A well-balanced Runge–Kutta discontinuous Galerkin method for the shallow water equations with flooding and drying

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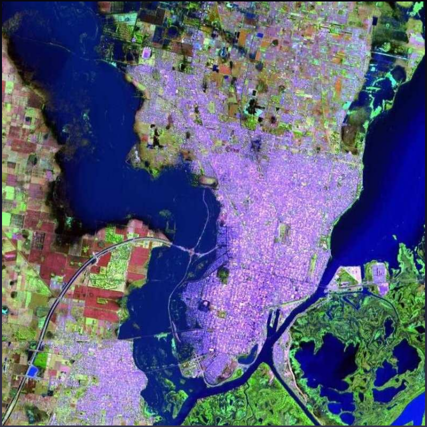
FVCA05 - Aussois, France
June 08-13 2008

Motivation I

Salado river (Argentina) before flooding (January 25, 2000)



After flooding (May 3, 2003)

Motivation I

Salado river flood, Santa Fe, Argentina, April-May 2003

Motivation II

- **Stratigraphy:** the study of rock strata, especially the distribution, deposition, and age of sedimentary rocks
- Interactions between processes over a large range of spatial and temporal scales (petrol industry, geophysics, etc)

Outline

Mathematical model: shallow water equations

Numerical solution: discontinuous Galerkin method

Applications

Conclusions and outlook

Mathematical model

Shallow water equations (SWE)

Leading order model for hydrodynamics in river, ocean, and coastal flows, among other cases of engineering and scientific interest

Conservative form

Let the bounded domain $\Omega \subset \mathbb{R}^2$, and let $T > 0$ be the simulation time

$$\begin{aligned} \frac{\partial \mathbf{U}}{\partial t} + \nabla \cdot F(\mathbf{U}) &= S \quad \text{in} \quad \Omega \times]0, T[, \\ \mathbf{U}(\mathbf{x}, 0) &= \mathbf{U}_0(\mathbf{x}) \quad \text{in} \quad \Omega, \end{aligned}$$

- completed with suitable boundary conditions on $\partial\Omega \times]0, T[$
- $\mathbf{U} = (h \;\; hu \;\; hv)^T$ is the vector of state variables, h water depth, $h\mathbf{u} = (hu, hv)^T$ discharge of the flow; $\mathbf{u} = (u, v)^T$ flow velocity field

Mathematical model

Flux

$$F(\mathbf{U}) = \begin{pmatrix} uh & vh \\ u^2h + gh^2/2 & vuh \\ uvh & v^2h + gh^2/2 \end{pmatrix}$$

Source

$$S(\mathbf{U}, b) = \begin{pmatrix} 0 \\ -gh\partial_x b - S_{b_x} \\ -gh\partial_y b - S_{b_y} \end{pmatrix}$$

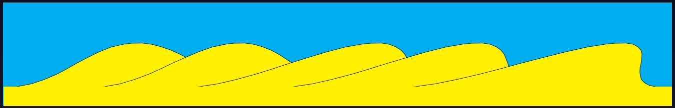
- b topography (measured from a reference altitude)
- g gravity acceleration
- $\mathbf{S}_b = (S_{b_x}, S_{b_y})^T$ closure for friction.

Numerical schemes for SWE

- Finite differences (FD)
- Finite volumes (FV) [Audusse *et al.*, 2004; Brufau *et al.*, 2004, Benkhaldoun *et al.*, 2007]
- Finite elements (FEM)
  - Stabilized continuous FE methods [Hervouet, 2007; TELEMAC Modelling System, 1998]
- Discontinuous Galerkin methods (DG) [Dawson and Proft, 2004; Bokhove, 2005; Kubatko *et al.*, 2006; Tassi *et al.*, 2007; Ern *et al.*, 2008]

Discontinuous Galerkin methods

- DG combine ideas **high-resolution FD/FV** methods within a **FEM** framework

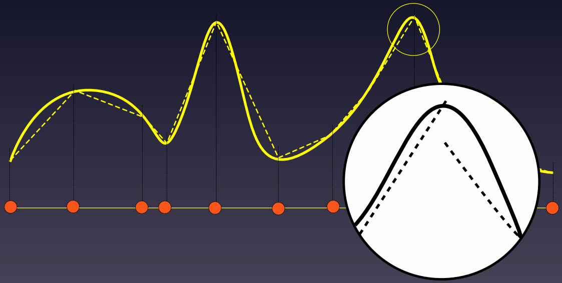


- DG captures shocks or sharp fronts more efficiently than FEM
- General solution spaces:
  - freedom in the choice of the base
  - freedom in building the mesh
  - *hp*$-$adaptivity easily implemented
- **Locally conservative**: conservation of a transported quantity is satisfied on a local or elemental level

Discontinuous Galerkin methods

- Approximated functions are **discontinuous** across the finite element boundaries
- We consider the following approximation space made up of **polynomial functions**:

$$\mathcal{V}_{\mathrm{h}} = \left\{ v_{\mathrm{h}} \in L^2 ; v_{\mathrm{h}}|_K \in \mathbb{P}_p(K), p \in \mathbb{N} \right\}$$



Space discretization

For all $K \in \mathcal{T}_{\rm h}$, we multiply $\partial_t \mathbf{U} + \nabla \cdot F = S$ by $v_{\rm h} \in \mathcal{V}_{\rm h}$, integrate over K, and apply Green's formula:

Find $\mathbf{U}_{\rm h} \in C^1([0,T], \mathcal{V}_{\rm h})$ *such that* $\forall t \in]0,T[, \forall K \in \mathcal{T}_{\rm h}, \forall v_{\rm h} \in \mathcal{V}_{\rm h}$,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_K \mathbf{U}_{\rm h} v_{\rm h} \,\mathrm{d}K + \int_{\partial K} \widehat{F}(\mathbf{U}_{\rm h}^l, \mathbf{U}_{\rm h}^r, \mathbf{n}_K) v_{\rm h} \,\mathrm{d}\sigma - \int_K \boldsymbol{\nabla} v_{\rm h} \cdot F(\mathbf{U}_{\rm h}) \,\mathrm{d}K = \int_K \mathbf{S} v_{\rm h} \,\mathrm{d}K,$$

where:

- $\mathbf{U}_{\rm h}^l$ and $\mathbf{U}_{\rm h}^r$ values of $\mathbf{U}_{\rm h}$ at left (l) and right (r) of face σ
- Weak inter-element links established by introducing the **numerical flux** $\widehat{F}(\mathbf{U}_{\rm h}^l, \mathbf{U}_{\rm h}^r)$
- HLLC, Roe, kinetic flux, etc. [Tassi *et al.*, 2007; Ern *et al.*, 2008]

Time discretization

Flow field approximation (equal order) and choice of test functions

$$\mathbf{U}_\mathrm{h}|_K = \sum_{j=1}^{N_b} \widehat{U}_{j,k}\psi_j, \quad \text{and} \quad v_\mathrm{h}|_K = \psi_i,$$

Time discretization

- The DG space discretized equations can be written as:

$$\mathcal{M}\,\frac{\mathrm{d}\widehat{\mathbf{U}}}{\mathrm{d}t} = \mathcal{L}(\widehat{\mathbf{U}}),$$

  $\mathcal{M}$ block diagonal mass matrix, $\mathcal{L} : \mathbb{R}^N \to \mathbb{R}^N$

- Time integration: schemes TVD Runge–Kutta (explicit).

Well-balanced DG scheme

Flow-at-rest

- $h + b = C$ and $h\mathbf{u} = 0$
- The DG scheme proposed is not well-balanced: it does not preserve steady-states at rest

DG scheme with flux modification [Ern *et al.*, 2008]

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_K \mathbf{U}_\mathrm{h} v_\mathrm{h}\,\mathrm{d}K + \int_{\partial K} \widehat{F}(\check{\mathbf{U}}_\mathrm{h}^l, \check{\mathbf{U}}_\mathrm{h}^r, \mathbf{n}_K) v_\mathrm{h}\,\mathrm{d}\sigma - \int_K \nabla v_\mathrm{h} \cdot F(\mathbf{U}_\mathrm{h})\,\mathrm{d}K = \int_K \mathbf{S} v_\mathrm{h}\,\mathrm{d}K + \int_{\partial K} v_\mathrm{h} \delta_{K\,\mathrm{h}}\,\mathrm{d}\sigma,$$

- Modified state $\check{\mathbf{U}}_\mathrm{h}^l, \check{\mathbf{U}}_\mathrm{h}^r, \check{h}_\mathrm{h}$ (hydrostatic reconstruction water height, see [Audusse *et al.*, 2004])
- Flux modification: $\delta_{K\,\mathrm{h}} = (0, g/2(\check{h}_\mathrm{h}{}^2|_K - h_\mathrm{h}^2|_K)\mathbf{n}_K)$

Flooding and drying

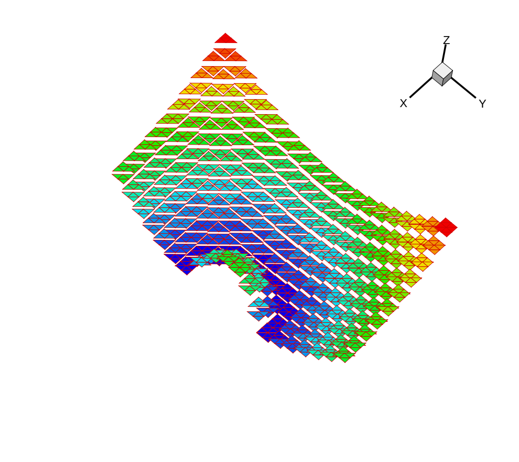
- Flooding and drying: prevent the discrete water height from taking negative values
- We propose a *slope modification* technique based on the idea of *threshold* value
- For each $K \in \mathcal{T}_h$ with non-admissible fields, water height h and discharges $h\mathbf{u}$ replaced by corrected $\hat{h}$ and $\widehat{h\mathbf{u}}$
- Steps:
  1. Detection based on mean water height
  2. If $p \geqslant 2$: fields h, $h\mathbf{u}$ restricted to $\mathbb{P}_1$ by L^2 orthogonal projections
  3. Slope modification of h and $h\mathbf{u}$ which **preserves mass**
  4. Restriction on maximal velocity also enforced

Parabolic bowl

- To assess the capacity of the method to treat flooding and drying [Thacker, 1981; Ern *et al.*, 2008]
- Consider a paraboloid of revolution $b(\mathbf{x}) = \alpha r^2$, $r^2 = |\mathbf{x}|$, $\alpha > 0$ and circular paraboloidic water volume initially at rest and subject to gravity

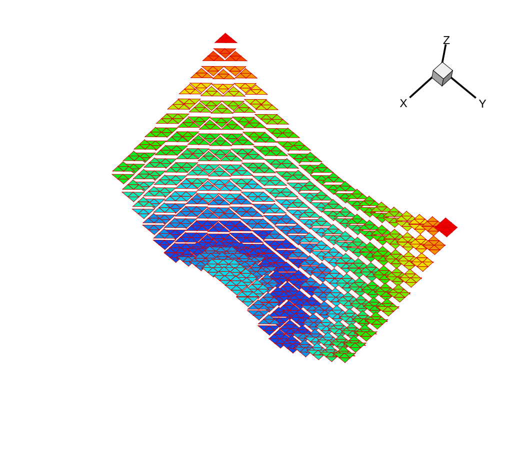
- L^2-norm error:
  - First half period (flooding): 0.9 ($p = 0$); 1.4 ($p = 1$); 1.5 ($p = 2$)
  - Second half period (drying): algorithm does not perform well

Parabolic bowl



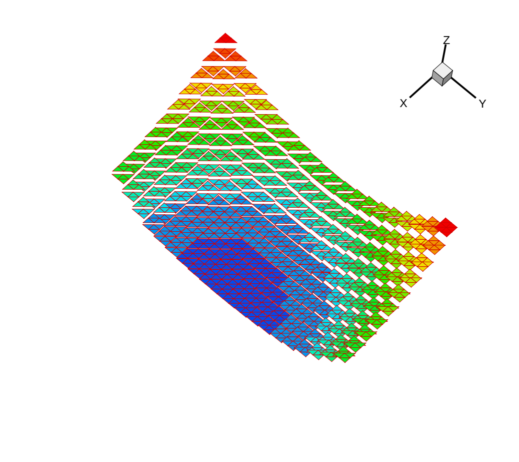
$p0$

Parabolic bowl



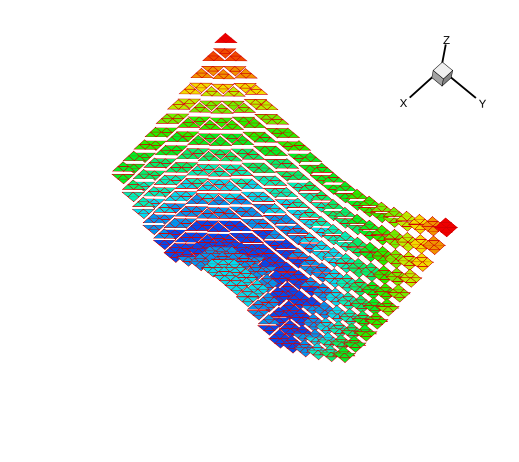
*p*0

Parabolic bowl



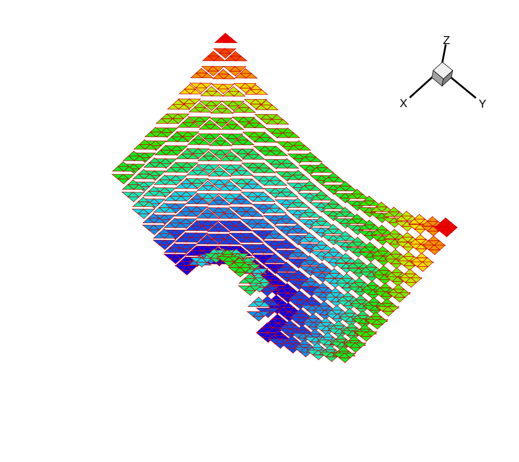
*p*0

Parabolic bowl



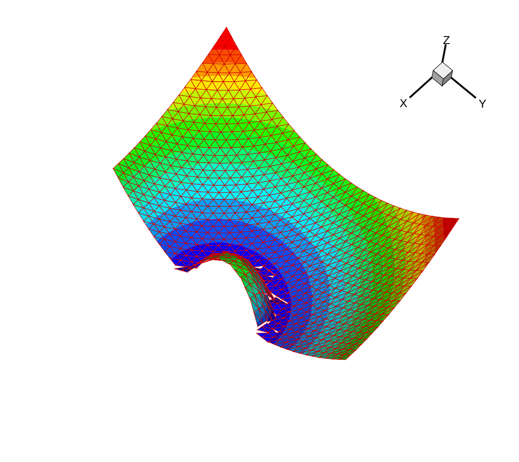
*p*0

Parabolic bowl



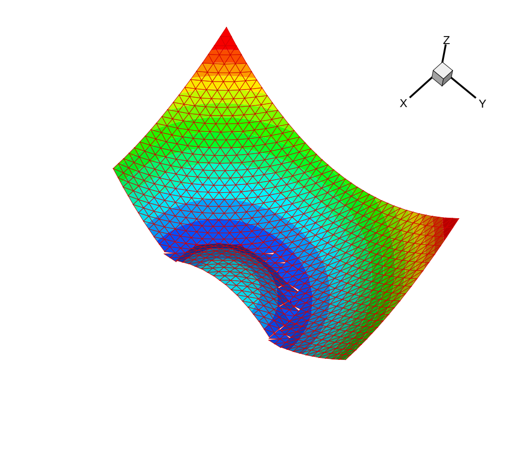
$p0$

Parabolic bowl



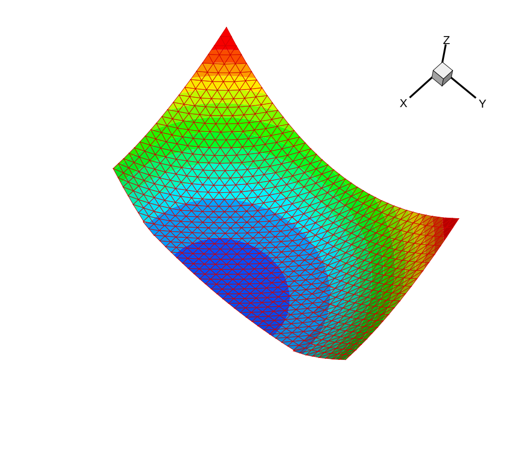
*p*1

Parabolic bowl



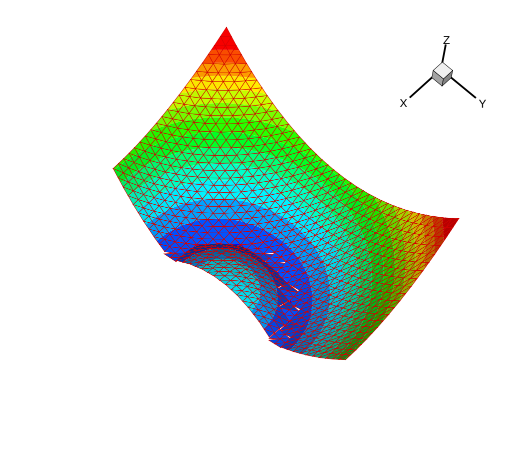
$p1$

Parabolic bowl



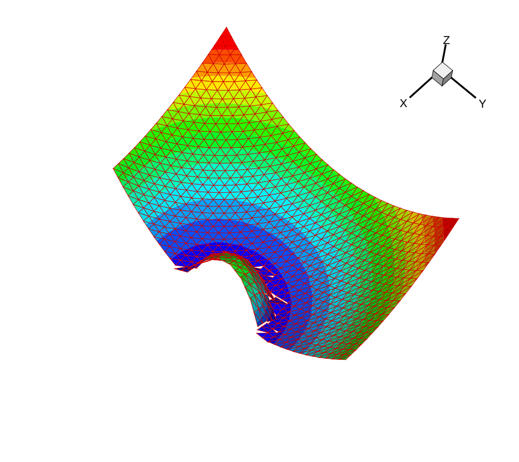
$p1$

Parabolic bowl



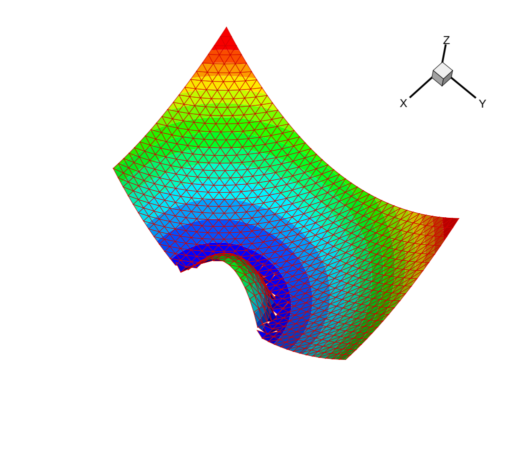
*p*1

Parabolic bowl



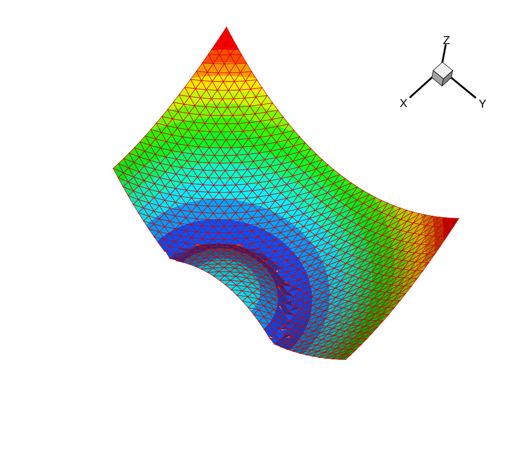
$p1$

Parabolic bowl



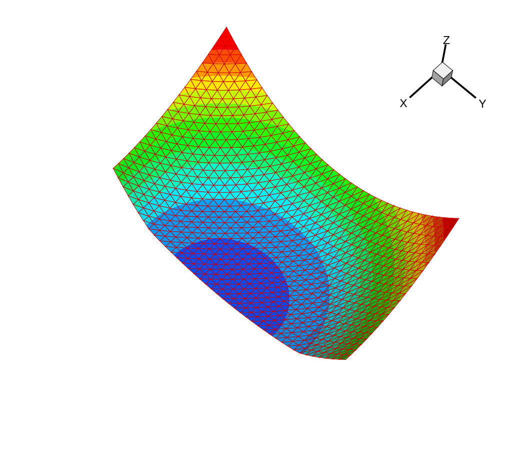
p2

Parabolic bowl



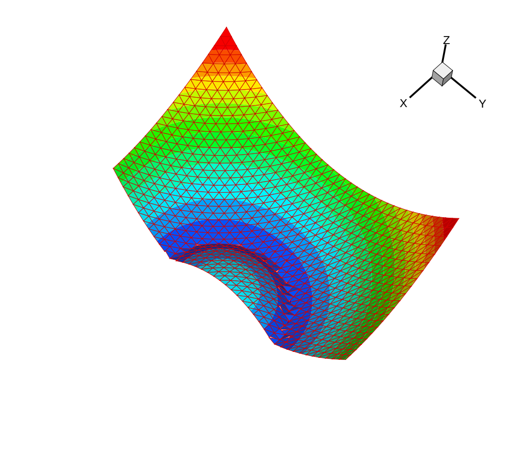
*p*2

Parabolic bowl



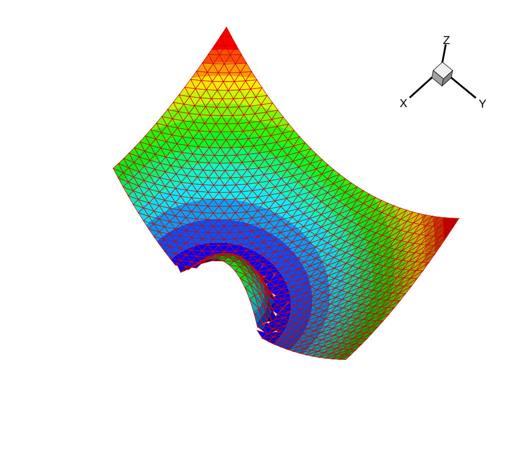
*p*2

Parabolic bowl



p2

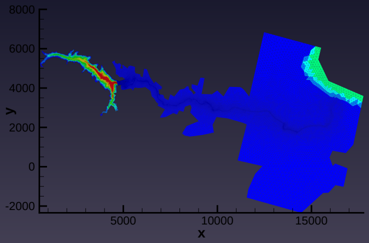
Parabolic bowl



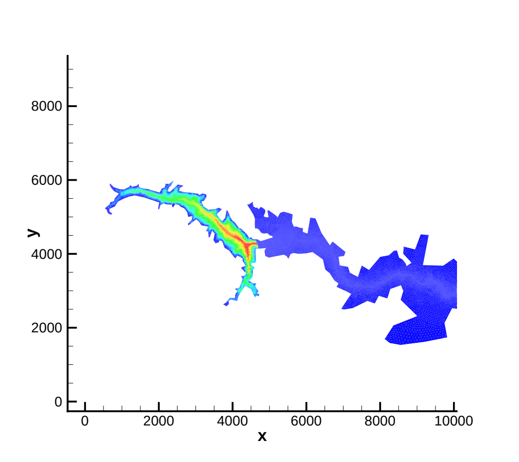
p2

Malpasset dambreak

- Malpasset dambreak: topography and data provided by EDF R&D [Hervouet, 2007]
- Closure for friction: $\mathbf{S}_b = -\frac{g|h\mathbf{u}|}{K^2 h^{7/3}}$, with $K = 30$
- Fully explicit treatment of friction term can lead to instabilities[Brufau *et al.*, 2004]: semi-implicit treatment

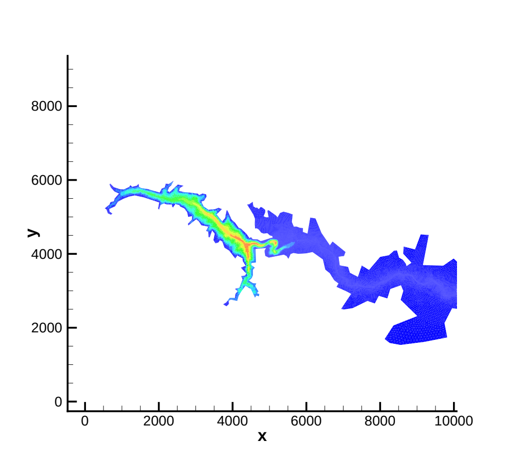


Malpasset dambreak



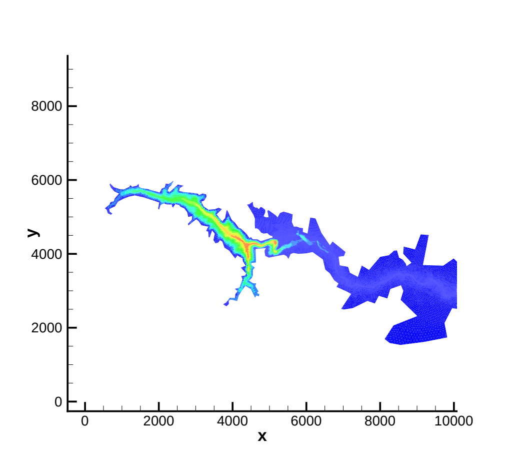
$t = 0s$

Malpasset dambreak



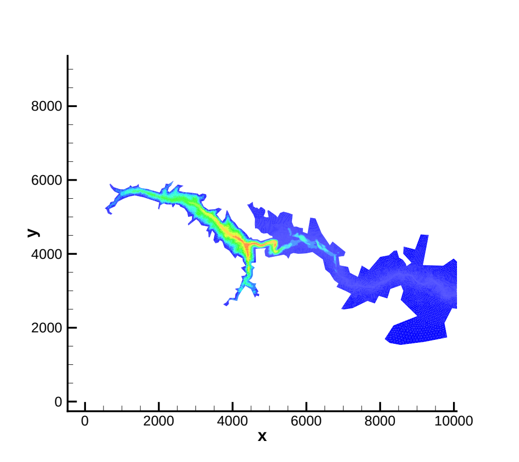
$t = 100s$

Malpasset dambreak



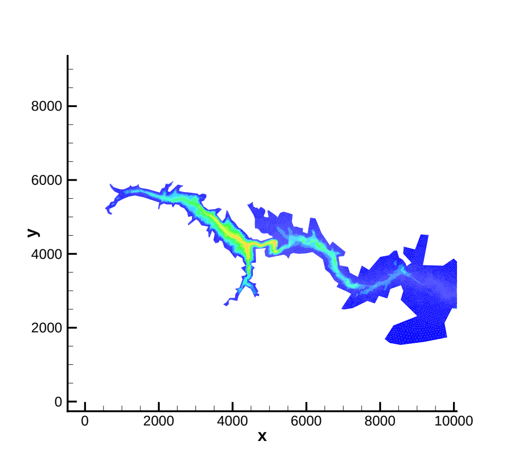
$t = 150s$

Malpasset dambreak



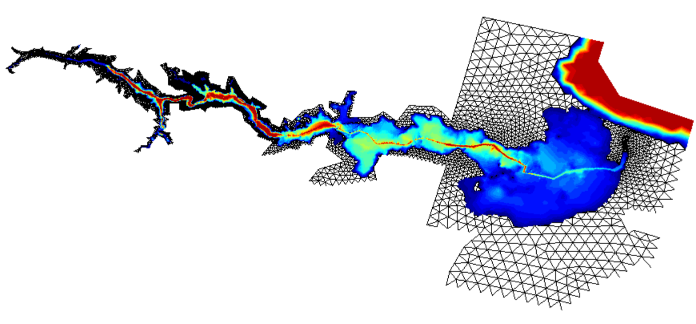
$t = 200s$

Malpasset dambreak



$t = 500s$

Malpasset dambreak



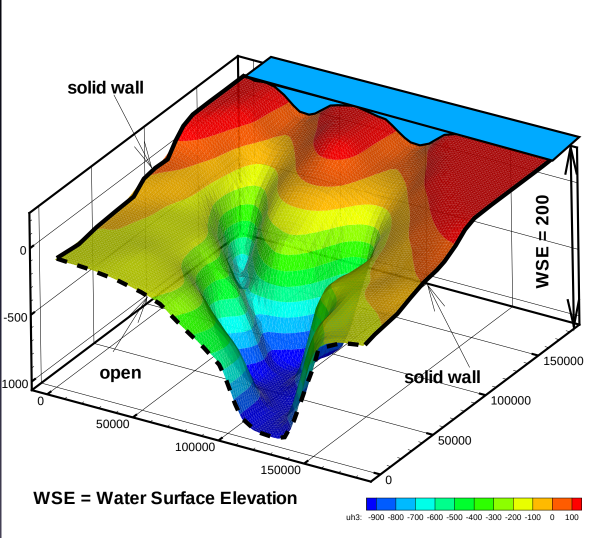
$t = 2500s; p = 1$

Steady flow solutions for stratigraphy

- Work in cooperation with IFP (Roland Masson and Didier Granjeon)
- Rectangular domain $\Omega = [200 \times 200]$ km^2
- Steady-state criterion:

$$||h^{n+1} - h^n||_2 \leq \varepsilon_{ss}||h^n||_2, \quad \varepsilon_{ss} \approx \mathcal{O}(10^{-5})$$

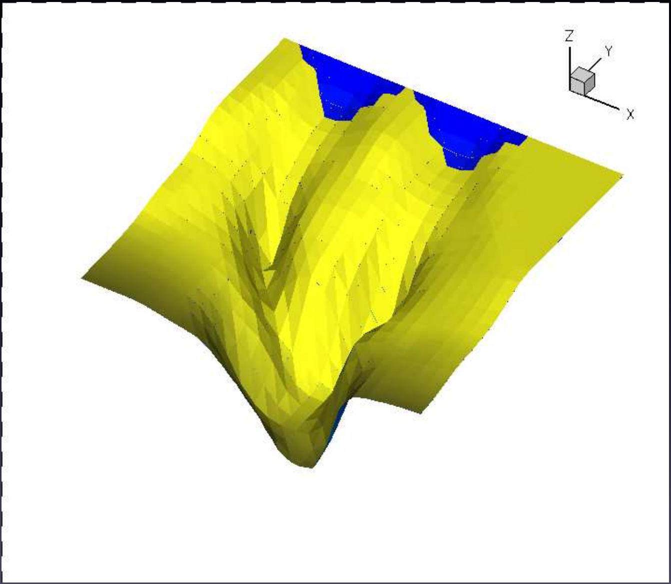
Steady flow solutions for stratigraphy



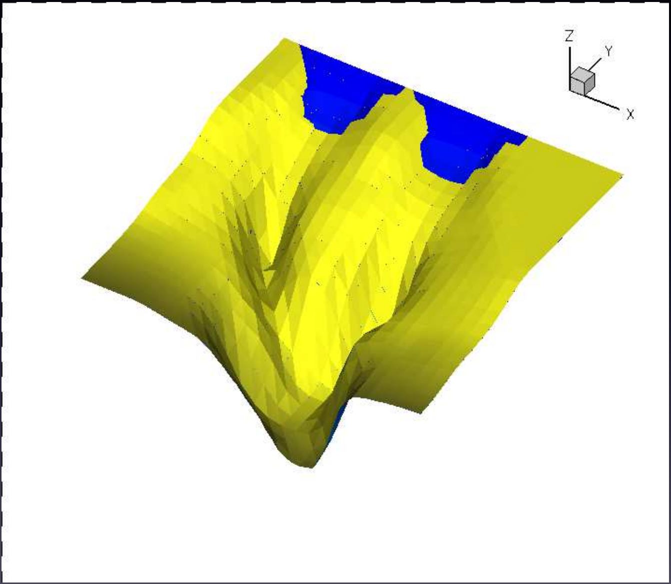
Steady flow solutions for stratigraphy

$p0$

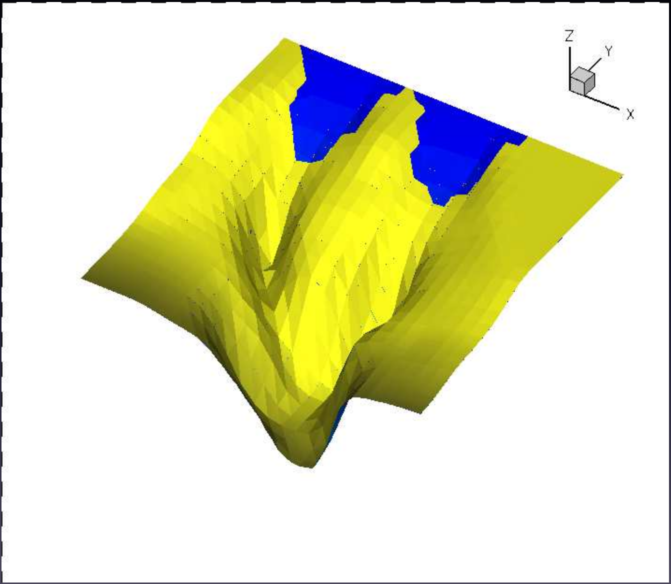
Steady flow solutions for stratigraphy



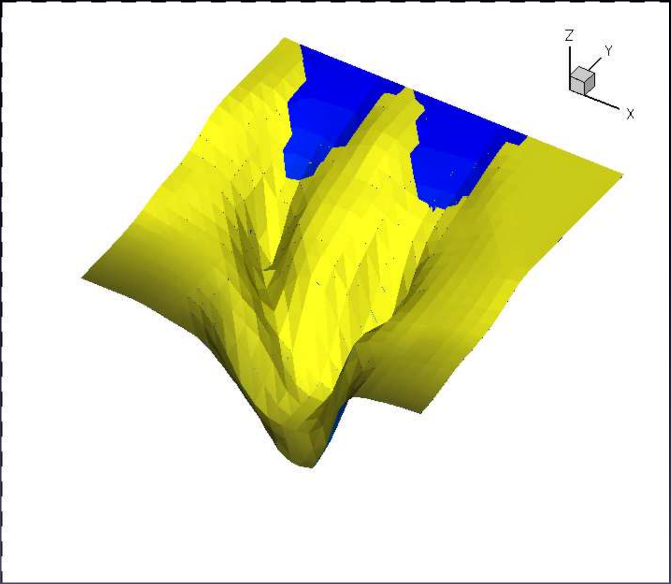
Steady flow solutions for stratigraphy



Steady flow solutions for stratigraphy



Steady flow solutions for stratigraphy



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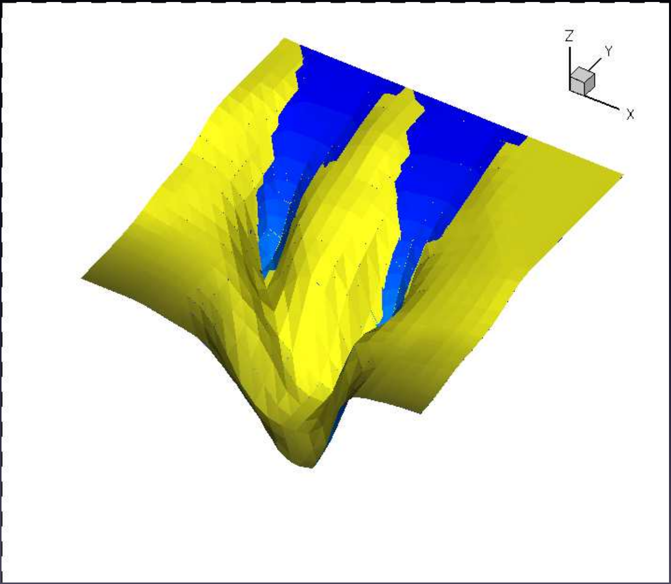

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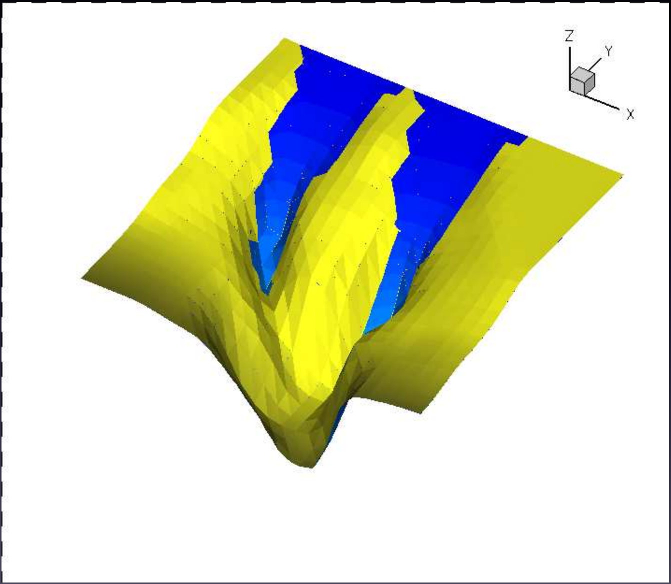

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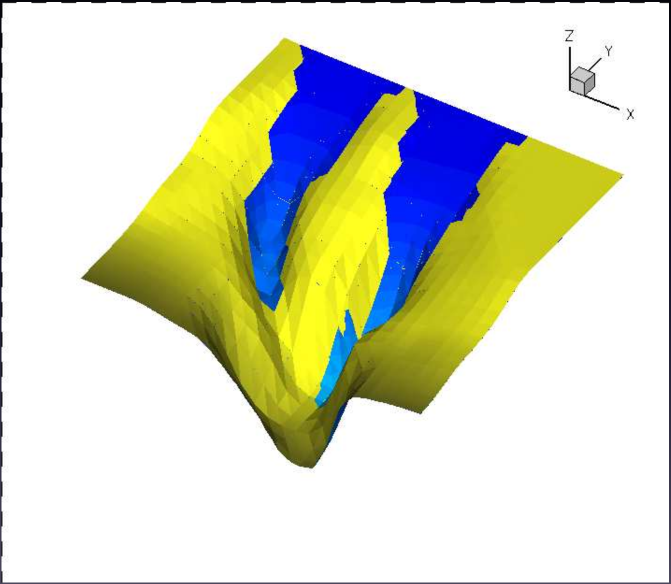

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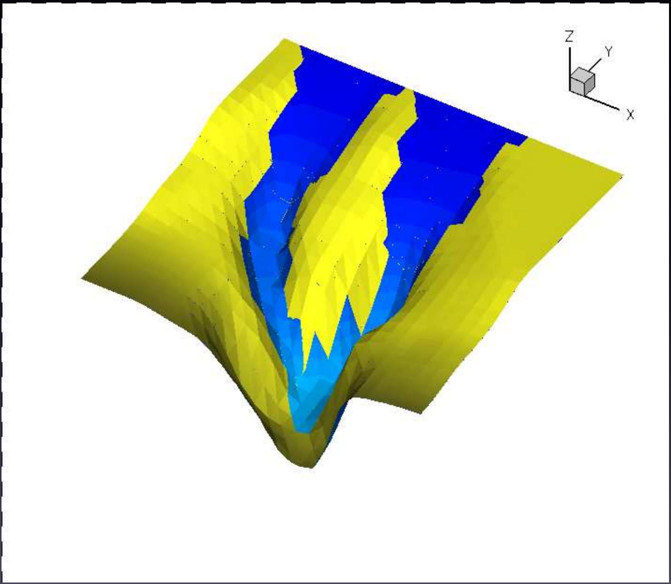

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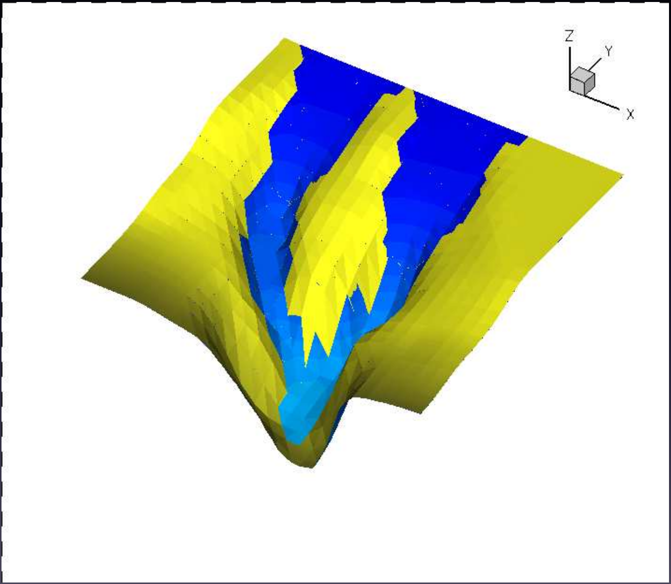
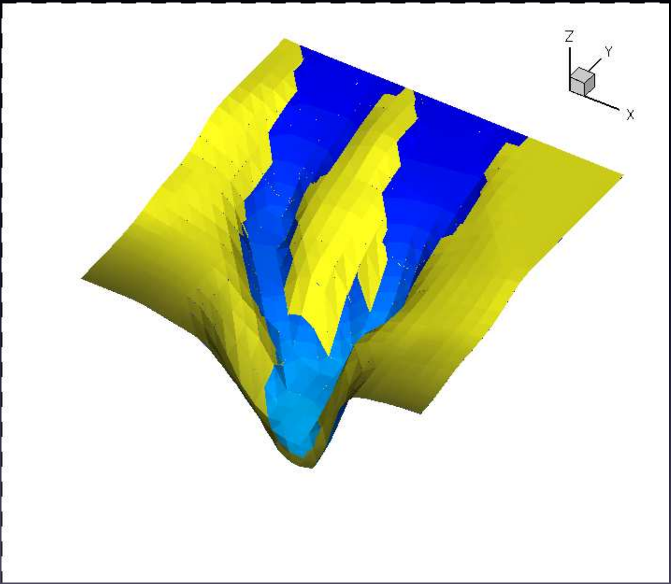
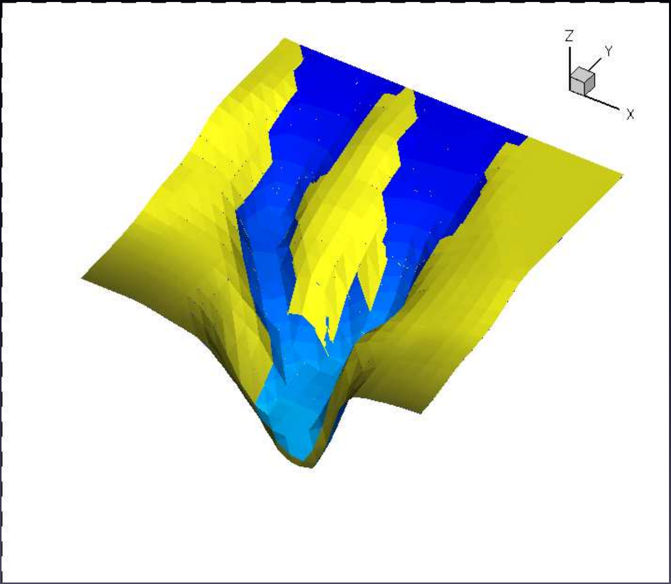


Steady flow solutions for stratigraphy



Steady flow solutions for stratigraphy



Morphodynamic modelling

- Prediction of changes of the topography can be done by integrating a mathematical model by modules.
- **Multi–scale problem**, different physical mechanisms acting according to their time response.
- Relevant mechanisms: [Tassi P., 2007; Tassi *et al.*, 2008]
  - **Hydrodynamics**, conservation laws of mass and momentum.
  - **Sediment transport**, predictors for river sediment carrying capacity.
  - **Bed evolution**, conservation law for sediment mass.

Morphodynamic modelling

Exner' equation of conservation of bed sediment

Bed evolution model: conservation law for sediment mass and closure relationship for sediment transport.

Scaled equation

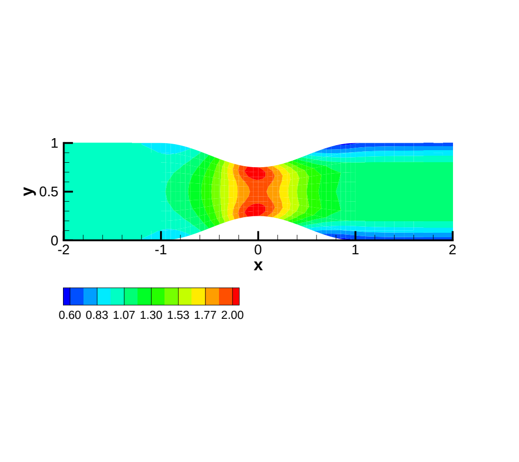
$$\frac{\partial b}{\partial t} + \nabla \cdot \mathbf{q}(\mathbf{u}, b) = 0,$$

Closure for sediment transport:

$$\mathbf{q} = (q_x, q_y)^T = |\mathbf{u}|^{\beta} \left(\frac{\mathbf{u}}{|\mathbf{u}|} - \kappa \nabla b \right),$$

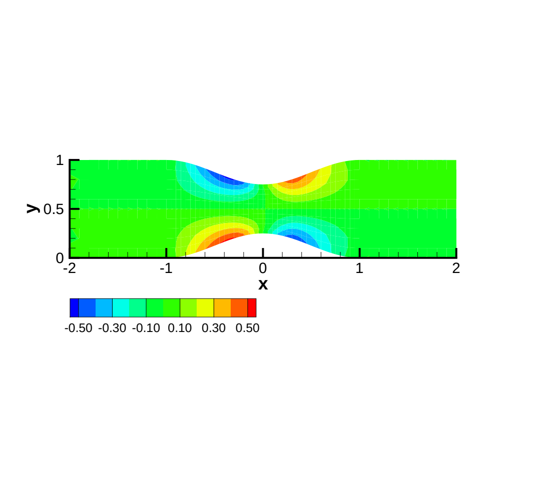
β and κ coefficients, and downhill rolling sediment.

Morphodynamic modelling



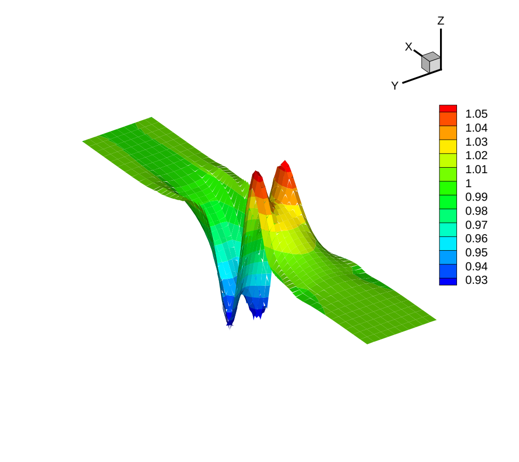
Streamwise discharge $hu(\mathbf{x})$

Morphodynamic modelling



Crosswise discharge $hv(\mathbf{x})$

Morphodynamic modelling



Topography evolution $b(\mathbf{x})$

Morphodynamic modelling

$b(\mathbf{x})$

Conclusions and outlook

- We have designed and implemented a well-balanced, robust and efficient DG scheme for the shallow water equations;
- Our scheme is able to simulate the flow over complex topography and dry lands;
- A simple algorithm to deal with flooding and drying is proposed;
- We presented the evolution of a loose sediment bed in shallow flows using DG methods;
- Subsequent work: topography evolution/sediment transport in large domains.

Questions?

Ern A., Piperno S. and Djadel K.(2008)
A well-balanced Runge–Kutta Discontinuous Galerkin method for the Shallow-Water Equations with flooding and drying.
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Numerical modelling of river processes: flow and river bed deformation
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Thank You