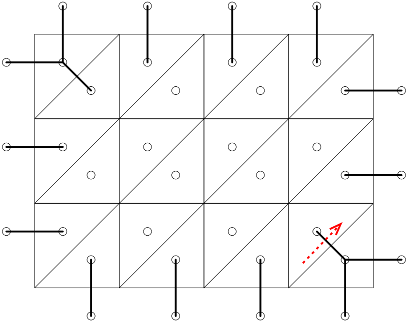

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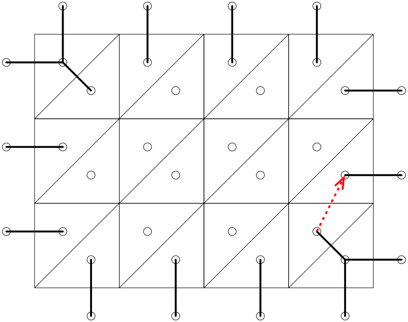
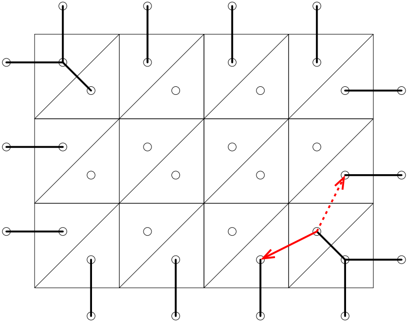
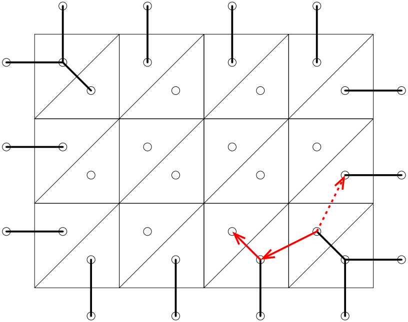


Degrees of freedom for u



Degrees of freedom for **u**



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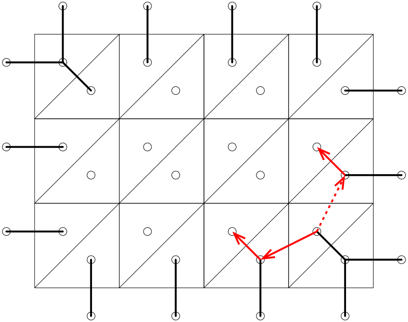

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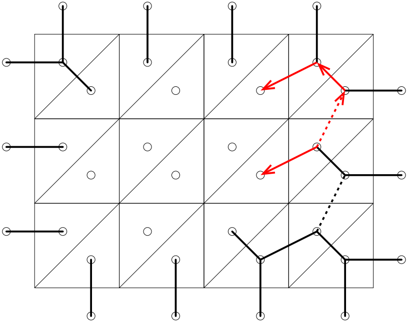
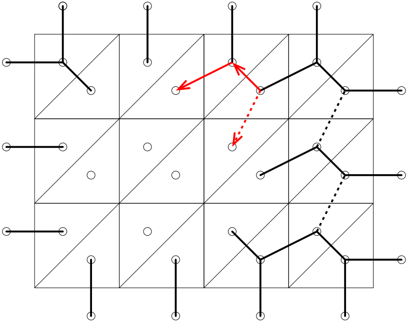


Degrees of freedom for u



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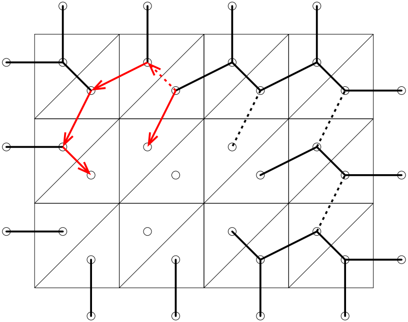

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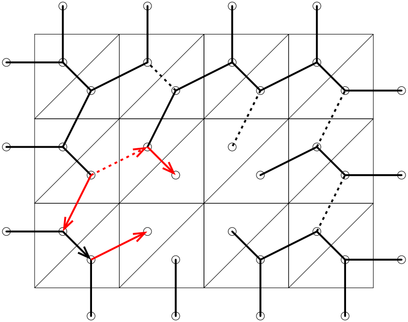

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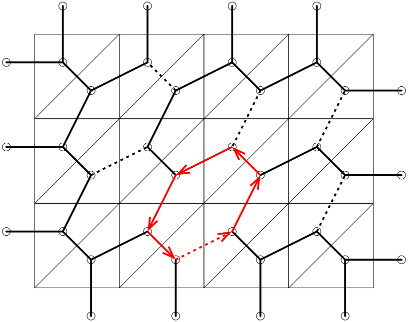

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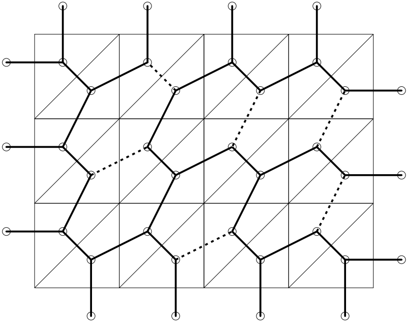

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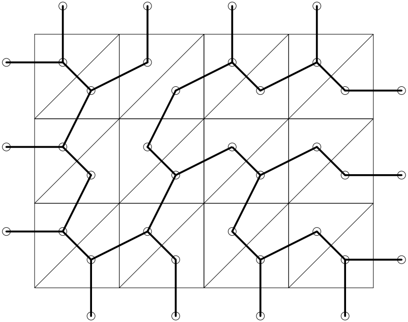

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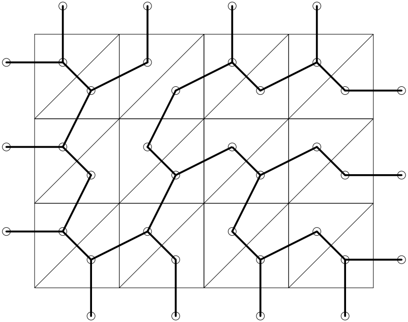

Low Mach:

$$f_{m\times n} = (m-1)(n-1) = \mathcal{O}(mn)$$

High Mach:

$$f_{m\times n} = 4mn = \mathcal{O}(mn)$$

Summary: asymptotic results

Euler equations	Roe on $\square$-grid	Roe on $\triangle$-grid
$\nabla p^{(1)} = 0$	$\nabla p^{(1)} \sim u_{xx}^{(0)} \Delta x$	$\Delta p^{(1)} = \Delta U^{(0)} = 0$
no viscosity	art. viscosity $\mathcal{O}(\Delta x/\mathrm{M})$	art. viscosity $\mathcal{O}(\Delta x)$

Conclusions

If you want to do low Mach number flow, you can ...

- use preconditioning techniques on arbitrary grids (Turkel, Guillard, Viozat, ...)
- use projection methods (Klein, Wesseling, Dick, ...)
- or use the Roe scheme[1] on triangular grids

[1] not HLL

Thank you for your attention