An Asymptotic Preserving Scheme for Euler Equations with Gravity and Friction

Nicolas Seguin

Laboratoire Jacques-Louis Lions
Université Pierre et Marie Curie – Paris 6

FVCA 5, Aussois

Context of the work: Coupling of thermohydraulic models

Collaboration between

Laboratoire Jacques-Louis Lions, Université Pierre et Marie Curie – Paris 6
Filipa Caetano, Christophe Chalons, Frédéric Coquel,
Edwige Godlewski, Frédéric Lagoutière, Pierre-Arnaud Raviart, NS

CEA Saclay (French nuclear agency)
Annalisa Ambroso, Benjamin Boutin, Thomas Galié,
Samuel Kokh, Jacques Segré

Also participating
EDF R&D

Jean-Marc Hérard, Olivier Hurisse

Outline of the talk

- **Asymptotic behaviors for hyperbolic systems with source term**
  - Different asymptotic behaviors
  - Adapted numerical schemes
- **Asymptotic preserving schemes**
  - The case of the telegrapher's equations
  - Godunox-type schemes and approximate Riemann solvers
  - Euler equations with gravity and friction
- **Numerical experiments**
  - Well-balanced property
  - Asymptotic preserving case

Asymptotic behaviors

Hyperbolic systems with source term

$$\begin{cases} \partial_t U + \partial_x F(U,V) = 0 \\ \partial_t V + \partial_x G(U,V) = \alpha R(U,V) \end{cases}$$

Stiff source term: $\alpha \gg 1$ (Friction, chemical reactions, external forces...)

Different asymptotic behaviors

- Stationary solutions
  $\partial_t (U,V) = 0$
- Equilibrium solutions
  $R(U,V) = 0$
- Asymptotic solutions
  $1 \ll \alpha < \infty \rightarrow$ rescaling

$\boxed{\text{How to reproduce these behaviors at the numerical level}}$

Asymptotic behaviors: stationary solutions

Hyperbolic systems with source term

$$\begin{cases} \partial_t U + \partial_x F(U,V) = 0 \\ \partial_t V + \partial_x G(U,V) = \alpha R(U,V) \end{cases}$$

Stationary solutions

$$\partial_t U = \partial_t V = 0$$

Well-balanced schemes
Verify at the numerical level

$$\begin{cases} \partial_x F(U,V) = 0 \\ \partial_x G(U,V) = \alpha R(U,V) \end{cases} \qquad \text{for all } \alpha > 0$$

Asymptotic behaviors: relaxation

Hyperbolic systems with source term

$$\begin{cases} \partial_t U + \partial_x F(U,V) = 0 \\ \partial_t V + \partial_x G(U,V) = \alpha R(U,V) \end{cases}$$

Relaxation

$$R(U,V) = 0 \quad \Longleftrightarrow \quad V = V_{eq}(U)$$

Relaxation schemes
Verify at the numerical level

$$\partial_t U + \partial_x F(U, V_{eq}(U)) = 0$$

Asymptotic behaviors: rescaling

Hyperbolic systems with source term

$$\begin{cases} \partial_t U + \partial_x F(U,V) = 0 \\ \partial_t V + \partial_x G(U,V) = \alpha R(U,V) \end{cases}$$

Rescaling $\longrightarrow$ PDE system independent of α

$$(\mathscr{A}) \qquad \begin{cases} V = V(U, \partial_x U, \ldots) \\ \partial_t U + \partial_x F(U, V(U, \partial_x U, \ldots)) = 0 \end{cases}$$

Asymptotic preserving schemes ([Jin], [Klar]...)
Same rescaling leads to a numerical scheme for system $(\mathscr{A})$

The telegrapher's equations and the asymptotic limit

Linear hyperbolic system

$$(\mathscr{S})\quad \begin{cases} \partial_t \tau - \partial_x u = 0 \\ \partial_t u - \partial_x \tau = -\alpha u \end{cases}$$

Long time behavior: $\boxed{t = \alpha s}$ and $\boxed{v = \alpha u}$

Drop high order terms in α^{-1}

$$\begin{cases} \alpha^{-1}\partial_s \tau - \alpha^{-1}\partial_x v = 0 \\ \alpha^{-2}\partial_s v - \partial_x \tau = -v \end{cases} \quad \Longrightarrow \quad \begin{cases} \partial_s \tau - \partial_x v = 0 \\ v = \partial_x \tau \end{cases}$$

We then obtain

$$(\mathscr{A})\quad \begin{cases} \partial_s \tau - \partial^2_{xx} \tau = 0 \\ v = \partial_x \tau \end{cases}$$

Asymptotic preserving scheme: approximate $(\mathscr{S})$ and $(\mathscr{A})$ for all α $(\gg 1)$

Numerical approximation of the telegrapher's equations

Semi-discrete scheme: flux $\partial_x f \approx \Delta_x f^*$ source term $\alpha u \approx \alpha \tilde{u}$

$$\begin{cases} \partial_t \tau - \Delta_x u^* = 0 \\ \partial_t u - \Delta_x \tau^* = -\alpha \tilde{u} \end{cases}$$

Long time behavior: $\boxed{t = \alpha s}$

$$\begin{cases} \alpha^{-1} \partial_s \tau - \Delta_x u^* = 0 \\ \alpha \tilde{u} = \Delta_x \tau^* + \mathcal{O}(\alpha^{-1}) \end{cases}$$

Drop term $\mathcal{O}(\alpha^{-1})$

$$\begin{cases} \partial_s \tau - \Delta_x (\alpha u^*) = 0 \\ (\alpha \tilde{u}) = \Delta_x \tau^* \end{cases}$$

AP scheme for the telegrapher's equations

$$\begin{cases} \partial_s \tau - \Delta_x(\alpha u^*) = 0 \\ (\alpha \tilde{u}) = \Delta_x \tau^* \end{cases}$$

In general,

$$\boxed{\Delta \tilde{u} = \Delta u^* + \mathcal{O}(\Delta x)}$$

We then obtain ($v = \alpha u$)

$$(\mathscr{A}) \quad \begin{cases} \partial_s \tau - \Delta^2_{xx} v^* + \boxed{\alpha} \, \mathcal{O}(\Delta x) = 0 \\ \tilde{v} = \Delta_x \tau \end{cases}$$

Convergent scheme but very diffusive when $\alpha \gg 1$...

AP scheme for the telegrapher's equations

$$\begin{cases} \partial_s \tau - \Delta_x(\alpha u^*) = 0 \\ (\alpha \tilde{u}) = \Delta_x \tau^* \end{cases}$$

Choose $\tilde{u}$ s. t.

$$\boxed{\Delta \tilde{u} = \Delta u^* + \alpha^{-1}\, \mathcal{O}(\Delta x)}$$

We then obtain ($v = \alpha u$)

$$(\mathscr{A}) \quad \begin{cases} \partial_s \tau - \Delta^2_{xx} v^* + \boxed{1}\, \mathcal{O}(\Delta x) = 0 \\ \tilde{v} = \Delta_x \tau \end{cases}$$

Addition of numerical diffusion, but independent of α !

AP scheme for the telegrapher's equations

$$\begin{cases} \partial_s \tau - \Delta_x(\alpha u^*) = 0 \\ (\alpha \tilde{u}) = \Delta_x \tau^* \end{cases}$$

Choose

$$\boxed{\tilde{u} = u^*}$$

We then obtain ($v = \alpha u$)

$$(\mathscr{A}) \quad \begin{cases} \partial_s \tau - \Delta^2_{xx} v^* + \boxed{0} \, \mathcal{O}(\Delta x) = 0 \\ v^* = \Delta_x \tau \end{cases}$$

Asymptotic scheme independent of α !!

Euler equations with gravity and friction

Linear friction (also works with non linear friction)

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p) = \rho g - \alpha \rho u \\ \partial_t(\rho E) + \partial_x(u(\rho E + p)) = \rho g u - \alpha \rho u^2 \end{cases}$$

with $E = \varepsilon + u^2/2$

Long time behavior:

$\boxed{t = \alpha s}$ and $\boxed{v = \alpha u}$

Asymptotic system

$$\begin{cases} \partial_s \rho + \partial_x(\rho g - \partial_x p]) = 0 \\ \partial_s(\rho \varepsilon) + \partial_x(v(\rho \varepsilon + p)) = \rho g v - \rho v^2 \\ v = g - \rho^{-1} \partial_x p \end{cases}$$

Numerical methods

Conservative hyperbolic system

$$\partial_t U + \partial_x F(U) = 0$$

Godunov-type schemes by Harten-Lax-Van Leer

$$U_i^{n+1} = \frac{1}{\Delta x}\left(\int_0^{\Delta x/2} \mathcal{U}(x/\Delta t; U_{i-1}^n, U_i^n)\, dx + \int_{-\Delta x/2}^{0} \mathcal{U}(x/\Delta t; U_i^n, U_{i+1}^n)\, dx\right)$$

where $\mathcal{U}(x/t; U_l, U_r)$ is an approximate Riemann solver:
Self-similar function satisfying the consistency property

$$\int_{-\Delta x/2}^{\Delta x/2} \mathcal{U}(x/\Delta t; U_l, U_r)\, dx = \frac{\Delta x}{2}(U_l + U_r) + \Delta t\big(F(U_r) - F(U_l)\big)$$

(integration over the staggered cell $[-\Delta x/2, \Delta x/2] \times [0, T]$ of the PDE)

Numerical methods

Hyperbolic system with source term

$$\partial_t U + \partial_x F(U) = S(U)$$

Godunov-type schemes by Harten-Lax-Van Leer (see also Gallice)

$$U_i^{n+1} = \frac{1}{\Delta x}\left(\int_0^{\Delta x/2} \mathcal{U}(x/\Delta t; U_{i-1}^n, U_i^n)\, dx + \int_{-\Delta x/2}^{0} \mathcal{U}(x/\Delta t; U_i^n, U_{i+1}^n)\, dx\right)$$

where $\mathcal{U}(x/t; U_l, U_r)$ is an approximate Riemann solver:
Self-similar function satisfying the consistency property

$$\int_{-\Delta x/2}^{\Delta x/2} \mathcal{U}(x/\Delta t; U_l, U_r)\, dx = \frac{\Delta x}{2}(U_l + U_r) + \Delta t\big(F(U_r) - F(U_l)\big) + \Delta x \Delta t \tilde{S}(\Delta x, U_l, U_r)$$

where $\tilde{S}(0, U, U) = S(U)$.

Construction of the asymptotic preserving scheme

The construction of the numerical scheme follows the following steps:

- The approximate Riemann solver:
  - Write the PDE system with source term in Lagrangian coordinates
  - Construct an approximate Riemann solver, with $\boxed{\tilde{u} = u^*}$
  - Go back to Eulerian coordinates
- Define the associated Godunov-type scheme by the previous formula

NB

- The scheme can be re-written as a finite volume scheme
- The scheme is asymptotic preserving
- The numerical scheme is entropy satisfying
- Also works with non linear friction terms

Numerical experiments

Comparison with the splitting method:

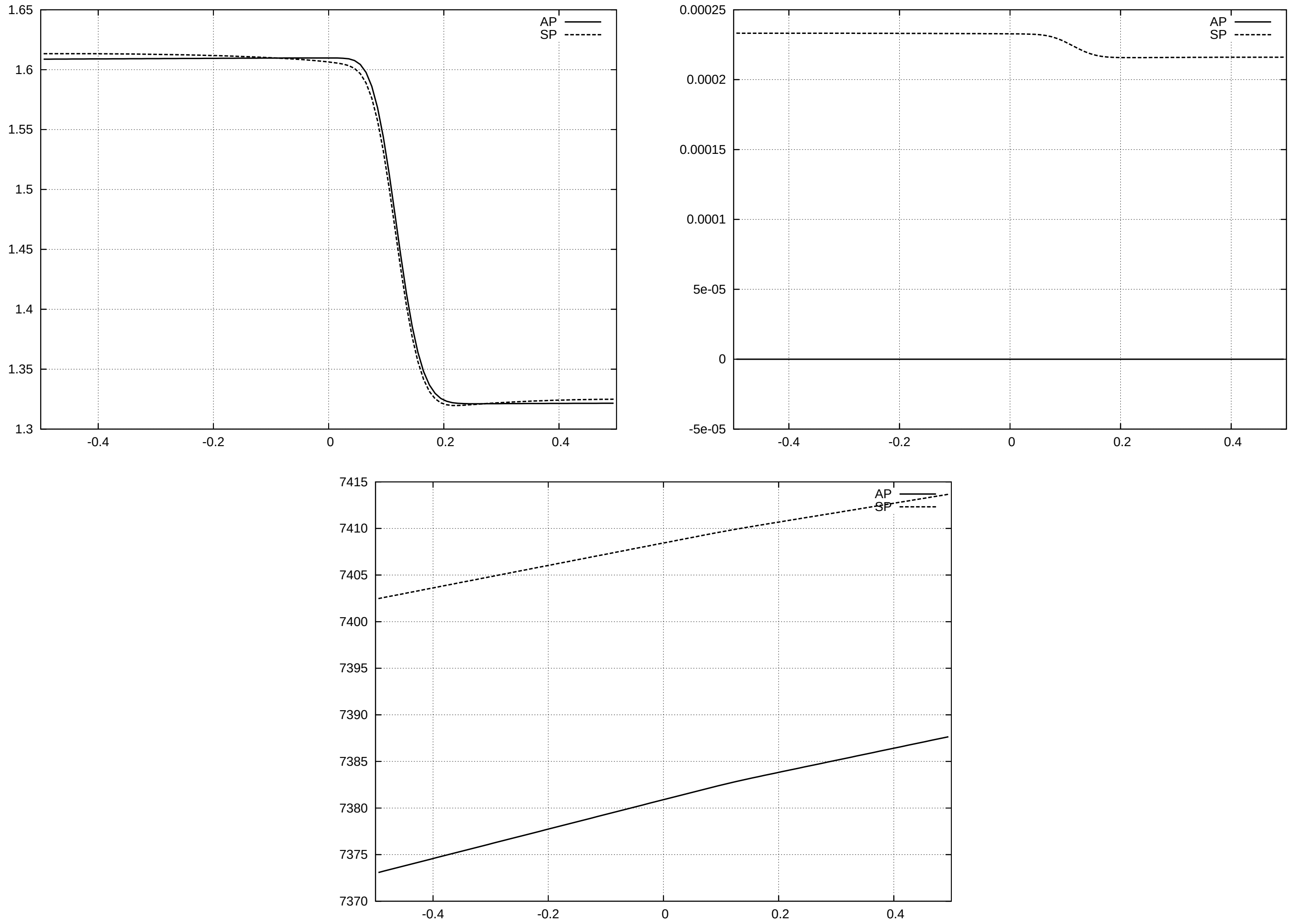
- HLLC scheme for the PDE part
- implicit scheme for the source term

First test: Convergence toward a stationary solution
($\alpha = 10^4 s^{-1}$, $g = 9.81 m/s^{-2}$)

Second test: Asymtotic behavior and mesh sensibility
($\alpha = 10^6 s^{-1}$, $g = 9.81 m/s^{-2}$)

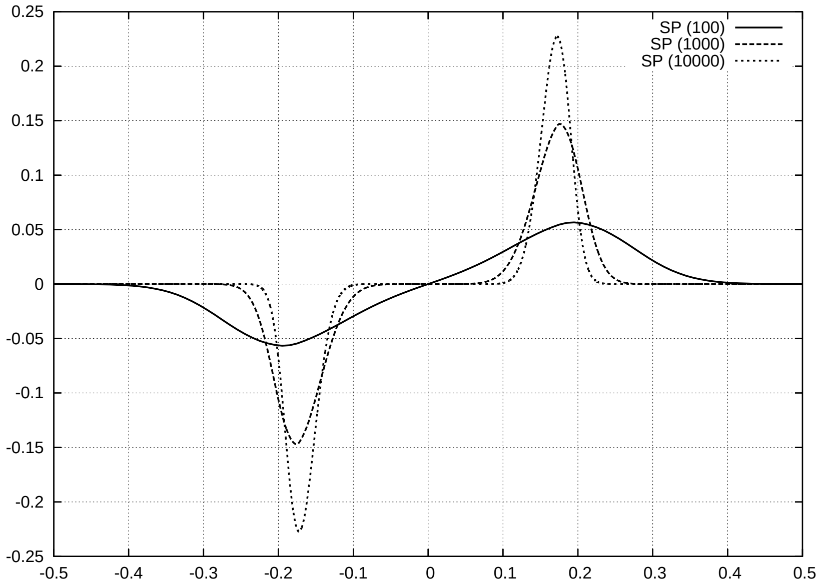
Convergence towards a stationary solution

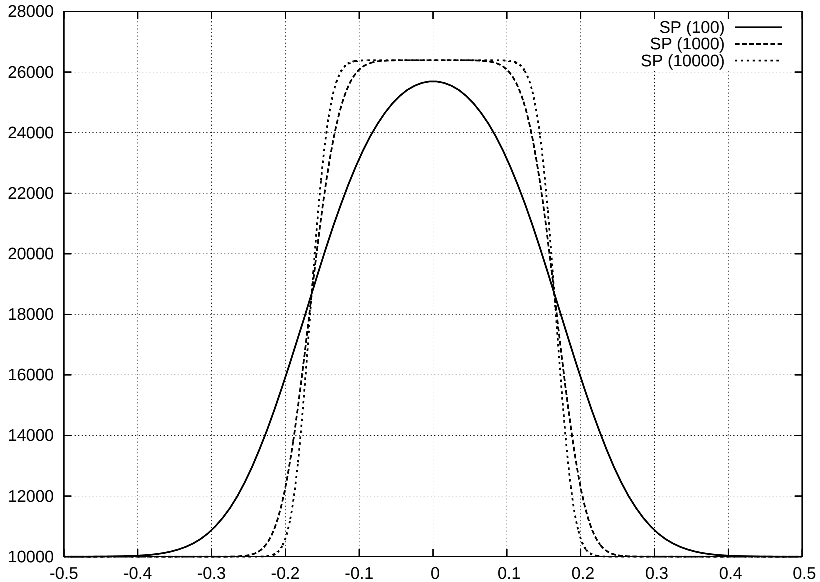
Riemann problem with wall BC's (long time): (ρ, u, p)



Asymtotic behavior and mesh sensibility: splitting method

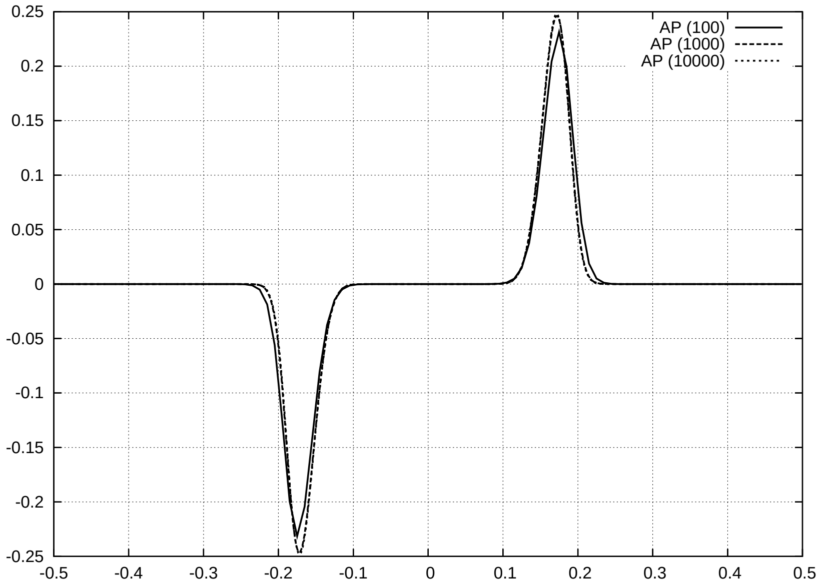
Centered step with periodic BC (long time): (u, p)

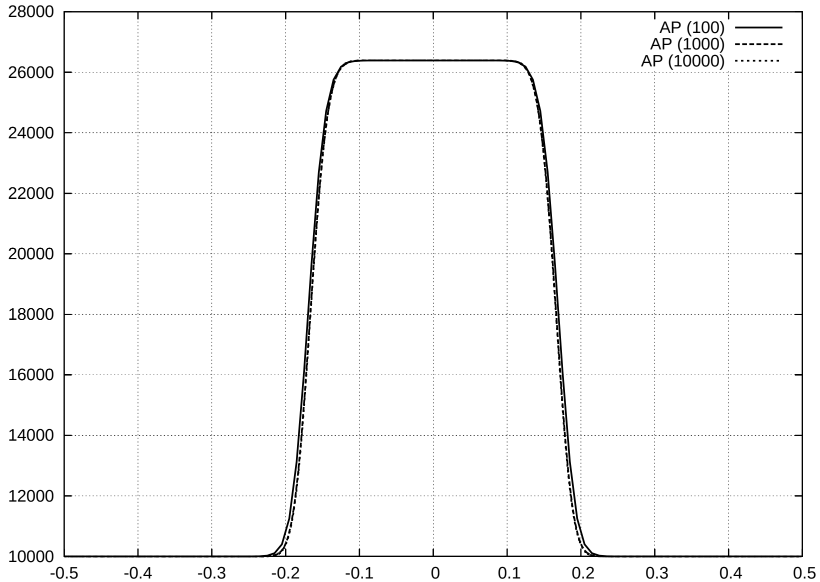




Asymtotic behavior and mesh sensibility: AP scheme

Centered step with periodic BC (long time): (u, p)





Conclusion

Asymptotic preserving scheme for Euler equations with gravity and friction

- Godunov-type scheme
- Numerical diffusion independent of α
- Explicit scheme: hyperbolic CFL condition (positivity, entropy)

Work in progress and perspectives

- Large spatial variations of α
  $\longrightarrow$ model coupling (see E. Godlewski, FVCA5, Thursday)
- Two-phase flows with drag force (see A. Ambroso, FVCA5, Tuesday)
- Deeper analysis (non uniform mesh, multi-D)