SUSHI for a Bingham Flow Model

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Online Presentation

Aim of the presentation

- In this talk, we briefly present the Bingham fluids and their properties.
- We consider a simple Bingham Flow type equation in a Cylinder.
- We establish a finite volume scheme approximating this problem and prove its convergence.

Plan of the presentation

Some references

1. W. Aboussi, F. Benkhaldoun, A. Aberqi, and A. Bradji. Homogeneous incompressible Bingham viscoplastic as a limit of bi-viscosity fluids. In preparation.
2. J. Baranger, A. Machmoum. Existence of approximate solutions and error bounds for viscoelastic fluid flow: characteristics method. Comput. Methods Appl. Mech. Engrg. 148/1-2, 39–52 (1997).
3. E-J. Dean, R. Glowinski, G. Guidoboni. Review on the numerical simulation of Bingham visco-plastic flow: Old and new results. Journal of Non-Newtonian Fluid Mechanics 142/1-3, 36–62 (2007).
4. R. Eymard, T. Gallouët, R. Herbin. Discretization of heterogeneous and anisotropic diffusion problems on general nonconforming meshes. IMA J. Numer. Anal. 30/4, 1009–1043 (2010).

Non-Newtonian fluids

- As well known, the classical form of Navier Stokes equation is restricted to fluids whose stress-strain relationship is linear. This category of fluids is called Newtonian fluids and they have a simple molecular structure (e.g. water, air, and alcohol).
- To study more complex fluids, such as molten plastics, biological fluids, paints, and greases, etc., it is necessary to consider a generalized Navier Stokes system that models the behavior of fluids whose viscosity depends on the rate of deformation (i.e., non-Newtonian fluids).

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This complex behavior is translated into a mathematical complexity which gives rise to complex stress-strain laws, such as the Carreau-Yasuda, Bingham, power law, Cross, Casson, Herschel-Bulkley, etc..

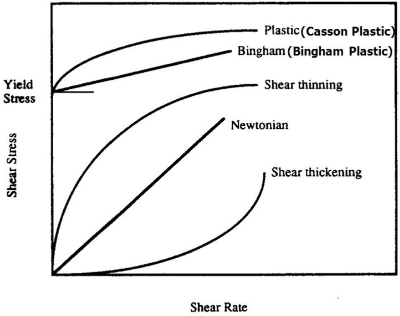

Figure: Examples of Non-Newtonian fluid models.

- Among the various classes of non-Newtonian materials, those exhibiting viscoplastic properties are particularly interesting by their ability to strain only if the stress rate exceeds a minimum value.
- The most commonly used model to account for this particular behavior is the Bingham model.

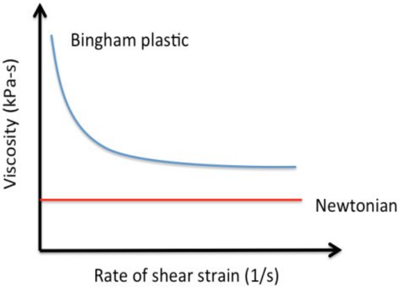

Figure: Bingham viscosity.

Bingham Navier Stokes equations

Let Ω be a smooth domain in $\mathbb{R}^d$ and Ω_T the open set $\Omega \times (0,T)$, where $T > 0$ is the final time.

$$\begin{cases} \partial_t u + (u \cdot \nabla) u - \nabla \cdot (\tau(Du)) + \nabla p = f & \text{in } \Omega_T, \\ \nabla \cdot u = 0 & \text{in } \Omega_T. \end{cases} \tag{1}$$

Here u is the velocity vector, p is the pressure, and τ is the stress tensor where the strain tensor is defined as

$$Du = \frac{1}{2}(\nabla u + \nabla u^t),$$

and $f : \Omega_T \to \mathbb{R}^d$ represents the external forces (such as gravity).

The system [1] is equipped with the following initial condition

$$u(\cdot, 0) = u_0 \quad \text{in } \Omega, \tag{2}$$

and the homogeneous Dirichlet boundary condition

$$u = 0 \quad \text{on } \partial\Omega \times (0, T)\cdot \tag{3}$$

The Bingham stress strain constitutive law is defined as

$$\begin{cases} \tau(Du) = 2\mu Du + \sqrt{2} g \frac{Du}{|Du|} & \text{if } \mid \tau \mid > g, \\ \\ Du = 0 & \text{if } \mid \tau \mid \leq g\cdot \end{cases} \tag{4}$$

Here μ is the viscosity, g is the yield stress and $\mid A \mid^2 = A : A$ where the inner product is defined as $A : B = \sum_{i,j} A_{ij} B_{ij}$.

Axial flow in infinitely cylinder

Let us consider the following simplified Bingham flow in cylinders introduced in Glowinski et al. (2007):

$$\rho u_t(\boldsymbol{x}, t) - \mu \Delta u(\boldsymbol{x}, t) - g \nabla \cdot \left(\frac{\nabla u}{|\nabla u|} \right) (\boldsymbol{x}, t) = C(\boldsymbol{x}, t), \quad (\boldsymbol{x}, t) \in \Omega \times (0, T), \tag{5}$$

with initial and Dirichlet boundary conditions:

$$u(\boldsymbol{x}, 0) = u^0(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega \quad \text{and} \quad u(\boldsymbol{x}, t) = 0, \quad (\boldsymbol{x}, t) \in \partial\Omega \times (0, T). \tag{6}$$

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Regularization problem

As we can see that there is no sense to the previous equation when $\nabla u = 0$. A regularize problem can be given by, for $(\boldsymbol{x}, t) \in \Omega \times (0, T)$

$$\rho \partial_t u_\varepsilon(\boldsymbol{x}, t) - \mu \Delta u_\varepsilon(\boldsymbol{x}, t) - g \nabla \cdot \left(\frac{\nabla u_\varepsilon}{\sqrt{\varepsilon^2 + |\nabla u_\varepsilon|^2}} \right)(\boldsymbol{x}, t) = C(\boldsymbol{x}, t), \tag{7}$$

where

$$u_\varepsilon(\boldsymbol{x}, 0) = u^0(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega \quad \text{and} \quad u_\varepsilon(\boldsymbol{x}, t) = 0, \quad (\boldsymbol{x}, t) \in \partial\Omega \times (0, T). \tag{8}$$

The parameters μ and g denote respectively the viscosity and plasticity.

Existence and uniqueness of the regularization problem

It is proved in Glowinski et al. [3] that the regularization problem can be written in the following variational inequation formulation: $u(t) \in H_0^1(\Omega)$, for $a.e.$ $t \in (0,1)$, for all $v \in H_0^1(\Omega)$:

$$\begin{aligned}\rho \int_\Omega \partial_t u(\boldsymbol{x},t)(v(\boldsymbol{x}) - u(\boldsymbol{x},t))d\boldsymbol{x} + \mu \int_\Omega \nabla u(\boldsymbol{x},t)(\nabla v(\boldsymbol{x}) - \nabla u(\boldsymbol{x},t))d\boldsymbol{x} \\ + \quad g(j(v) - j(u)) \geq C \int_\Omega (v(\boldsymbol{x}) - u(\boldsymbol{x},t))d\boldsymbol{x},\end{aligned} \tag{9}$$

with

$$u(0) = u^0 \quad \text{and} \quad j(v) = \int_\Omega |\nabla v|(\boldsymbol{x})d\boldsymbol{x}. \tag{10}$$

Error estimate betwwen the original problem and its regularization

$$\|u(t) - u_\varepsilon(t)\|_{L^2(\Omega)} \leq \sqrt{\frac{\mathrm{gm}(\Omega)}{\mu\lambda_0}} \left(1 - \exp\left(-\frac{2\mu\lambda_0}{\rho}t\right)\right)^{\frac{1}{2}} \sqrt{\varepsilon}. \tag{11}$$

A simplified problem

A simplified problem

- Equation

$$u_t(\boldsymbol{x},t) - \alpha\Delta u(\boldsymbol{x},t) - \nabla\cdot\mathcal{F}(\nabla u)(\boldsymbol{x},t) = \mathcal{G}(\boldsymbol{x},t), \quad (\boldsymbol{x},t)\in\Omega\times(0,T), \tag{12}$$

where Ω is an open bounded polyhedral subset in $\mathbb{R}^d$, with $d\in\mathbb{N}^\star=\mathbb{N}\setminus\{0\}$, $T>0$, $\mathcal{F}$ is a function defined on $\mathbb{R}^d$ into $\mathbb{R}^d$, and $\mathcal{G}$ is a real function defined on $\Omega\times(0,T)$.

- Initial condition is given by, for a given function u^0 defined on Ω

$$u(0) = u^0. \tag{13}$$

- Homogeneous Dirichlet boundary conditions are given by

$$u(\boldsymbol{x},t) = 0, \qquad (\boldsymbol{x},t)\in\partial\Omega\times(0,T). \tag{14}$$

Assumptions on the data

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The function $\mathcal{F}$ is assumed to satisfy:

- $\mathcal{F} \in \mathcal{C}^1(\mathbb{R}^d)$. We set $\mathcal{F} = (F_1, \ldots, F_d)$, we assume that, for some $M > 0$,

$$F_1, \ldots, F_d \text{ and their first derivatives are bounded by } M. \tag{15}$$

- For all $\theta_1, \theta_2 \in \mathbb{R}^d$

$$(\mathcal{F}(\theta_1) - \mathcal{F}(\theta_2), \theta_1 - \theta_2)_{L^2(\Omega)^d} \geq 0. \tag{16}$$

Principles of Finite Volume methods

Principles of Finite Volume methods

- Subdivision of the spatial domain into subsets called Control Volumes.
- Integration of the equation to be solved over the Control Volumes.
- Approximation of the derivatives appearing after integration.

Nonconforming mesh

Definition (New mesh of Eymard et *al.*, IMAJNA 2010):

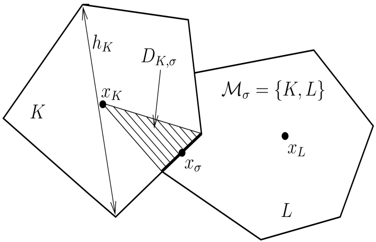

Figure: Notations for two neighbouring control volumes in $d = 2$

Nonconforming mesh

Main properties of this new mesh:

1. (mesh defined at any space dimension): $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$
2. (orthogonality property is not required): the classical orthogonality property is not required in this new mesh. But, additional discrete unknowns are required.
3. (convexity): the classical admissible mesh should satisfy that the control volumes are convex, whereas the convexity property is not required in this new mesh.

SUSHI scheme for Poisson equation

Principles of discretization for Poisson's equation:

1. Discrete unknowns: the space of solution as well as the space of test functions are in

$$\mathcal{X}_{\mathcal{D},0} = \{ \big((v_K)_{K\in\mathcal{M}} \,,\, (v_\sigma)_{\sigma\in\mathcal{E}} \big) \,,\;\; v_K, v_\sigma \in \mathbb{R},\;\; v_\sigma = 0,\, \forall \sigma \in \mathcal{E}_{\text{ext}} \}$$

2. Discretization of the gradient: the discretization of ∇ can be performed using a stabilized discrete gradient denoted by $\nabla_{\mathcal{D}}$, see Eymard et *al.* (IMAJNA, 2010):
    1. The discrete gradient $\nabla_{\mathcal{D}}$ is stable
    2. The discrete gradient $\nabla_{\mathcal{D}}$ is consistent.

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SUSHI scheme for Poisson equation

Discrete gradient

For $u \in \mathcal{X}_{\mathcal{D}}$, we define, for all $K \in \mathcal{M}$

$$\nabla_{\mathcal{D}} u(\boldsymbol{x}) = \nabla_K u + \left(\frac{\sqrt{d}}{d_{K,\sigma}} \left(u_\sigma - u_K - \nabla_K u \cdot (\boldsymbol{x}_\sigma - \boldsymbol{x}_K) \right) \right) \mathbf{n}_{K,\sigma}, \quad \text{a.e. } \boldsymbol{x} \in \mathcal{D}_{K,\sigma}, \tag{17}$$

where $\nabla_K u = \dfrac{1}{\mathrm{m}(K)} \displaystyle\sum_{\sigma \in \mathcal{E}_K} \mathrm{m}(\sigma) \left(u_\sigma - u_K \right) \mathbf{n}_{K,\sigma}$.

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SUSHI scheme for Poisson equation

Weak formulation for Poisson's equation:

Find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u(\boldsymbol{x}) \cdot \nabla v(\boldsymbol{x}) d\boldsymbol{x} = \int_\Omega f(\boldsymbol{x}) v(\boldsymbol{x}) d\boldsymbol{x}, \quad \forall v \in H_0^1(\Omega). \tag{18}$$

SUSHI (Scheme Using stabilized Hybrid Interfaces) for Poisson's equation:

Find $u_{\mathcal{D}} \in \mathcal{X}_{\mathcal{D},0}$ such that

$$\int_\Omega \nabla_{\mathcal{D}} u_{\mathcal{D}}(\boldsymbol{x}) \cdot \nabla_{\mathcal{D}} v(\boldsymbol{x}) d\boldsymbol{x} = \int_\Omega f(\boldsymbol{x}) v(\boldsymbol{x}) d\boldsymbol{x}, \quad \forall v \in \mathcal{X}_{\mathcal{D},0}. \tag{19}$$

SUSHI scheme for Poisson equation

Theorem

Assume that the exact solution u satisfies $u \in \mathcal{C}^2(\overline{\Omega})$. Then the following convergence result hold:

1. *H_0^1-error estimate*

$$\|\nabla u - \nabla_{\mathcal{D}} u_{\mathcal{D}}\|_{L^2(\Omega)^d} \leq Ch\|u\|_{2,\overline{\Omega}}. \tag{20}$$

2. *L^2-error estimate:*

$$\|u - \Pi_{\mathcal{M}} u_{\mathcal{D}}\|_{L^2(\Omega)} \leq Ch\|u\|_{2,\overline{\Omega}}. \tag{21}$$

Assumptions on the unknowns of the edges

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- We associate any $\sigma \in \mathcal{E}_{\text{int}}$ with a family of real numbers $\left(\beta_\sigma^K\right)_{K\in\mathcal{M}}$ (this family contains in general at most $d+1$ nonzero elements) such that

$$1 = \sum_{K\in\mathcal{M}} \beta_\sigma^K \qquad \text{and} \qquad \boldsymbol{x}_\sigma = \sum_{K\in\mathcal{M}} \beta_\sigma^K \boldsymbol{x}_K. \tag{22}$$

- We then use the finite volume space $\mathcal{H}_\mathcal{D} \subset \mathbb{L}^2(\Omega)$ of functions which are constant on each control volume K of $\mathcal{M}$.
- Then, for any $u \in \mathcal{H}_\mathcal{D}$, we set

$$u_\sigma = \sum_{K\in\mathcal{M}} \beta_\sigma^K u_K \ \ \forall \sigma \in \mathcal{E}_{\text{int}} \quad \text{and} \quad u_\sigma = 0 \ \ \forall \sigma \in \mathcal{E}_{\text{ext}}.$$

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Temporal mesh

Temporal mesh

The time discretization is performed with a uniform mesh with constant step $k = \dfrac{T}{N+1}$ with $N \in \mathbb{N} \setminus \{0\}$. We denote the mesh points by $t_n = nk$, for $n \in [\![0, N+1]\!]$. We denote by ∂^1 the discrete first time derivative given by

$$\partial^1 v^{j+1} = \frac{v^{j+1} - v^j}{k}.$$

Formulation of the scheme

Formulation of the scheme.

1. **Approximation of initial condition (13).** The discretization of initial condition (13) can be performed as: Find $u_{\mathcal{D}}^0 \in \mathcal{H}_{\mathcal{D}}$ such that for all $v \in \mathcal{H}_{\mathcal{D}}$

$$\left(\nabla_{\mathcal{D}} u_{\mathcal{D}}^0, \nabla_{\mathcal{D}} v\right) = -\left(\Delta u^0, v\right)_{\mathbb{L}^2(\Omega)}. \tag{23}$$

2. **Approximation of (12).** For any $n \in [\![0, N]\!]$, find $u_{\mathcal{D}}^{n+1} \in \mathcal{H}_{\mathcal{D}}$ such that, for all $v \in \mathcal{H}_{\mathcal{D}}$

$$\begin{aligned}&\left(\partial^1 u_{\mathcal{D}}^{n+1}, v\right)_{\mathbb{L}^2(\Omega)} + \alpha\left(\nabla_{\mathcal{D}} u_{\mathcal{D}}^{n+1}, \nabla_{\mathcal{D}} v\right)\\ &+ \quad \left(\mathcal{F}(\nabla_{\mathcal{D}} u_{\mathcal{D}}^{n+1}), \nabla_{\mathcal{D}} v\right)_{\mathbb{L}^2(\Omega)^d} = \left(\mathcal{G}(t_{n+1}), v\right)_{\mathbb{L}^2(\Omega)}.\end{aligned} \tag{24}$$

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Main results

Theorem 1

There exists a unique solution to (23)–(24) and the following error estimates hold:

- $L^\infty(L^2)$–error estimate:

$$\max_{n=0}^{n=N+1} \|u(t_n) - u_{\mathcal{D}}^n\|_{\mathbb{L}^2(\Omega)} \leq C(k+h). \tag{25}$$

- $L^2(H^1)$–error estimate:

$$\left(\sum_{n=0}^{n=N+1} k \|\nabla u(t_n) - \nabla_{\mathcal{D}} u_{\mathcal{D}}^n\|^2_{\mathbb{L}^2(\Omega)^d} \right)^{\frac{1}{2}} \leq C(k+h). \tag{26}$$

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Outline of the proof

Theorem 1

- The proof is based on a well developed *a priori* estimate.
- The proof will be detailed in the Full Paper Proceedings.

Conclusion and perspectives

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We considered a nonlinear cell centered finite volume scheme (the unknowns are located on the centers of the control volumes) for a simple Bingham Flow Type equation. This equation is a nonlinear parabolic. The space discretization is performed using SUSHI (Scheme using Stabilization and Hybrid Interfaces) whereas the time discretization is uniform. We first proved the existence and uniqueness of the approximate solution and we proved optimal error estimates, under assumption that the exact solution is smooth, in the discrete norms of $L^\infty(L^2)$ and $L^2(H^1)$.

Perspectives

- Consider a linearized scheme version for the one considered here.
- Extend this work to a thixotropic Bingham model.
- Study other non-Newtonian models as Herschel–Bulkley and Casson.
- Use Crank-Nicolson scheme to improve the order in time

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Thank you for your attention